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ALTITUDE OF 4-REGULAR CIRCULANTS

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Abstract

The *altitude* $\alpha(G)$ of a graph G is the largest integer k such that for each linear ordering f of its edges, G has a (simple) path P of length k for which f increases along its edge sequence. The altitude of G is bounded above by its chromatic index $\chi'(G)$. Sometimes G admits an edge ordering that realizes $\alpha(G)$ and corresponds to an edge colouring in $\chi'(G)$ colours. We obtain a necessary condition for this to be the case for 4-regular graphs with girth four and use this result to show that 4-regular triangle-free circulants have altitude three if and only if they are bipartite. We obtain exact values or bounds for the altitude of other 4-regular circulants.

Keywords: edge ordering, increasing paths, altitude, circulants, edge colourings

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1. Introduction

An *edge ordering* of a graph $G = (V, E)$ is a one-to-one function f from E to the set of positive integers. We denote the set of all edge orderings of G by $\mathcal{F}$. For $f \in \mathcal{F}$, a path of G for which f increases along the edge sequence is called an f*-ascent* of G, and a (k, f)*-ascent* if it has length k. A (k, f)-ascent is *maximal* if it cannot be extended to a $(k+1, f)$-ascent. The *height* $h(f)$ of f is the maximum length of an f-ascent. The *altitude* $\alpha(G)$ of G, first considered by Chvátal and Komlós [5] for complete graphs, is defined by

$$\alpha(G) = \min_{f \in \mathcal{F}} h(f).$$

Thus $\alpha(G)$ is the greatest integer k such that G has a (k, f)-ascent for each edge ordering $f \in \mathcal{F}$. Note that $\alpha(G) \geq 2$ if G has a vertex of degree at least two, and if H is a subgraph of G, then $\alpha(H) \leq \alpha(G)$.

For previous work on the altitude of graphs the reader is referred to [1, 2, 3, 6, 7, 8, 12, 13]. Bialostocki and Roditty [1] obtained a forbidden subgraph characterization of graphs with altitude two. The forbidden subgraphs, also called the minimal altitude

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three graphs, are the odd cycles and six small graphs (with orders ranging from five to seven). Although there does not exist such an elegant or indeed any characterization of graphs with $\alpha = 3$, a characterization of cubic graphs with girth at least five and $\alpha = 3$ is given in [11]. This class includes some snarks, while other snarks (for example the Petersen graph) have $\alpha = 4$.

Bounds for $\alpha(K_n)$ were obtained in [2], [3] and [8], and bounds for α for trees, planar graphs and graphs with bounded arboricity in [12]. It was proved in [11] that graphs with minimum degree at least four and girth at least five have $\alpha \geq 4$.

In this paper we continue the study of graphs with altitude three. We prove a necessary condition for 4-regular graphs with girth four to have edge orderings of height three that correspond to 4-edge colourings. We use this result to show that triangle-free 4-regular circulants have altitude three if and only if they are bipartite. We also investigate the altitude of 4-regular circulants with triangles.

2. Known results and more definitions

Upper bounds for α are established using the following method, also used in [2, 3, 6, 8, 12, 13]. We colour the edges of the graph (not necessarily obtaining a proper edge colouring) in $t \geq 2$ colours. Then we first label all the edges of one colour with consecutive integers, and then the edges of the next colour, etc. In any f-ascent, once we use edges with a next colour, we cannot use edges with one of the previous colours, because such edges have smaller labels. Let G_i be the subgraph of G induced by all edges coloured i. The above remarks, together with Vizing's Theorem on the chromatic index χ' (see e.g. [4, Corollary 8.19]), give the following elementary but useful results which have been used in the above-mentioned publications.

Proposition 1. (i) *For any graph* G, $\alpha(G) \leq \sum_{i=1}^{t} \alpha(G_i)$.

(ii) *If* G *has components* $G_1, ..., G_t$, *then* $\alpha(G) = \max_{i=1}^{t}\{\alpha(G_i)\}$.

(iii) *If* G_i *is* 1*-regular for each* i, *then* $\alpha(G_i) = 1$ *and hence* $\alpha(G) \leq t$.

(iv) *If* G *has maximum degree* Δ, *then* $\alpha(G) \leq \chi'(G) \leq \Delta + 1$.

When determining the height of an edge ordering f of G, it is not actually the labels of the edges of G under f that are of prime importance, but the **relative order** of the labels of the edges incident with each vertex. We therefore define a *bilabelling* g of G to be a labelling of the edges incident with each vertex u of G with the integers $1, ..., \deg u$, i.e. each edge $e = uv$ has a label $g_u(e) \in \{1, ..., \deg u\}$ at u and a label $g_v(e) \in \{1, ..., \deg v\}$ at v, and all integers in the set $\{1, ..., \deg u\}$ occur as labels of the edges incident with u. To simplify notation we write $g(uv)$ for $g_u(uv)$ and $g(vu)$ for $g_v(uv)$; that is,

- $g(uv)$ denotes the label of uv at u while $g(vu)$ denotes the label of uv at v.

Given an edge ordering f and bilabelling g of G, if g has the property that for each vertex u and all edges uv_i incident with u, $g(uv_i) < g(uv_j)$ if and only if $f(uv_i) < f(uv_j)$, $i, j \in \{1, ..., \deg u\}$, we call g the *essence* $\varepsilon(f)$ of f. A bilabelling of G is *essential* if it is the essence of some edge ordering f of G. Note that $\varepsilon(f)$ is uniquely determined by f, but an essential labelling may be the essence of several distinct edge orderings. An edge ordering f of $K_{2,3}$ of height three (in

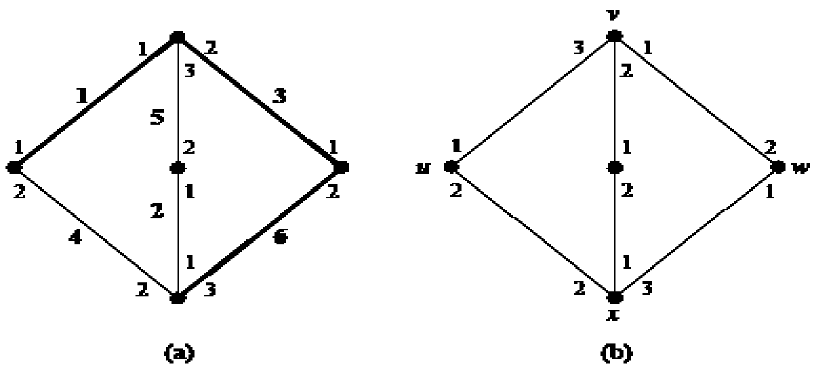

Figure 1: An edge ordering of $K_{2,3}$ and its essence, and a nonessential bilabelling of $K_{2,3}$

bold), a $(3,f)$-ascent and the essence of f are given in Figure 1(a). It is easy to see that the height of an edge ordering is determined by the height (defined in the obvious way) of its essence (see [11]).

For a bilabelling g, an edge uv is called a *switch* if $g(uv) \neq g(vu)$, otherwise it is called a *carrier*. In particular we say uv is a k - l *-switch* if $k \neq l$ and $g(uv) = k$, $g(vu) = l$, and a k *-carrier* if $g(uv) = g(vu) = k$. When we wish to emphasize that uv is a k - l *-switch* (k *-carrier*) with respect to the bilabelling g, we say that uv is a g - k - l *-switch* (g - k *-carrier*). For example, edge 4 in Figure 1(a) is a 2-carrier, and 3 is a 1-2-switch.

If H is a subgraph of G and g is a bilabelling of G, then the *bilabelling of* H *induced by* g is the bilabelling g_H of H obtained by relabelling the edges incident with each vertex u of H with the integers $1, 2, ..., \deg_H u$ such that $g_H(uv_i) < g_H(uv_j)$ whenever $g(uv_i) < g(uv_j)$.

Not all bilabellings are essential. For example, let $V(K_3) = \{u, v, w\}$ and let g be the bilabelling with $g(uv) = g(vw) = g(wu) = 1$, $g(vu) = g(wv) = g(uw) = 2$. Then any possible edge ordering f with essence g satisfies $f(uv) > f(vw) > f(wu) > f(uv)$, which is absurd. Essential labellings are characterized in [11].

Theorem 2. [11] *The bilabelling* g *of* G *is essential if and only if every cycle* C *of* G *with induced bilabelling* g_C *has a* g_C - 1 *-carrier.*

Using Theorem 2 it is easy to see that the bilabelling in Figure 1(b) is not essential, as the induced bilabelling of the cycle u, v, w, x has no 1-carrier.

We also need the following well-known results. By Vizing's Theorem the chromatic index χ' satisfies $\Delta \leq \chi' \leq \Delta + 1$. Graphs for which the lower (upper) bound holds are called class 1 (class 2) graphs respectively.

Theorem 3. (i) *Every nonempty regular graph of odd order is of class two.* (See for example [4, Exercise 8.23].)

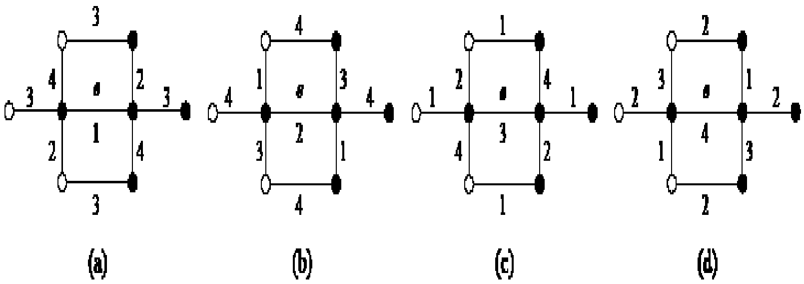

Figure 2: Configuration for each edge e of G

(ii) *Every regular bipartite graph is of class one.* (See for example [4, Theorem 9.18].)

3. Edge colourings

Edge orderings are in fact edge colourings (in which all edges receive different colours), hence essential labellings also correspond to edge colourings. We shall assume throughout that there is a linear order on the colours in any edge colouring. An essential labelling in which all edges are carriers is an edge colouring in Δ colours and the graph is of class one. Conversely, Theorem 2 shows that any edge colouring corresponds to an essential labelling. In particular, if G is an r-regular graph and c is an r-edge colouring of G, then c is an essential labelling in which all edges are carriers.

It is not known when a graph G admits an edge colouring in $\chi'(G)$ colours of height $\alpha(G)$. For 4-regular graphs with girth four we prove the following result, which we use in Section 4.2 to determine the altitude of some classes of 4-regular circulants.

Theorem 4. *If G is a 4-regular graph with girth four and $c : E(G) \to \{1,2,3,4\}$ is a 4-edge colouring of G with $h(c) = 3$, then G is bipartite.*

Proof. Assume G and c satisfy the hypothesis of the statement. Let F be the graph in Figure 2, where the only possible other edges of F join a white vertex to a black vertex. The *central edge* of F is the edge e joining the two grey vertices of degree four. We first prove

Lemma 4.1 *Each edge of G occurs as central edge of a subgraph of G isomorphic to F. For each colour of e, the colours of the other edges are as indicated.*

Proof of Lemma 4.1. Let $e = uv$ be any edge of G and suppose firstly that $c(e) = 1$. Then u (respectively v) is adjacent to vertices u_i (respectively v_i), $i = 2, 3, 4$, such that $c(uu_i) = c(vv_i) = i$ and $u_i \neq u_j$ because G is triangle-free. Now v, u, u_2 is a 2-ascent which extends to an increasing trail λ of length four since G is 4-regular

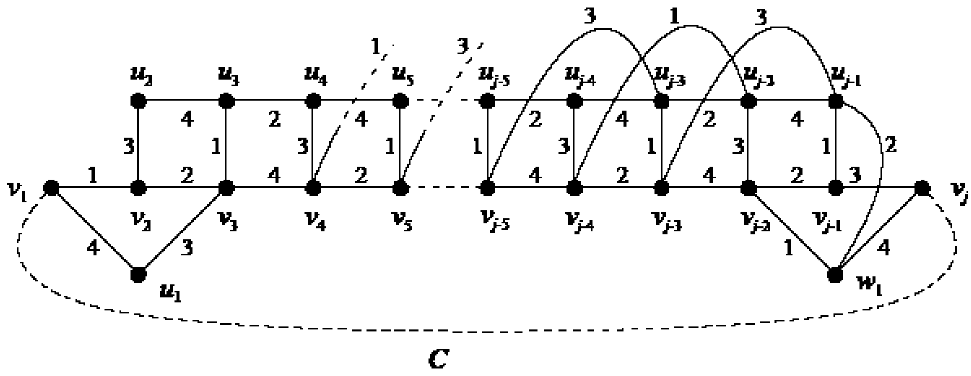

Figure 3: The edge w_1u_{j-1} cannot receive colour 2

and c is a 4-edge colouring. But $h(c) = 3$, hence λ is a 4-cycle and the only possibility is $\lambda = v, u, u_2, v_4, v$. Similarly, the 2-ascent u, v, v_2 extends to the 4-cycle u, v, v_2, u_4, v. Thus $G\langle\{u, v, u_2, u_3, u_4, v_2, v_3, v_4\}\rangle$ is isomorphic to F or to one of its permissible supergraphs. The cases where $c(e) \neq 1$ are similar. $\square$

Suppose to the contrary that G contains an odd cycle. Then G contains a smallest chordless odd cycle C of length at least five. Since $\alpha(C_{2n+1}) = 3$, C contains a 3-ascent λ. In this proof, when referring to a k-ascent by the colours $c_1, c_2, ..., c_k$ of its edges, we shall denote the k-ascent by $(c_1, c_2, ..., c_k)$. If $\lambda = (1, 2, 3)$, then by the regularity of G and the fact that c is a 4-edge colouring with $h(c) = 3$, C contains a chord coloured 4, which is not the case, so $\lambda \neq (1, 2, 3)$; similarly $\lambda \neq (2, 3, 4)$. Hence $\lambda = (1, 2, 4)$ or $\lambda = (1, 3, 4)$; the two cases being symmetric (upon reversing the order of the colours) we may assume without loss of generality that $\lambda = (1, 2, 4)$.

Let $v_1, v_2, ..., v_{2k+1}, v_1$ be the vertex sequence of C, where $c(v_1v_2) = 1$, $c(v_2v_3) = 2$, $c(v_3v_4) = 4$. The minimality of C and Lemma 4.1 imply the existence of vertices $u_i \in V(G) - C$, $i = 1, 2, 3, 4$, such that (see Figure 3)

$$\begin{aligned} u_1v_1, u_1v_3 \in E(G) &\text{ with } c(u_1v_3) = 3,\ c(u_1v_1) = 4; \\ u_2v_2 \in E(G) &\text{ with } c(u_2v_2) = 3; \\ u_3v_3, u_2u_3 \in E(G) &\text{ with } c(u_3v_3) = 1,\ c(u_2u_3) = 4; \\ u_4v_4, u_3u_4 \in E(G) &\text{ with } c(u_4v_4) = 3,\ c(u_3u_4) = 2. \end{aligned}$$

This forms the basis step for the inductive proof of the following statement:

Lemma 4.2 *Beginning with* v_2v_3, *the edges of* C *are coloured alternately* 2 *and* 4, *and for each vertex* v_i, $i \geq 2$, *there is a vertex* $u_i \in V(G) - C$ *such that* $u_iv_i \in E(G)$, *with* $c(u_iv_i) = 1$ *if* i *is odd and* $c(u_iv_i) = 3$ *if* i *is even. Moreover,* $u_iu_{i+1} \in E(G)$, *where* $c(u_iu_{i+1}) = 2$ *if* $c(v_iv_{i+1}) = 4$ *and* $c(u_iu_{i+1}) = 4$ *if* $c(v_iv_{i+1}) = 2$.

Proof of Lemma 4.2. Suppose the statement is true for some value of i, $2 \leq i \leq 2k - 1$, and let $j = i + 1$. First, consider the case $c(v_{j-2}v_{j-1}) = 2$ as in Figure 3.

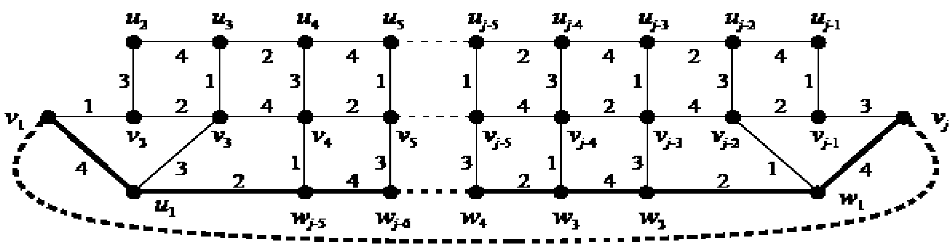

Figure 4: A shorter odd cycle than C shows that $c(v_{j-1}v_j) \neq 3$

Then j is even, so that $c(u_{j-1}v_{j-1}) = 1$. Therefore $c(v_{j-1}v_j) \in \{3, 4\}$. Suppose $c(v_{j-1}v_j) = 3$.

Then $\lambda : v_{j-2}, v_{j-1}, v_j$ is a 2-ascent, $\lambda = (2, 3)$, and v_{j-2} is incident with some edge e with $c(e) = 1$, while v_j is incident with an edge e' with $c(e') = 4$. However, since λ does not extend to a 4-ascent, there exists a vertex w_1 such that $e = (w_1 v_{j-2})$ and $e' = (w_1 v_j)$. Since C is chordless, $w_1 \in V(G) - C$ and since each vertex u_i, $2 \leq i \leq j-1$, is incident with an edge coloured 4, $w_1 \neq u_i$ for all $i \in \{2, 3, ..., j-1\}$.

Consider $\tau : w_1, v_{j-2}, v_{j-3}$. By Lemma 4.1, τ forms part of a 4-cycle $v_{j-2}, w_1, x, v_{j-3}, v_{j-2}$ with $c(w_1 x) = 2$ and $c(x v_{j-3}) = 3$. If $j > 6$, then $x \neq u_1$, and the only vertex already mentioned that could serve as x is u_{j-1}. Say this is the case. Then $\mu : v_{j-4}, v_{j-3}, u_{j-1}, u_{j-2}$ is a 3-ascent with $\mu = (2, 3, 4)$, so that $u_{j-2}v_{j-4} \in E(G)$ and $c(u_{j-2}v_{j-4}) = 1$. Likewise, $u_{j-3}v_{j-5} \in E(G)$ and $c(u_{j-3}v_{j-5}) = 3$. This process of adding edges $u_{j-t}v_{j-t-2}$ continues until we reach v_3, which already has degree four. So we have the 2-ascent $\eta : v_4, u_6, u_5$ with $\eta = (1, 2)$ which does not extend to a 4-cycle as prescribed by Lemma 4.1, a contradiction. Thus $x \neq u_{j-1}$.

So, to complete the 4-cycle $v_{j-2}, w_1, x, v_{j-3}, v_{j-2}$, a new vertex w_2 must be added for x, and $c(w_2 w_1) = 2$, $c(w_2 v_{j-3}) = 3$. See Figure 4. If w_2 lies on C, then $w_2 = v_k$ for some $k \geq v_{j+2}$ or $k = 1$. But then either $v_1, v_2, ..., v_{j-2}, w_1, v_k, v_{k+1}, ..., v_1$ or $v_j, w_1, v_k, v_{k-1}, ..., v_j$ is an odd cycle of length less than C. Hence $w_2 \notin V(C)$.

Once again we have a 2-ascent v_{j-4}, v_{j-3}, w_2 that has to be part of a 4-cycle, and evidently a new vertex $w_3 \notin V(C)$ must be added for this purpose, where $c(w_3 w_2) = 4$ and $c(w_3 v_{j-4}) = 1$. Repeating the argument above with $\tau' : w_3, v_{j-4}, v_{j-5}$ instead of τ we deduce that there exists a vertex w_4, not yet considered, such that $w_4 w_3, w_4 v_{j-5} \in E(G)$ and $c(w_4 w_3) = 2$, $c(w_4 v_{j-5}) = 3$. This process continues until w_{j-5} is added; note that $c(w_{j-5} v_4) = 1$. Now u_1, v_3, v_4 is a 2-ascent which extends to a 4-cycle in which some edge e incident with v_4 is coloured 1. Since $c(w_{j-5} v_4) = 1$, $e = w_{j-5} v_4$. Hence $u_1 w_{j-5} \in E(G)$ and $c(u_1 w_{j-5}) = 2$. But now $v_1, u_1, w_{j-5}, w_{j-6}, ..., w_2, w_1, v_j, v_{j+1}, ..., v_{2k+1}, v_1$ is a cycle in G of length $2k-1$, thus an odd cycle of length less than that of C, contradicting the choice of C as an odd cycle of shortest length.

So $c(v_{j-1}v_j) = 4$. By Lemma 4.1 there exists a vertex u_j such that $u_jv_j, u_ju_{j-1} \in E(G)$ and $c(u_jv_j) = 3, c(u_ju_{j-1}) = 2$. Again $u_j \in V(G) - C$ since C is chordless, and $u_j \neq u_i$ for any $i < j$ because each such u_i is incident with an edge coloured 2 or 3. This completes the first case of the proof.

Now consider the case $c(v_{j-2}v_{j-1}) = 4$, so j is odd. Then v_{j-1} is incident with edges coloured 3 and 4, so $c(v_{j-1}v_j) \in \{1, 2\}$. If $c(v_{j-1}v_j) = 1$, we proceed exactly as in the first part of the proof to obtain an odd cycle of length $2k-1$, a contradiction as before. Hence $c(v_{j-1}v_j) = 2$ and Lemma 4.1 once again asserts the existence of a vertex $u_j \in V(G) - (V(C) \cup \{u_1, u_2, ..., u_{j-1}\})$ such that $u_jv_j, u_ju_{j-1} \in E(G)$ and $c(u_jv_j) = 1$, $c(u_ju_{j-1}) = 4$. This completes the proof of Lemma 4.2. □

Thus, beginning with v_2v_3, the edges of C are coloured alternately 2 and 4. Since C is odd this means that $c(v_{2k+1}v_1) = 4$. But this is impossible as $c(v_1u_1) = 4$. ■

The necessary condition of bipartiteness in Theorem 4 is not sufficient for a 4-regular graph with girth four to have a 4-edge colouring of height three. In fact, we have verified by computer that the 4-regular bipartite graph of girth four obtained by deleting the edges of two disjoint 6-cycles from $K_{6,6}$ has altitude four, hence it has no bilabelling of any kind of height three.

4. Circulants

The *circulant* $C_p\langle a_1, \cdots, a_k\rangle$ with $0 < a_1 < ... < a_k < (p+1)/2$ has p vertices labelled $0, 1, ..., p-1$, and x and y are adjacent if and only if $x - y \equiv \pm a_i \pmod p$ for some i. Hattingh [9] determined the chromatic index of circulants and Heuberger [10] characterized bipartite circulants.

Theorem 5. *Let* $G = C_p\langle a_1, \cdots, a_k\rangle$ *and* $d = \gcd(a_1, ..., a_k, p)$.

- (i) Hattingh [9] *The graph* G *is of class one if and only if* p/d *is even.*
- (ii) Heuberger [10] *The graph* G *is bipartite if and only if there is a nonnegative integer* l *such that* 2^l *divides each of* $a_1, ..., a_k$, $2^{l+1}|p$ *but* $2^{l+1} \nmid a_i$ *for each* $i \in \{1, ..., k\}$.

We only consider connected circulants, that is, those circulants $C_p\langle a_1, \cdots, a_k\rangle$ with $d = \gcd(a_1, ..., a_k, p) = 1$, for otherwise $C_p\langle a_1, \cdots, a_k\rangle$ consists of d disjoint copies of the circulant $C_{p/d}\langle a_1/d, \cdots, a_k/d\rangle$. Hence in the case of 4-regular circulants, we consider $C_p\langle a, b\rangle$ with $p \geq 6$, $a < b$ and $\gcd(a, b, p) = 1$. Then by Theorem 5 (i), $C_p\langle a, b\rangle$ is of class one if and only if p is even, and by Theorem 5 (ii), $C_p\langle a, b\rangle$ is bipartite if and only if p is even and a and b are odd.

Drawings of circulants usually depict the vertices arranged in a regular polygon. For our purposes it will be more convenient to arrange the vertices of 4-regular triangle-free circulants as lattice points of a rectangular grid.

- Let k_1, k_2 and l be the smallest positive integers such that $k_1a \equiv lb \equiv 0 \pmod p$ and $k_2a \equiv b \pmod p$. Let $k = \min\{k_1, k_2\}$ and $q = \operatorname{lcm}(a, b, p)$.

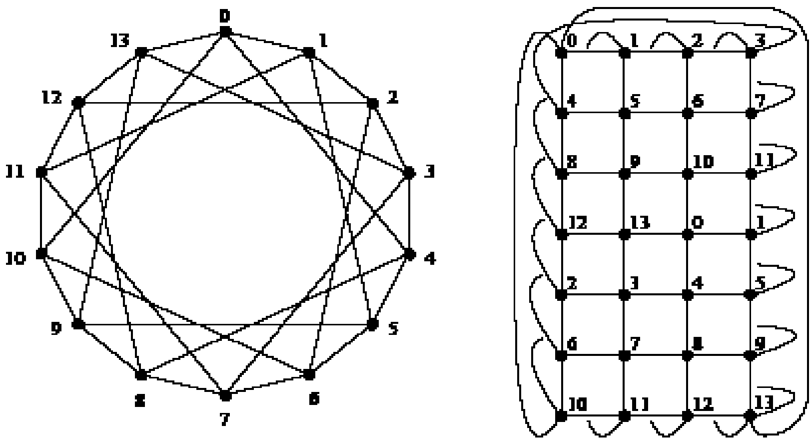

Figure 5: Two representations of $C_{14}\langle 1,4\rangle$

See Figure 5 for two representations of $C_{14}\langle 1,4\rangle$, where $k = k_2 = 4$. Each vertex appears q/p times in the grid, which is arranged as follows, with arithmetic modulo p.

- The vertices in the first row are $0, a, 2a, ..., (k-1)a$,

  those in the second row are $b, b+a, b+2a, ..., b+(k-1)a$,

  etc., and those in the last row are $(l-1)b, (l-1)b+a, (l-1)b+2a, ..., (l-1)b+(k-1)a$.

- The vertices in each row and column form a path as indicated, with the following additional edges.

  Whenever $k = k_2$, the last vertex in each row is adjacent to the first vertex of the next row (the last vertex in the last row being adjacent to the first vertex in the first row).

  When $k = k_1$, the last vertex in each row is adjacent to the first vertex of that row.

  In either case the last vertex in each column is adjacent to the first vertex of that column.

Let H_2 be the graph obtained by joining a vertex of C_4 to a vertex of K_2; this is one of the minimal altitude three graphs in [1], so any graph which contains H_2 as subgraph has altitude at least three. It is easy to see that $C_p\langle a,b\rangle$ contains H_2 and so $\alpha(C_p\langle a,b\rangle) \geq 3$.

In Section 4.1 we show that bipartite 4-regular circulants and the (connected) circulants $C_{2n}\langle a,2a\rangle \cong C_{2n}\langle 1,2\rangle$ have altitude three. In Section 4.2 we consider triangle-free 4-regular circulants and show that these graphs have altitude at least four. It follows that if $C_p\langle a,b\rangle$ is triangle-free, then $\alpha(C_p\langle a,b\rangle) = 3$ if and only if it is bipartite. We briefly consider the remaining 4-regular circulants with triangles in Section 4.3. To avoid confusion with the labels of bilabellings we henceforth denote the vertices $0, ..., p-1$ of $C_p\langle a, b\rangle$ by $v_0, ..., v_{p-1}$.

4.1 Circulants with altitude three

If $C_p\langle a, 2a\rangle$ is connected, then $\gcd(a,p) = 1$ and $C_p\langle a, 2a\rangle \cong C_p\langle 1, 2\rangle$, which has triangles. We now consider the graphs $C_{2n}\langle 1, 2\rangle$, $n \geq 3$, and the bipartite graphs $C_{2n}\langle 2l+1, 2m+1\rangle$, $n \geq 3$, $0 \leq l \leq m < (n-1)/2$.

Theorem 6. (i) *If* $n \geq 3$ *and* $0 \leq l \leq m < (n-1)/2$, *then* $\alpha(C_{2n}\langle 2l+1, 2m+1\rangle) = 3$.
(ii) *If* $n \geq 3$, *then* $\alpha(C_{2n}\langle 1, 2\rangle) = 3$.

Proof. (i) We define a bilabelling, in this case an edge colouring, of $G = C_{2n}\langle 2l+1, 2m+1\rangle$ of height three. For each $i \in \{0, ..., n-1\}$, let $c(v_{2i}v_{2i+2m+1}) = 1$, $c(v_{2i}v_{2i+2l+1}) = 2$, $c(v_{2i+1}v_{2i+2m+2}) = 3$ and $c(v_{2i+1}v_{2i+2l+2}) = 4$ (see Figure 6(a) for the corresponding edge colouring of $C_{12}\langle 1, 3\rangle$). Since G has no odd cycles, c is a proper 4-edge colouring of G. If there is a $(4, c)$-ascent, then its edges are labelled, in order, $1, 2, 3, 4$. But any such edge sequence results in a 4-cycle, which is not allowed. Hence $h(c) = 3$ and since $\alpha(G) \geq 3$, it follows that $\alpha(G) = 3$.

(ii) Note that $C_{2n}\langle 1, 2\rangle$ decomposes into a hamilton cycle and two disjoint copies of C_n. If n is even, colour the edges of the hamilton cycle alternately 2 and 3, and those of the two disjoint even cycles alternately 1 and 4 (see Figure 6(b) for the edge colouring of $C_{12}\langle 1, 2\rangle$). It is routine to check that every increasing walk of length four contains a cycle.

So assume n is odd, say $n = 2k+1$. We describe a bilabelling ε that does not correspond to a 4-edge colouring of $G = C_{4k+2}\langle 1, 2\rangle$. (It gives a 5-edge colouring.) With the hamilton cycle of G drawn as a regular $4k+2$-gon, ε is symmetric with respect to reflection in the line through the centres of the edges v_0v_{4k+1} and $v_{2k}v_{2k+1}$. For the odd cycle $C : v_0, v_2, ..., v_{4k}, v_0$ and beginning with v_0v_2, let the edges be alternately 1-carriers and 4-carriers, except that $v_{4k}v_0$ is a 3-carrier. Label the odd cycle $v_{4k+1}, v_{4k-1}, ..., v_1, v_{4k+1}$ to be a mirror image of the labelling of C. For the hamilton cycle, let v_0v_{4k+1} be a 4-carrier, v_0v_1 and $v_{4k+1}v_{4k}$ 2-1-switches (thus $\varepsilon(v_0v_1) = \varepsilon(v_{4k+1}v_{4k}) = 2$ and $\varepsilon(v_1v_0) = \varepsilon(v_{4k}v_{4k+1}) = 1$) and v_1v_2 and $v_{4k}v_{4k-1}$ 2-carriers. Beginning with v_2v_3, the remaining $4k-3$ edges of the hamilton cycle are labelled, in sequence, 3-carrier, 2-3-switch, 2-carrier, 3-2-switch; since $4k-3 \equiv 1 \pmod 4$, the sequence ends with the 3-carrier $v_{4k-2}v_{4k-1}$. See Figure 6(c) and (d) for the labellings of $C_{10}\langle 1, 2\rangle$ and $C_{14}\langle 1, 2\rangle$. Using Theorem 2 it is easy to check that ε is essential. It is also straightforward to see that all increasing walks of length four contain cycles. ■

4.2 Triangle-free circulants with altitude at least four

Now we consider connected 4-regular triangle-free non-bipartite circulants $C_p\langle a, b\rangle$. For simplicity we assume that $a = 1$; the proof for other values is similar. We first consider circulants with the additional property that their only 4-cycles are those of the form $v_k, v_{k+a}, v_{k+a+b}, v_{k+b}, v_k$ for each $k \in \mathbb{Z}_p$. These 4-cycles

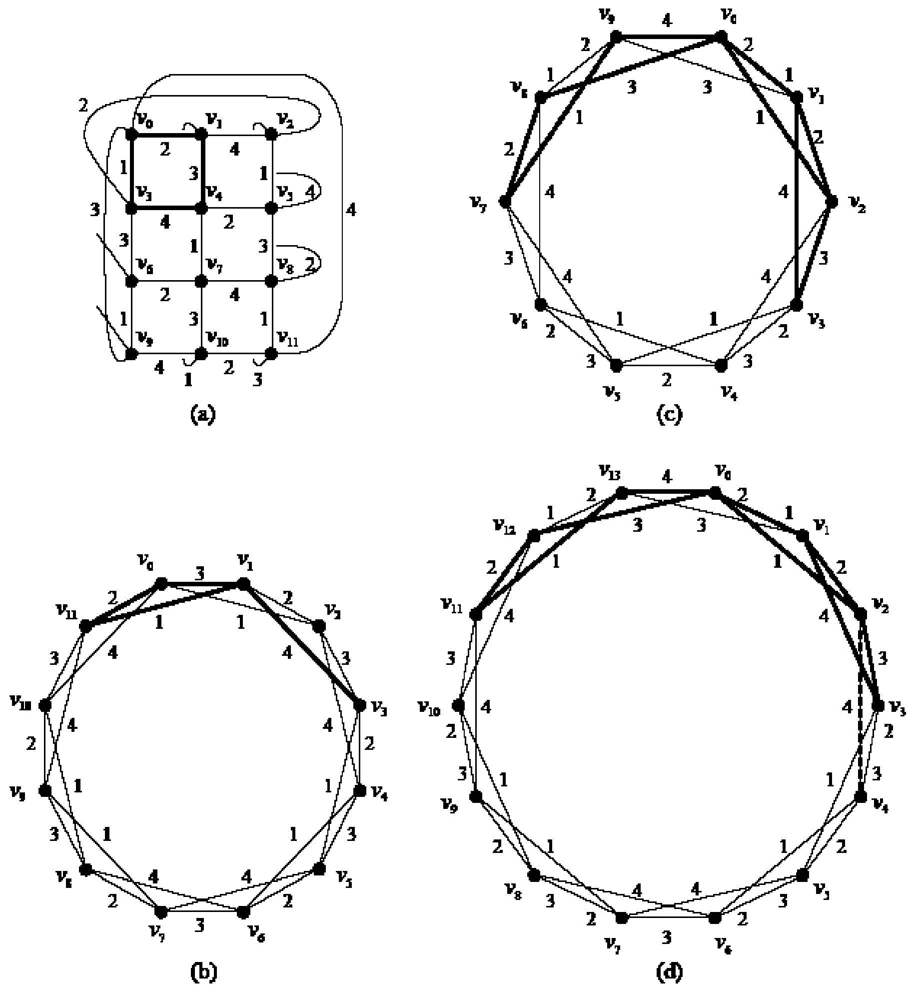

Figure 6: Essential labellings of circulants of height three

necessarily occur in all circulants of degree at least three and correspond to squares of the grid representation; thus we refer to them as *squares*. It is routine to verify the following remark.

Remark 7. *The 4-regular triangle-free non-bipartite circulants $C_p\langle 1,b\rangle$ whose only 4-cycles are squares, are precisely the graphs*

(A) $C_{2n}\langle 1,2m\rangle$, $2\le m<(n-1)/2$, $m\ne n/3,n/4$

$[m=1$ *or* $m=n/3$ *gives triangles;* $m=(n-1)/2$ *or* $m=n/4$ *gives* 4*-cycles other than squares*$]$

(B) $C_{2n+1}\langle 1,2m\rangle$, $2\le m<n/2$, $m\ne n/3$, $m\ne (n+1)/3$

$[m=1$ *or* $m=n/2$ *gives triangles;* $m=n/3$ *or* $m=(n+1)/3$ *gives* 4*-cycles other than squares*$]$

(C) $C_{2n+1}\langle 1,2m+1\rangle$, $2\le m<(n-1)/2$, $m\ne (n-1)/3$

$[m=(n-1)/3$ *or* $m=(n-1)/2$ *gives triangles;* $m=1$ *gives* 4*-cycles*$]$.

Theorem 8. *If G is a 4-regular triangle-free non-bipartite circulant whose only 4-cycles are squares, then $\alpha(G)\ge 4$.*

Proof. Suppose to the contrary that $\alpha(G)=3$ and let ε be an essential bilabelling of G of height three. By Theorem 4 not all edges are carriers. We proceed to show that there are no switches; the resulting contradiction will prove the theorem. We first show that there are no 1-2-switches; the proof that there are no other switches will then follow easily. Consider the grid representation of G.

Suppose there is an ε-1-2-switch e. We only consider the case where e is a horizontal edge; although G is not necessarily edge-transitive the case where e is vertical is similar. Assume without loss of generality that $e=v_0v_1$, where $\varepsilon(v_0v_1)=1$, $\varepsilon(v_1v_0)=2$. Then each edge $v'v_0\ne e$ has higher label at v_0 than e, and some edge $v''v_1$ incident with v_1 has $\varepsilon(v_1v'')=1$. In particular, v'',v_1,v_0,v' is a $(3,\varepsilon)$-ascent for each such v'. We consider two cases, depending on the edge v_1v''. Bear in mind that the squares are the only 4-cycles.

Case 1 $v''=v_2$ (see Figure 7(a)). To avoid the $(4,\varepsilon)$-ascents v_2,v_1,v_0,v',u, where

$$u\in\begin{cases}\{v_{-b-1},v_{-b+1},v_{-2b}\} & \text{if} \quad v'=v_{-b}\\ \{v_{-b-1},v_{-2},v_{b-1}\} & \text{if} \quad v'=v_{-1}\\ \{v_{b-1},v_{b+1},v_{2b}\} & \text{if} \quad v'=v_b\end{cases},$$

$$\varepsilon(v_{-b}v_0)=\varepsilon(v_{-1}v_0)=\varepsilon(v_bv_0)=4.$$

By symmetry we may assume without loss of generality that

$$\varepsilon(v_0v_b)<\varepsilon(v_0v_{-b}).$$

Then for each vertex $v^*\ne v_0$ incident with v_b, the path v^*,v_b,v_0,v_{-b} is a $(3,\varepsilon)$-ascent because $\varepsilon(v_bv^*)<\varepsilon(v_bv_0)=4$. To avoid the $(4,\varepsilon)$-ascents (u,v^*,v_b,v_0,v_{-b}), where

$$u\in\begin{cases}\{v_1,v_{b+2},v_{2b+1}\} & \text{if} \quad v^*=v_{b+1}\\ \{v_{2b+1},v_{3b},v_{2b-1}\} & \text{if} \quad v^*=v_{2b}\\ \{v_{-1},v_{2b-1},v_{b-2}\} & \text{if} \quad v^*=v_{b-1}\end{cases},$$

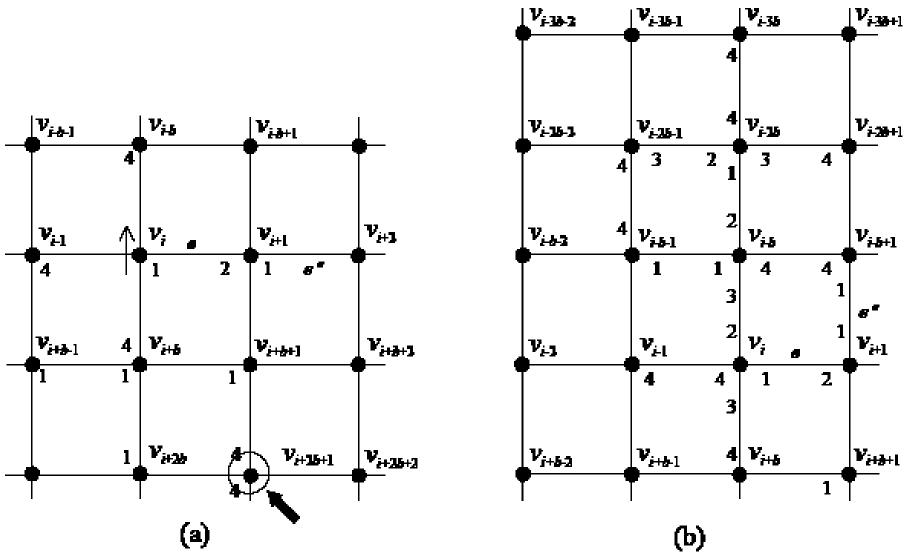

Figure 7: No 1-2-switches are allowed

$$\varepsilon(v_{b+1}v_b) = \varepsilon(v_{2b}v_b) = \varepsilon(v_{b-1}v_b) = 1.$$

Note that $v_b, v_{b-1}, v_{-1}, v_0$ is a $(3, \varepsilon)$ -ascent. Thus

$$\varepsilon(v_b v_{b-1}) < \varepsilon(v_b v_{b+1})$$

to avoid the $(4, \varepsilon)$ -ascent $v_{b+1}, v_b, v_{b-1}, v_{-1}, v_0$. Similarly,

$$\varepsilon(v_b v_{b-1}) < \varepsilon(v_b v_{2b}),$$

so it follows that $\varepsilon(v_b v_{b-1}) = 1$, that is, $v_b v_{b-1}$ is a 1-carrier. But now, to avoid the $(4, \varepsilon)$ -ascents

$$v_{b-1}, v_b, v_{b+1}, v_{2b+1}, w, \text{ where } w \in \{v_{2b+2}, v_{3b+1}, v_{2b}\},$$

and

$$v_{b-1}, v_b, v_{2b}, v_{2b+1}, y, \text{ where } y \in \{v_{2b+2}, v_{3b+1}, v_{b+1}\},$$

it follows that

$$\varepsilon(v_{2b+1}v_{2b}) = \varepsilon(v_{2b+1}v_{b+1}) = 4,$$

which is impossible since adjacent edges have different labels at their common incident vertex. Therefore the 1-2-switch e and its incident edge $v_1 v''$ labelled 1 do not occur in the same row; similarly they do not occur in the same column.

Case 2 Without loss of generality, $v'' = v_{-b+1}$ (see Figure 7(b)). Then, arguing as in Case 1, $\varepsilon(v_{-1}v_0) = \varepsilon(v_b v_0) = 4$, while $\varepsilon(v_{-b}v_0) \geq 3$ (as the increasing sequence $v_{-b+1}, v_1, v_0, v_{-b},\ v_{-b+1}$ is a 4-cycle and thus allowed). Also, considering the 3-ascent v_{-b+1}, v_1, v_0, v_b, it is evident that $v_{-b+1}v_1$ is a 1-carrier.

Suppose $\varepsilon(v_{-b}v_0) = 4$. To ensure that $v_1, v_{-b+1}, v_{-b}, v_0, v_b$ and $v_1, v_{-b+1}, v_{-b}, v_0, v_{-1}$ are not 4-ascents, v_0v_{-b} is a 4-carrier. But then, depending on the ordering of the edges at v_{b-1}, either $v_{-1}, v_{b-1}, v_b, v_0, v_{-b}$ or $v_b, v_{b-1}, v_{-1}, v_0, v_{-b}$ is a 4-ascent, a contradiction.

Hence we assume that $\varepsilon(v_{-b}v_0) = 3$, in which case $\varepsilon(v_{-b}v_{-b+1}) = 4$. Considering the 3-ascent $v_1, v_0, v_{-b}, v_{-b+1}$, and to avoid 4-ascents, it follows that $v_{-b}v_{-b+1}$ is a 4-carrier.

To ensure that $v_{2b}, v_b, v_0, v_{-b}, v_{-b+1}$ does not form a 4-ascent,

$$\varepsilon(v_0v_b) > \varepsilon(v_0v_{-b}).$$

Similarly,

$$\varepsilon(v_0v_{-1}) > \varepsilon(v_0v_{-b}),$$

so that $\varepsilon(v_0v_{-b}) = 2$. Note that $v_{-2b}, v_{-b}, v_0, v_b$ and $v_{-b-1}, v_{-b}, v_0, v_b$ form $(3,\varepsilon)$-ascents. Hence to avoid $(4,\varepsilon)$-ascents,

$$\varepsilon(v_{-2b}v_{-b}) = \varepsilon(v_{-b-1}v_{-b}) = 1.$$

Now $v_{-b}, v_{-b-1}, v_{-1}, v_0$ forms a $(3,\varepsilon)$-ascent, which implies that

$$\varepsilon(v_{-b}v_{-b-1}) = 1 \text{ and } \varepsilon(v_0v_{-1}) = 4;$$

that is, $v_{-b-1}v_{-b}$ is a 1-carrier and $v_{-1}v_0$ is a 4-carrier, the latter implying that v_0v_b is a 3-4-switch. Then $\varepsilon(v_{-b}v_{-2b}) = 2$, so $v_{-b}v_{-2b}$ is a 1-2-switch. The same argument as before now shows that the cycle $v_{-b}, v_{-2b}, v_{-2b-1}, v_{-b-1}, v_{-b}$ has the same labelling as the cycle $v_1, v_0, v_{-b}, v_{-b+1}, v_1$. So, $v_{-b}v_{-2b}$ is a 2-1-switch, $v_{-2b}v_{-2b-1}$ is a 2-3-switch, $v_{-2b-1}v_{-b-1}$ is a 4-carrier and $v_{-b-1}v_{-b}$ is a 1-carrier. By again repeating previous arguments we also find that $v_{-2b}v_{-3b}$ is a 4-carrier like v_0v_{-1}, and $v_{-2b}v_{-2b+1}$ is a 3-4-switch like v_0v_{+b}. However, now $v_1, v_{-b+1}, v_{-2b+1}, v_{-2b}, v_{-3b}$ is a $(4,\varepsilon)$-ascent, a contradiction. This concludes Case 2.

The above two cases show that there are no 1-2-switches. Since each argument for an increasing sequence has a dual argument for a decreasing sequence, it follows that there are also no 3-4-switches.

Suppose v_0v_1 is a 1-3- or a 1-4-switch, where $\varepsilon(v_0v_1) = 1$. Then for each $x \in \{v_{1-b}, v_2, v_{1+b}\}$, x, v_1, v_0, v_{-1} is a 3-ascent, and to avoid extension to a 4-ascent,

$$\varepsilon(v_{1-b}v_1) = \varepsilon(v_2v_1) = \varepsilon(v_{1+b}v_1) = 1.$$

But one of $\varepsilon(v_1v_{1-b})$, $\varepsilon(v_1v_2)$ or $\varepsilon(v_1v_{1+b})$ is equal to 2, thus forming a 1-2-switch, which is impossible. By symmetry there is no 2-4-switch.

Finally, suppose v_0v_1 is a 2-3-switch, where $\varepsilon(v_0v_1) = 2$. Then two of the sequences v_1, v_0, v_{-b}, v_1, v_0, v_{-1} and v_1, v_0, v_b, as well as two of v_{-b+1}, v_1, v_0, v_2, v_1, v_0 and v_{b+1}, v_1, v_0, are 2-ascents. But then (to avoid a 4-ascent) one of the edges $v_{-b+1}v_1$, v_2v_1 or $v_{b+1}v_1$ is a 1-2-switch, the final contradiction which proves the theorem. ■

We have proved results similar to Theorem 8 for circulants with 4-cycles other than squares. The proofs, in all cases similar to that of Theorem 8, are omitted here, but available on request from any of the authors.

Theorem 9. (i) *If* $G = C_{2n}\langle 1, 2m\rangle$, $2m = n - 1 \geq 4$ *, then* $\alpha(G) \geq 4$.
(ii) *If* $G = C_{2n}\langle 1, 2m\rangle$, $2m = 2n/4 \geq 4$ *, then* $\alpha(G) \geq 4$.

We also state the next result without proof.

Theorem 10. (i) *If* $G = C_{2n+1}\langle 1, 2n/3\rangle$ *or* $G = C_{2n+1}\langle 1, 2(n+1)/3\rangle$, $n \geq 5$ *, then* $\alpha(G) \geq 4$.
(ii) *If* $G = C_{2n+1}\langle 1, 3\rangle$, $n \geq 5$ *, then* $\alpha(G) \geq 4$.

We summarize the results on triangle-free 4-regular circulants below. Recall that if $C_p\langle a, b\rangle$ is connected, then $\gcd(a, b, p) = 1$, so $C_p\langle a, b\rangle$ is bipartite if and only if p is even and a and b are odd, and of class one if and only if p is even.

Corollary 11. (i) *If* $C_{2n}\langle a, b\rangle$ *is connected and triangle-free, then it has altitude three if and only if it is bipartite, otherwise it has altitude four.*
(ii) *If* $C_{2n+1}\langle a, b\rangle$ *is connected and triangle-free, then* $4 \leq \alpha(C_{2n+1}\langle a, b\rangle) \leq 5$.

Proof. (i) The result for bipartite graphs was proved in Theorem 6. If $C_{2n+1}\langle a, b\rangle$ is not bipartite then the result follows from Theorems 1 (iv) , 5 (i) , 8 and 9.

(ii) Theorems 1 (iv) , 8 and 10. ■

If $3 \leq a < b$ and $\gcd(a, b) = 1$, the grid representation shows that $C_{ab}\langle a, b\rangle \cong C_a \times C_b$. Although not all products of cycles are circulants, the same proofs as those for Theorems 6 (i) , 8 and 9 also establish the following results.

Proposition 12. (i) *If* $n \geq m \geq 4$ *, then* $\alpha(C_m \times C_n) = 3$ *if* m *and* n *are both even,* $\alpha(C_m \times C_n) = 4$ *if exactly one of* m *and* n *is even, and* $4 \leq \alpha(C_m \times C_n) \leq 5$ *if* m *and* n *are both odd.*
(ii) *If* $n \geq m \geq 2$, m *and* n *not both equal to* 2 *, then* $\alpha(P_m \times P_n) = 3$.

4.3. Circulants with triangles

We proved in Section 4.1 that $\alpha(C_{2n}\langle 1, 2\rangle) = 3$, $n \geq 3$. In general, it is more difficult to determine the altitude of graphs with triangles than of those without. We next consider $C_{2n+1}\langle 1, 2\rangle$, $n \geq 3$. (It is easy to see that if $G = C_5\langle 1, 2\rangle = K_5$, then $\alpha(G) = 3$.)

Theorem 13. (i) *If* $G = C_{4n+3}\langle 1, 2\rangle$, $n \geq 1$ *, then* $3 \leq \alpha(G) \leq 4$ *, and for any* $e \in E(G)$, $\alpha(G - e) = 3$.
(ii) *If* $G = C_{4n+1}\langle 1, 2\rangle$, $n \geq 2$ *, then* $3 \leq \alpha(G) \leq 4$.

Proof. (i) We give a bilabelling ε of $G - e$ of height three for each $e = uv$. Since the edges incident with u , and those incident with v , in $G - e$ are labelled $1, 2, 3$, we may then define $\varepsilon(uv) = \varepsilon(vu) = 4$ to obtain a bilabelling of G of height at most four. By symmetry we consider only the edges $e = v_0v_1$ and $e = v_0v_2$.

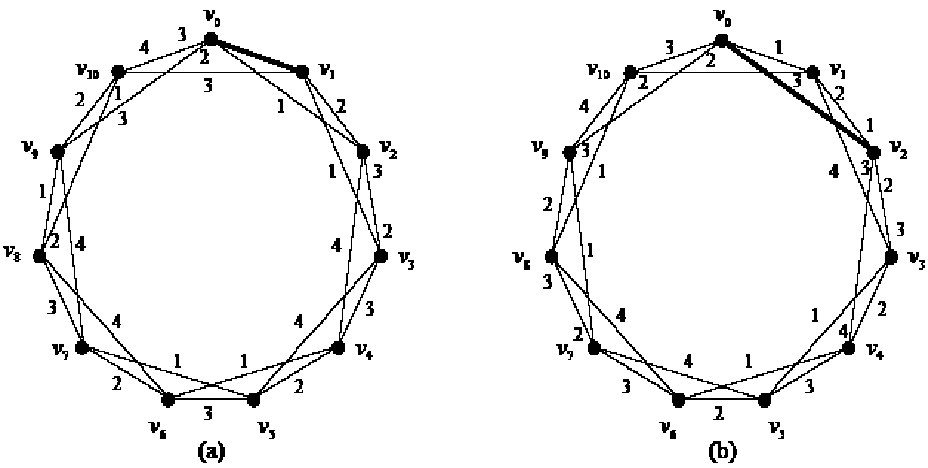

Figure 8: Bilabellings of $C_{11}\langle 1,2\rangle - v_0v_1$ and $C_{11}\langle 1,2\rangle - v_0v_2$ of height three

For $G - v_0v_1$, first consider the path $v_1, v_2, ..., v_{4n+2}, v_0$. Let v_1v_2 and $v_{4n+1}v_{4n+2}$ be ε-2-carriers, $v_{4n}v_{4n+1}$ a 1-carrier, v_2v_3 a 3-2-switch and $v_{4n+2}v_0$ a 4-3-switch. Then, beginning with v_3v_4 and ending with $v_{4n-1}v_{4n}$, let these edges be alternately 3-carriers and 2-carriers. See Figure 8(a) for $C_{11}\langle 1,2\rangle$. Now consider the hamilton cycle $v_0, v_2, ..., v_{4n+1}, v_0$. Beginning with v_0v_2 and ending with $v_{4n-2}v_{4n}$, let these edges be alternately 1-carriers and 4-carriers. Label $v_{4n}v_{4n+2}$ to be a 2-1-switch and $v_{4n+2}v_1$ to be a 3-carrier. Then beginning with v_1v_3 and ending with $v_{4n-1}v_{4n+1}$, let these edges again be alternately 1-carriers and 4-carriers. Finally, let $v_{4n+1}v_0$ be a 3-2-switch.

For $G - v_0v_2$ and the hamilton cycle $v_0, v_1, ..., v_{4n+2}, v_0$, define ε as follows. Label its subpath $v_{4n-1}, v_{4n}, ..., v_3$ with 2-3, 2-2, 4-4, 3-3, 1-1, 2-1, 2-3, where the notation i-j means that $\varepsilon(v_kv_{k+1}) = i$ and $\varepsilon(v_{k+1}v_k) = j$. Then from v_3v_4 to $v_{4n-2}v_{4n-1}$, let these edges be alternately 2-carriers and 3-carriers. For the path $v_2, v_4, ..., v_{4n+1}, v_0$, let v_2v_4 be a 3-4-switch, while $v_{4n+2}v_1$ and v_0v_{4n+1} are 2-3-switches. Then let $v_4v_6, ..., v_{4n}v_{4n+2}$ be alternately 1-carriers and 4-carriers, while $v_1v_3, ..., v_{4n-1}v_{4n+1}$ are alternately 4-carriers and 1-carriers.

It is routine to verify that these bilabellings are essential and have height three.

(ii) In this case we give a bilabelling ε of $G - v_0v_2$ of height three, from which it again follows that $3 \leq \alpha(G) \leq 4$. The bilabelling is illustrated in Figure 9 for $C_{13}\langle 1,2\rangle - v_0v_2$. For the hamilton cycle $v_0, v_1, ..., v_{4n}, v_0$, let v_2v_3 be a 3-4-switch and all the other edges alternately 2-carriers and 3-carriers, where v_3v_4 is a 3-carrier. For the hamilton path $v_2, v_4, ..., v_{4n-1}, v_0$, let the edges on the subpath $v_2, v_4, ..., v_{4n}, v_1$ be alternately 1-carriers and 4-carriers (v_2v_4 is a 1-carrier), v_1v_3 a 1-2-switch and the edges on the subpath $v_3, v_5, ..., v_{4n-1}, v_0$ alternately 1-carriers and 4-carriers, where v_3v_5 is a 1-carrier. Again it is easy to establish that ε is essential and has height three. ■

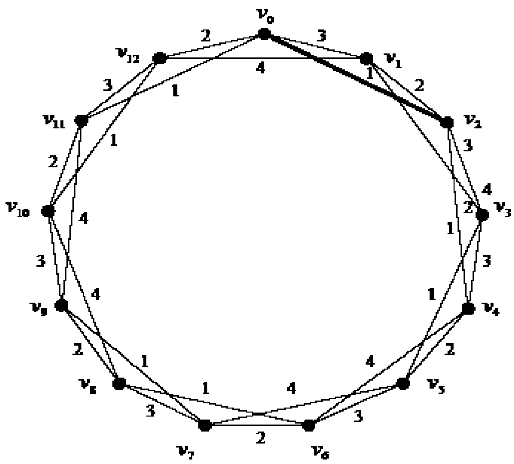

Figure 9: A bilabelling of $C_{13}\langle 1,2\rangle - v_0v_2$ of height three

For $3 \le n \le 16$ we have verified by computer that $\alpha(C_{2n+1}\langle 1,2\rangle) = 4$ and $\alpha(C_{2n+1}\langle 1,2\rangle - v_0v_1) = \alpha(C_{2n+1}\langle 1,2\rangle - v_0v_2) = 3$ and we therefore formulate

Conjecture 1 If $n \ge 3$, then $C_{2n+1}\langle 1,2\rangle$ is a minimal altitude four graph.

The other 4-regular circulants $C_p\langle 1,b\rangle$ that contain triangles are, as stated in Remark 7, $C_{2n}\langle 1,2n/3\rangle$ ($n \ge 6$), $C_{2n+1}\langle 1,n\rangle$, ($n \ge 3$), $C_{2n+1}\langle 1,(2n+1)/3\rangle$ ($n \ge 4$) and $C_{2n+1}\langle 1,n\rangle$ ($n \ge 3$). For small values of n we have also shown by computer that these graphs have altitude four. Thus we further conjecture:

Conjecture 2 If G is a connected 4-regular circulant that contains triangles and $G \ncong C_{2n}\langle 1,2\rangle$, K_5, then $\alpha(G) = 4$.

5. Other open problems

The set of minimal altitude three graphs consists of the odd cycles and six small graphs, given in [1]. Some minimal altitude four graphs are known, for example the Petersen graph, $K_6 - e$, $K_{2,5}$ (the only bipartite one known), and the two graphs obtained from $K_{2,2,3}$ by deleting two adjacent edges, the first pair joining a vertex of degree five to two other vertices of degree five, and the second pair joining a vertex of degree five to one other vertex of degree five and a vertex of degree four. In addition, K_7 is a minimal altitude five graph [2]. The altitude of these graphs and their subgraphs with one edge removed have all been determined by computer.

Problem 1 Determine infinite classes of minimal altitude four graphs.

Problem 2 As stated in Corollary 11, if $C_{2n+1}\langle a,b\rangle$ is connected and triangle-free, then $4 \leq \alpha(C_{2n+1}\langle a,b\rangle) \leq 5$. Determine the exact value of $\alpha(C_{2n+1}\langle a,b\rangle)$. Are any of these graphs minimal with respect to altitude?

Problem 3 The condition in Theorem 4 is necessary but not sufficient for a 4-regular graph of girth four to have a 4-edge colouring of height three. Characterize 4-regular bipartite graphs of girth four with 4-edge colourings of height three. Is this class the same as the class of all 4-regular bipartite graphs of girth four with $\alpha = 3$?

Problem 4 In general, which graphs G have $\chi'(G)$-edge colourings of height $\alpha(G)$?
(It is easy to see that class one cubic graphs have this property, and so do class two cubic graphs with girth at least five [11].)

Problem 5 Cubic graphs with $\alpha = 4$ are of class two, but some class two cubic graphs have $\alpha = 3$. Class two cubic graphs with $\alpha = 4$ and girth at least five were characterized in [11]. Characterize class two cubic graphs with girth and altitude equal to four. Do there exist class two cubic graphs with $\alpha = 4$ that contain triangles?

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References

[1] A. Bialostocki and Y. Roditty, A monotone path in an edge-ordered graph, *Internat. J. Math. & Math. Sci.* **10** (1987), 315-320.

[2] A. P. Burger, E. J. Cockayne and C. M. Mynhardt, Altitude of small complete and complete bipartite graphs, submitted.

[3] A. R. Calderbank, F. R. K. Chung and D. G. Sturtevant, Increasing sequences with non-zero block sums and increasing paths in edge-ordered graphs, *Discrete Math.* **50** (1984), 15-28.

[4] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Third Edition, Chapman & Hall, London, 1996.

[5] V. Chvátal and J. Komlós, Some combinatorial theorems on monotonicity, *Canad. Math. Bull.* **14** (1971), 151-157.

[6] E. J. Cockayne and C. M. Mynhardt, Altitude of $K_{3,n}$, *J. Combin. Math. Combin. Comput.,* to appear.

[7] E. J. Cockayne and C. M. Mynhardt, Complete Bipartite Graphs with Altitude Four, submitted.

[8] R. L. Graham and D. J. Kleitman, Increasing paths in edge ordered graphs, *Period. Math. Hungar.* **3** (1973), 141-148.

[9] J. H. Hattingh, The edge-chromatic number of a circulant, *Quaestiones Math*. **11**(4) (1988), 371–381.

[10] C. Heuberger, On planarity and colorability of circulant graphs, *Discrete Math*. *268* (2003), 153-169.

[11] C. M. Mynhardt, A. P. Burger, T. C. Clark, B. Falvai and N. D. R. Henderson, Altitude of regular graphs with girth at least five, submitted.

[12] Y. Roditty, B. Shoham and R. Yuster, Monotone paths in edge-ordered sparse graphs, *Discrete Math.* **226** (2001), 411-417.

[13] R. Yuster, Large monotone paths in graphs with bounded degree, *Graphs Combin.* **17** (2001), 579-587.