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Vertices belonging to all or to no minimum double dominating sets in trees

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Abstract

In a graph $G = (V, E)$, a vertex dominates itself and all its neighbors. A double dominating set of G is a dominating set that dominates every vertex of G at least twice. In this paper, we characterize vertices that are in all or in no minimum double dominating sets in trees.

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1. Introduction and preliminary results

For a simple graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For a set S, we denote by $\langle S \rangle$ the *subgraph* induced by the vertices of S. A set $S \subseteq V$ is a *dominating set* if for each vertex $v \in V - S$, $N(v) \cap S \neq \emptyset$. For more treatment on domination in graphs, see [7, 8].

A set S is a *double-dominating set*, abbreviated DDS, if for every vertex $v \in V$, $|N[v] \cap S| \geqslant 2$, that is, v is in S and has at least one neighbor in S or v is in $V - S$ and has at least two neighbors in S. The *double-domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality of a double-dominating set of G. We call a double-dominating set of cardinality $\gamma_{\times 2}(G)$ a $\gamma_{\times 2}(G)$*-set.* Double domination was introduced by Harary and Haynes [6] and studied for example in ([1, 3, 6])

For a property P of a vertex subset of a graph G, let $\mu_P(G)$ denote the minimum (or maximum) cardinality of a set with the property P. Many researchers were interested in characterizing the vertices of G that are in all or in no set with the property P and cardinality $\mu_P(G)$. Indeed, Hammer et.al., [5] have characterized those vertices in a graph for independent sets with maximum cardinalities, Mynhardt [9] has characterized the vertices in all or in no minimum dominating set of trees and Cockayne et.al., [4] have characterized the set of vertices contained in all or in no total dominating set of trees (a total dominating set is a dominating set S where $\langle S \rangle$ has no isolated vertex).

In this paper, we investigate vertices belonging to all or to no minimum double-dominating set in a tree. Let us give some definition and notation.

For a tree T we define the sets $\mathcal{A}_{\times 2}(T)$ and $\mathcal{N}_{\times 2}(T)$ by

$$\mathcal{A}_{\times 2}(T) = \{v \in V(T) \mid v \text{ is in every } \gamma_{\times 2}(T)\text{-set}\}, \text{ and}$$
$$\mathcal{N}_{\times 2}(T) = \{v \in V(T) \mid v \text{ is in no } \gamma_{\times 2}(T)\text{-set}\}.$$

The *degree* of a vertex v, denoted by $\deg_G(v)$, is the number of vertices adjacent to v and the *diameter* of G is $diam(G) = \max\{d(x,y) \mid x, y \in V(G)\}$ where $d(x,y)$ is the length of the shortest path between x and y. Specifically, for a vertex v in a rooted tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A *leaf* is a vertex of degree one, while a *support vertex* is a vertex adjacent to a leaf. We denote the set of leaves of T by $L(T)$ and the set of support vertices by $S(T)$. If T is a tree T rooted at a vertex v, then we denote by $L(v)$ the set of leaves of T distinct from v, that is, $L(v) = D(v) \cap L(T)$. Also, a vertex of degree at least three is called a *branch vertex*, and we denote by $B(T)$ the set of such vertices. For a vertex branch w of T_v, we define $P^j(w)$ as the set of leaves $u \in L(w)$ such that $d(w,u) \equiv j \pmod 3$ with $j = 0, 1, 2,$ and every vertex of the $w - u$ path different to w has degree at most two.

We give below some useful observations.

Observation 1. *In any graph G, every DDS contains all support and pendent vertices.*

Observation 2. *If P_n is a path with $n \geqslant 2$, then $\gamma_{\times 2}(P_n) = 2n/3 + 1$ if $n \equiv 0$ $(mod\ 3)$ and $2\lceil n/3 \rceil$ for otherwise.*

Observation 3. *A path P_n with $n \equiv 2 (\mathrm{mod}\, 3)$ has a unique minimum double dominating set.*

The following lemma will be used in the next section.

Lemma 4. *Let T' be a tree and v a vertex of $V(T')$. If T is a tree obtained from T' by adding a path $P_3 = xyz$ and an edge ux, where u is any leaf of T' such that $v \notin N[u]$, then*

(a) $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2$;

(b) $v \in \mathcal{A}_{\times 2}(T')$ *if and only if* $v \in \mathcal{A}_{\times 2}(T)$;

(c) $v \in \mathcal{N}_{\times 2}(T')$ *if and only if* $v \in \mathcal{N}_{\times 2}(T)$.

Proof. (a) By Observation 1, u and its support vertex are in every $\gamma_{\times 2}(T')$-set. Such a set can be extended to a DDS of T by adding $\{y, z\}$, so $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 2$. On the other hand, if D is any $\gamma_{\times 2}(T)$-set, then by Observation 1, $z, y \in D$. Now if $x \notin D$, then $D' = D \cap V(T')$ contains u and is a DDS of T'. If $x \in D$, then $|D'| = |D| - 3$ and x can be replaced in D with u if $u \notin D$ or with its support vertex in T', say w, if $u \in D$. In both cases, the resulting set is a DDS of T' of cardinality $|D| - 2$. Thus $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 2$, implying that $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2$.

(b) Suppose that $v \notin \mathcal{A}_{\times 2}(T')$. Let D' be a $\gamma_{\times 2}(T')$-set that does not contain v. Since $u \in D'$, $D' \cup \{y, z\}$ is a $\gamma_{\times 2}(T)$-set that does not contain v, and so $v \notin \mathcal{A}_{\times 2}(T)$. Conversely, assume that $v \in \mathcal{A}_{\times 2}(T')$ and let D be any $\gamma_{\times 2}(T)$-set with $D' = D \cap V(T')$. If $x \notin D$, then D' is an DDS of T' with $|D'| = |D| - 2$. Hence, D' is a $\gamma_{\times 2}(T')$-set with $v \in D' \subset D$. If $x \in D$, then as discussed in (a), x can be replaced by u or w. Consequently, the resulting set is a $\gamma_{\times 2}(T)$-set that contains v since $v \neq u$ and w. Therefore $v \in \mathcal{A}_{\times 2}(T)$.

(c) Suppose that $v \notin \mathcal{N}_{\times 2}(T')$. Let D' be a $\gamma_{\times 2}(T')$-set that contains v. Clearly, $D' \cup \{y, z\}$ is a $\gamma_{\times 2}(T)$-set containing v so $v \notin \mathcal{N}_{\times 2}(T)$. Conversely, suppose that $v \in \mathcal{N}_{\times 2}(T')$ and let D be any $\gamma_{\times 2}(T)$-set with $D' = D \cap V(T')$. If $x \notin D$, then D' is a $\gamma_{\times 2}(T')$-set. Now if $x \in D$, then as seen in proof of item (a), x can be replaced by u or w. Thus the resulting set minus z and y is a $\gamma_{\times 2}(T')$-set. Now since $v \neq u$ and w and $v \in \mathcal{N}_{\times 2}(T')$, we deduce that $v \in \mathcal{N}_{\times 2}(T)$.

2. The Pruning of a Tree

Let us first define for a tree T rooted at v the set $W^*(T_v)$ by:

$$W^*(T_v) = \left\{w^* \in C(v) |\ D(w^*) \cap B(T_v) = \emptyset,\ \left|P^2(w^*)\right| \geq 2 \text{ and } P^0(w^*) \cup P^1(w^*) = \emptyset\right\}$$

The following straightforward observations will be useful for the next.

Observation 5. *Let T be a tree rooted at v with $|W^*(T_v)| \geq 2$, $C(v) - W^*(T_v) \neq \emptyset$, and let $w^* \in W^*(T_v)$. Then $v \in \mathcal{A}_{\times 2}(T_v)$ (resp. $\mathcal{N}_{\times 2}(T_v)$) if and only if $v \in \mathcal{A}_{\times 2}(T'_v)$ (resp. $\mathcal{N}_{\times 2}(T'_v)$) where T' is the tree obtained from T_v by removing, for every vertex $z \in W^*(T_v) - \{w^*\}$, z and all its descendants, that is $T'_v = T_v - \bigcup_{z \in W^*(T_v) - \{w^*\}} T_z$.*

Observation 6. *Let T be a tree rooted at v and $w^* \in W^*(T_v)$ with $\left|P^2(w^*)\right| \geq 3$. Then $v \in \mathcal{A}_{\times 2}(T_v)$ (resp. $\mathcal{N}_{\times 2}(T_v)$) if and only if $v \in \mathcal{A}_{\times 2}(T'_v)$ (resp. $\mathcal{N}_{\times 2}(T'_v)$) where T'_v is the tree obtained from T_v with replacing $D[w^*]$ with a P_5 of center w^* .*

In order to characterize the sets $\mathcal{A}_{\times 2}(T)$ and $\mathcal{N}_{\times 2}(T)$ for any nontrivial tree T, we will use a technique called *tree pruning* introduced by Mynhart [9] and used later by Cockayne, Henning and Mynhardt [4].

Let v be a vertex of a nontrivial tree T that is neither a support vertex nor a leaf. Using the process described below, with respect to the root v on every branch vertex, the tree T_v is transformed to another tree $\bar{T}_v$, called the pruning of T_v, in which every vertex $u \notin W^*(T_v) \cup \{v\}$ has degree at most two. As a consequence, the properties of the vertex v to be in $\mathcal{A}_{\times 2}(T)$ or $\mathcal{N}_{\times 2}(T)$ will be preserved in $\bar{T}_v$.

Let $T = T_v$ be a nontrivial tree rooted at a vertex v. If every vertex $u \notin W^*(T) \cup \{v\}$ has degree at most two then $\bar{T}_v = T_v$. Otherwise, let u be a branch vertex at maximum distance from v. Then apply the following process:

- If $\left|P^1(u)\right| \geq 1$, then delete $D(u)$ and attach a path P_1 at u.
- If $\left|P^2(u)\right| \geq 1$, $\left|P^0(u)\right| \geq 1$ and $P^1(u) = \emptyset$, then delete $D(u)$ and attach a path P_1 at u.
- If $\left|P^2(u)\right| \geq 2$ and $P^0(u) \cup P^1(u) = \emptyset$, then
  - If $u \in C(v)$, then delete $D(u)$ and attach two paths P_2 at u.
  - If $d(v, u) = 2$ and $p(u) \notin B(T)$, then delete $D(u)$ and attach a path P_2 at u.
  - If either $d(v, u) \geq 3$ or $d(u, v) = 2$ and $p(u) \in B(T)$, then delete $D[u]$.
- If $\left|P^0(u)\right| \geq 2$ and $P^1(u) \cup P^2(u) = \emptyset$, then delete $D(u)$ and attach a path P_3 at u.

To illustrate this technique, we consider the tree in Figure 1.(a) where v, w, u, x, y and z are the branch vertices of T. At this step, w is the the branch vertex at maximum distance from v, since $\left|P^1(w)\right| \geq 1$, we delete $D(w)$ and we attach a path P_1 at w. Then since y is the branch vertex at maximum distance from v and $\left|P^1(y)\right| \geq 1$, we delete $D(y)$ and we attach a P_1 at y. Now it remains three branch vertices z, u and x at distance one from v. Let us consider z. Since $z \in C(v)$, $P^1(z) \cup P^0(z) = \emptyset$ and $|P^2(z)| \geq 2$, we delete $D(z)$ and we attach two paths P_2 at z. Now since $|P^0(x)| \geq 1$, $|P^2(x)| \geq 1$ and $P^1 = \emptyset$, then we delete $D(x)$ and we attach a P_1 at x. Finally since $|P^0(u)| \geq 2$ and $P^1(u) \cup P^2(u) = \emptyset$, we delete $D(u)$ and we attach P_3 at u.

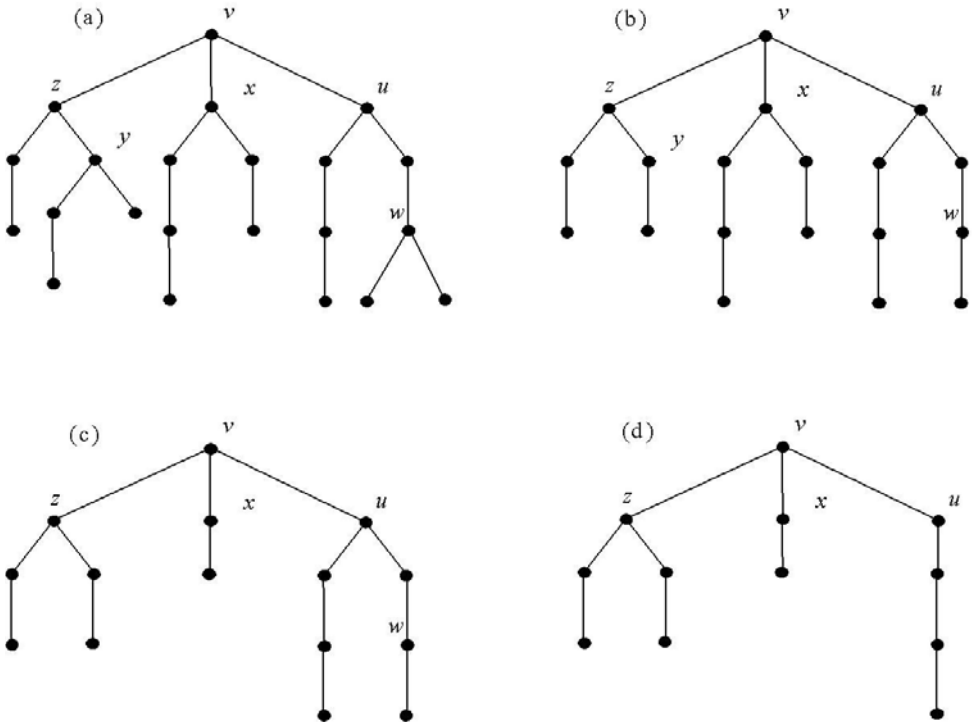

Figure 1. The prunning of a tree rooted at v.

Lemma 7. *Let T be a tree rooted at v and $u \neq v$ a branch vertex at maximum distance from v with $k_1 = |P^1(u)|$, $k_2 = |P^2(u)|$ and $k_3 = |P^0(u)|$. If*

(1) $k_1 \geq 1$, let T' be the tree obtained from T by deleting $D(u)$ and attaching a P_1 to u.

(2) $k_2 \geq 1$, $k_3 \geq 1$ and $k_1 = 0$, let T' be the tree obtained from T by deleting $D(u)$ and attaching a P_1 to u.

(3) $k_2 \geq 2$, $k_1 + k_3 = 0$ and $u \in C(v)$, let T' be the tree obtained from T by deleting $D(u)$ and attaching two paths P_2 to u.

(4) $k_2 \geq 2$, $k_1 + k_3 = 0$, $d(v,u) = 2$ and $p(u) \notin B(T)$, let T' be the tree obtained from T by deleting $D(u)$ and attaching a P_2 to u.

(5) $k_2 \geq 2$, $k_1 + k_3 = 0$, and either $d(v,u) \geqslant 3$ or $p(u) \in B(T) - \{v\}$, let T' be the tree obtained from T by deleting $D[u]$.

(6) $k_3 \geq 2$ and $k_1 + k_2 = 0$, let T' be the tree obtained from T by deleting $D(u)$ and attaching a P_3 to u.

Then for each case we have:

(a) $v \in \mathcal{A}_{\times 2}(T)$ *if and only if* $v \in \mathcal{A}_{\times 2}(T')$.

(b) $v \in \mathcal{N}_{\times 2}(T)$ *if and only if* $v \in \mathcal{N}_{\times 2}(T')$.

Proof. We first note that Lemma 4 allows us to reduce the tree T_v by replacing every $u-x$ path of T with a $u-x$ path of length j, where $j = 3, 1, 2$ if $x \in P^i(u)$, $i = 0, 1, 2$, respectively. So we may assume that every leaf of T_u is at distance at most three from u.

Let a_i, $t_j u_j$ and $x_k y_k z_k$ be paths of order 1, 2 and 3 respectively, attached to u where $a_i, u_j, z_k \in L(T) \cap D(u)$, for $0 \le i \le k_1$, $0 \le j \le k_2$ and $0 \le k \le k_3$. We consider the following cases:

Case 1. $k_1 \ge 1$.

Let $T' = T - (D(u) - \{a_1\})$. Since every $\gamma_{\times 2}(T')$-set can be extended to a DDS of T by adding the set $X = \{a_i\} \cup \{t_j, u_j\} \cup \{y_k, z_k\}$ where $i \in \{2, ..., k_1\}$, $j \in \{1, ..., k_2\}$ and $k \in \{1, ..., k_3\}$, $\gamma_{\times 2}(T) \le \gamma_{\times 2}(T') + (k_1 - 1) + 2k_2 + 2k_3$. On the other hand, if D is a $\gamma_{\times 2}(T)$-set with $D' = D \cap T'$ then it is a routine matter to check that $D' = D - X$ is a DDS of T'. Thus $\gamma_{\times 2}(T') \le \gamma_{\times 2}(T) - (k_1 - 1) - 2k_2 - 2k_3$ implying the equality.

(a) Suppose that $v \in \mathcal{A}_{\times 2}(T')$ and let D be a $\gamma_{\times 2}(T)$-set. As seen before $D' = D - X$ is a $\gamma_{\times 2}(T')$-set. Thus $v \in D' \subset D$ and hence $v \in \mathcal{A}_{\times 2}(T)$. Conversely, suppose that $v \in \mathcal{A}_{\times 2}(T)$ and let S' be a $\gamma_{\times 2}(T')$-set. Then $S = S' \cup X$ is a $\gamma_{\times 2}(T)$-set, and so $v \in S$. Now since $v \notin D[u]$, $v \in S'$ and hence $v \in \mathcal{A}_{\times 2}(T')$.

(b) Suppose that $v \in \mathcal{N}_{\times 2}(T')$ and let D be an arbitrary $\gamma_{\times 2}(T)$-set. We have seen that $D' = D - X$ is a $\gamma_{\times 2}(T')$-set. So $v \notin D'$ and since $v \notin D[u]$, $v \in \mathcal{N}_{\times 2}(T)$. Conversely, suppose that $v \in \mathcal{N}_{\times 2}(T)$ and let S' be a $\gamma_{\times 2}(T')$-set. Since $S = S' \cup X$ is a $\gamma_{\times 2}(T)$-set, $v \notin S$. Since $v \notin D[u]$, $v \notin S'$ and hence $v \in \mathcal{N}_{\times 2}(T')$.

The proof of item (b) will be omitted for the next since it is similar to the proof of item (a)

Case 2. $k_2 \ge 1$, $k_3 \ge 1$ and $k_1 = 0$.

Let $T' = T - (D(u) - \{t_1\})$. Then $\gamma_{\times 2}(T) \le \gamma_{\times 2}(T') + 2(k_2 - 1) + 2k_3 + 1$ because every $\gamma_{\times 2}(T')$-set can be extended to a DDS of T by adding the set $Y = \{u_1, t_j, u_j\} \cup \{y_k, z_k\}$ where $j \in \{2, ..., k_2\}$ and $k \in \{1, ..., k_3\}$. Let D be an arbitrary $\gamma_{\times 2}(T)$-set. If $u \in D$, then $D' = D - Y$ is a DDS of T'. If $u \notin D$, then by minimality $k_3 = 1$, and so $x_1 \in D$. It follows that $D' = (D - (\{u_1, t_j, u_j\} \cup \{x_1, y_1, z_1\})) \cup \{u\}$ where $2 \leqslant j \leqslant k_2$ is also a DDS of T'. Both cases yield $|D'| = |D| - 2(k_2 - 1) - 2k_3 - 1$. Thus $\gamma_{\times 2}(T') \le \gamma_{\times 2}(T) - 2(k_2 - 1) - 2k_3 - 1$ implying the equality.

(a) Suppose that $v \in \mathcal{A}_{\times 2}(T')$ and let D be a $\gamma_{\times 2}(T)$-set. We have seen above that $D' = \{u\} \cup (D - (\{u_1, t_j, u_j\} \cup \{x_1, y_k, z_k\}))$ where $2 \leqslant j \leqslant k_2$ and $1 \leqslant k \leqslant k_3$ is a $\gamma_{\times 2}(T')$-set. Thus $v \in D'$ and since $v \notin D[u]$, $v \in D$. It follows that $v \in \mathcal{A}_{\times 2}(T)$. Conversely, suppose that $v \in \mathcal{A}_{\times 2}(T)$ and let S' be a $\gamma_{\times 2}(T')$-set. Then $S = S' \cup \{u_1, t_j, u_j\} \cup \{y_k, z_k\}$ where $2 \leqslant j \leqslant k_2$ and $1 \leqslant k \leqslant k_3$ is a $\gamma_{\times 2}(T)$-set, so $v \in S$. Since $v \notin D[u]$, $v \in S'$, and so $v \in \mathcal{A}_{\times 2}(T')$.

Case 3. $k_2 \ge 2$, $k_1 + k_3 = 0$ and $u \in C(v)$. Let $T' = T - (D(u) - \{t_1, u_1, t_2, u_2\})$.

This case follows from lemma 4 and observation 6.

Case 4. $k_2 \ge 2$, $k_1 + k_3 = 0$, $d(v, u) = 2$ and $p(u) \notin B(T)$. Let $T' = T - (D(u) - \{t_1, u_1\})$.

Then $\gamma_{\times 2}(T) \le \gamma_{\times 2}(T') + 2(k_2 - 1)$ since any $\gamma_{\times 2}(T')$-set is extended to a DDS of T by adding the set $X = \{t_j, u_j\}$ where $2 \le j \le k_2$. Now let D be an arbitrary $\gamma_{\times 2}(T)$-set. If $u \in D$, then $D-X$ is a DDS of T'. If $u\notin D$, then $p(u)$ must be in D and hence $D-X$ is a DDS of T', so $\gamma_{\times 2}(T')\le\gamma_{\times 2}(T)-2(k_2-1)$. Thus we have $\gamma_{\times 2}(T')=\gamma_{\times 2}(T)-2(k_2-1)$.

(a) Suppose that $v\in\mathcal{A}_{\times 2}(T')$ and let D be an arbitrary $\gamma_{\times 2}(T)$-set. We know that $D'=D-X$ is a $\gamma_{\times 2}(T')$-set. Then $v\in D'\subset D$ and $v\in\ \mathcal{A}_{\times 2}(T)$. Conversely, suppose that $v\in\mathcal{A}_{\times 2}(T)$ and let S' be any $\gamma_{\times 2}(T')$-set. We have seen that $S=S'\cup X$ is a $\gamma_{\times 2}(T)$-set, so $v\in S$. Since $v\notin D[u]$, then $v\in S'$ and $v\in\ \mathcal{A}_{\times 2}(T')$.

Case 5. $k_2\ge 2$, $k_1+k_3=0$, and either $d(v,u)\ge 3$ or $p(u)\in B(T)-\{v\}$. Let $T'=T-D[u]$.

Clearly any $\gamma_{\times 2}(T')$-set can be extended to a DDS of T by adding the set $X=\{t_j,u_j\}$ where $j\in\{1,...,k_2\}$, and so $\gamma_{\times 2}(T)\le\gamma_{\times 2}(T')+2k_2$. Now let D be a $\gamma_{\times 2}(T)$-set. If $u\notin D$, then $D'=D-X$ is a DDS of T'. Assume now that $u\in D$. If $p(u)\notin D$, then $D''=(D-(X\cup\{u\}))\cup\{p(u)\}$ is a DDS of T', else ($p(u)\in D$), then by minimality $D'''=(D-(X\cup\{u\}))\cup\{x\}$ is a DDS of T' where $x\in N(p(u))-\{v,u\}$, and hence $\gamma_{\times 2}(T')\le\gamma_{\times 2}(T)-2k_2$. It follows that $\gamma_{\times 2}(T')=\gamma_{\times 2}(T)-2k_2$.

(a) Suppose that $v\in\mathcal{A}_{\times 2}(T')$ and let D be any $\gamma_{\times 2}(T)$-set. We showed before depending on D that one of D',D'' or D''' is a $\gamma_{\times 2}(T')$-set. Since $v\notin\{p(u),x\}$, v is in one of D',D'' or D''', so $v\in D$ and $v\in\mathcal{A}_{\times 2}(T)$. Conversely, suppose that $v\in\mathcal{A}_{\times 2}(T)$ and let S' be any $\gamma_{\times 2}(T')$-set. We have seen that $S=S'\cup X$ is a $\gamma_{\times 2}(T)$-set, so $v\in S$. Since $v\notin D[u]$, then $v\in S'$ and hence $v\in\ \mathcal{A}_{\times 2}(T')$.

Case 6. $k_3\ge 2$ and $k_1+k_2=0$. Let $T'=T-(D(u)-\{x_1,y_1,z_1\})$.

Let S' be a $\gamma_{\times 2}(T')$-set. If $u\in S'$, then $S=S'\cup X$ where $X=\{y_k,z_k\}$ with $k\in\{2,...,k_3\}$ is a DDS of T. Else, $x_1\in S'$ and since u is dominated twice by x_1 and $p(u)$, then $S=(S'-\{x_1\})\cup X\cup\{u\}$ is a DDS of T. Thus $\gamma_{\times 2}(T)\le\gamma_{\times 2}(T')+2(k_3-1)$. Now let D be any $\gamma_{\times 2}(T)$-set. If $u\in D$ then without loss of generality $x_k\notin D$ for $k\in\{2,...,k_3\}$, and so $D'=D-X$ is a DDS of T'. If $u\notin D$, then by the minimality of D, $k_3=2$ and $\{x_1,x_2\}\subset D$. Thus $D''=(D-\{x_2,y_2,z_2\})\cup\{u\}$ is a DDS of T' of cardinality $\gamma_{\times 2}(T)-2(k_3-1)$ where $k_3=2$. It follows that $\gamma_{\times 2}(T')\le\gamma_{\times 2}(T)-2(k_3-1)$ and so $\gamma_{\times 2}(T')=\gamma_{\times 2}(T)-2(k_3-1)$.

(a) Suppose that $v\in\mathcal{A}_{\times 2}(T')$ and let D be an arbitrary $\gamma_{\times 2}(T)$-set. Then depending on whether u is contained in D or no, D' or D'' respectively is a $\gamma_{\times 2}(T')$-set. Since $v\notin D[u]$, $v\in D'$ (or D'') and hence $v\in D$. It follows that $v\in\mathcal{A}_{\times 2}(T)$. Conversely, suppose that $v\in\mathcal{A}_{\times 2}(T)$ and let S' be a $\gamma_{\times 2}(T')$-set. As seen above S is a $\gamma_{\times 2}(T)$-set, so $v\in S$. Since $v\notin D[u]$, $v\in S'$, so $v\in\ \mathcal{A}_{\times 2}(T')$. $\square$

3. Characterizations

The next theorem gives a necessary and sufficient condition for a vertex v of a nontrivial tree T to be in $\mathcal{A}_{\times 2}(T_v)$ (resp. in $\mathcal{N}_{\times 2}(T_v)$).

Theorem 8. *Let T be a tree rooted at v such that $\deg(u)\le 2$ for every $u\notin W^*(T)\cup\{v\}$. Then*

a) $v\in\mathcal{A}_{\times 2}(T)$ *if and only if at least one of the following conditions is verified:*

- v *is a support vertex;*
- v *is a leaf;*
- $|P^1(v)|\ge 2$;
- $|P^0(v)|\ge 3$;

- $\left|P^1(v)\right| = 1$ *and* $\left|P^0(v)\right| \in \{1,2\}$;
- $\left|P^1(v)\right| = 1$, $W^*(T) \neq \emptyset$ *and* $P^2(v) \cup P^0(v) = \emptyset$;
- $\left|P^0(v)\right| = 2$ *and* $\left|P^2(v)\right| \geq 1$.

b) $v \in \mathcal{N}_{\times 2}(T)$ *if and only if* $\left|P^2(v)\right| \geq 2$ *and* $P^1(v) \cup P^0(v) = \emptyset$.

Proof. By Observation 1, the theorem is valid if v is a support vertex or a leaf. So suppose that v is neither a support vertex nor a leaf. By Lemma 4, the tree T_v can be reduced to a tree T_v^* by replacing each $v-b$ path of T with a $v-b$ path of length j where $j = 3,4,2$ if $b \in P^i(v)$ and $i = 0,1,2$, respectively. Likewise for every $w^* \in W^*(T_v)$ we replace every w^*-b path where $b \in P^2(w^*)$ with a path of length two. Thus every leaf of T_v^* is at distance 2, 3 or 4 from v and hence by Lemma 4, $v \in \mathcal{A}_{\times 2}(T_v)$ (resp. $\mathcal{N}_{\times 2}(T_v)$) if and only if $v \in \mathcal{A}_{\times 2}(T_v^*)$ (resp. $\mathcal{N}_{\times 2}(T_v^*)$).

We first show the sufficient condition. Let D be an arbitrary $\gamma_{\times 2}(T)$-set.

Case 1. $\left|P^1(v)\right| \geq 2$.

Let u and x be two vertices of $P^1(v)$ where $P_u = vu_1u_2u_3u$ and $P_x = vx_1x_2x_3x$. Then by Observation 1, $\{u,u_3,x,x_3\} \subset D$. If $v \notin D$, then $\{u_1,u_2,x_1,x_2\} \subset D$. In this case $D' = \{v\} \cup D - \{u_2,x_2\}$ is a DDS of T of cardinality $|D|-1$, a contradiction. Thus $v \in D$ and so $v \in \mathcal{A}_{\times 2}(T)$.

Case 2. $\left|P^0(v)\right| \geq 3$.

Let u,x and w be three vertices of $P^0(v)$ where $P_u = vu_1u_2u$, $P_x = vx_1x_2x$ and $P_w = vw_1w_2w$. Then $\{u,u_2,x,x_2,w,w_2\} \subset D$. Now if $v \notin D$, then $\{u_1,x_1,w_1\} \subset D$, and hence $D' = \{v\} \cup D - \{x_1,w_1\}$ is a DDS of T of cardinality $|D|-1$, a contradiction. Thus $v \in D$ and so $v \in \mathcal{A}_{\times 2}(T)$.

Case 3. $\left|P^1(v)\right| = 1$ and $\left|P^0(v)\right| \in \{1,2\}$.

Let u and x be two vertices of $P^1(v)$ and $P^0(v)$ where $P_u = vu_1u_2u_3u$ and $P_x = vx_1x_2x$. Then $\{u,u_3,x,x_2\} \subset D$. Now if $v \notin D$, then $\{u_1,u_2,x_1\} \subset D$, and hence $\{v\} \cup D - \{u_2,x_1\}$ is a DDS of T of cardinality less than $|D|$, a contradiction. It follows that $v \in D$ and so $v \in \mathcal{A}_{\times 2}(T)$.

Case 4. $\left|P^1(v)\right| = 1$, $W^* \neq \emptyset$ and $P^0(v) \cup P^2(v) = \emptyset$.

Let $u \in P^1(v)$ and $w^* \in W^*(T)$ where $P_u = vu_1u_2u_3u$. Clearly w^* is dominated twice by its children. Now assume that $v \notin D$. Then $V(P_u) \cup D[w^*] \subset D$, but then $\{v\} \cup D - \{u_2,w^*\}$ is a DDS of T of cardinality less than $|D|$, a contradiction. It follows that $v \in D$ and so $v \in \mathcal{A}_{\times 2}(T)$.

Case 5. $\left|P^0(v)\right| = 2$ and $\left|P^2(v)\right| \geq 1$.

Let $u,x \in P^0(v)$ and $w \in P^2(v)$ where $P_u = vu_1u_2u$, $P_x = vx_1x_2x$ and $P_w = vw_1w$. Then $\{u,u_2,x,x_2,w,w_1\} \subset D$. Now if $v \notin D$, then $\{u_1,x_1\} \subset D$ which implies that $\{v\} \cup D - \{x_1,u_1\}$ is a DDS of T of cardinality $|D|-1$, a contradiction. Thus $v \in D$ and so $v \in \mathcal{A}_{\times 2}(T)$.

Case 6. $\left|P^2(v)\right| \geq 2$ and $P^1(v) \cup P^0(v) = \emptyset$.

Note that $W^*(T)$ may be non empty and every vertex (if any) of $W^*(T)$ is dominated twice by its children. Then D is a unique $\gamma_{\times 2}(T)$-set that does not contain v. It follows that $v \in \mathcal{N}_{\times 2}(T)$.

In addition to the previous cases, we consider the following five cases to prove the necessary condition.

Case 7. $\left|P^0(v)\right| = 2$ and $\left|P^1(v)\right| = \left|P^2(v)\right| = 0$.

We distinguish between two cases:

If $W^*(T) = \emptyset$, then T is a path $P_7 = uu_2u_1vx_1x_2x$. By Observation 2, $\gamma_{\times 2}(P_7) = 2\left\lceil \frac{7}{3} \right\rceil = 6$. Clearly $\{u, u_2, u_1, x_2, x_1, x\}$ and $\{u, u_2, u_1, v, x_2, x\}$ are two $\gamma_{\times 2}(T)$-sets, and so $v \notin \mathcal{A}_{\times 2}(T) \cup \mathcal{N}_{\times 2}(T)$.

If $W^*(T) \neq \emptyset$, then by Observation 5, we may assume that $W^*(T) = \{w^*\}$. Thus T is formed by a path $P_7 = uu_2u_1vx_1x_2x$ and a path P_5 of center w^* by joining v and w^*. Let D be a $\gamma_{\times 2}(T)$-set and suppose that $v \in \mathcal{A}_{\times 2}(T)$. Then one of w^*, x_1 or u_1 is in D. Thus $\{x_1, u_1\} \cup D - \{v, w^*, x_1, u_1\}$ is a $\gamma_{\times 2}(T)$-set that does not contain v, a contradiction. Suppose now that $v \in \mathcal{N}_{\times 2}(T)$. Then $\{x_1, u_1\} \subset D$, and hence $\{v\} \cup D - \{u_1\}$ is a $\gamma_{\times 2}(T)$-set containing v, a contradiction. It follows that $v \notin \mathcal{A}_{\times 2}(T) \cup \mathcal{N}_{\times 2}(T)$.

Case 8. $\left|P^1(v)\right| = 1$, $\left|P^2(v)\right| \geq 1$ and $P^0(v) = \emptyset$.

Let $u \in P^1(v)$ where $P_u = vu_1u_2u_3u$. Then every $\gamma_{\times 2}(T)$-set must contain two vertices among v, u_1, u_2. Thus $v \notin \mathcal{A}_{\times 2}(T) \cup \mathcal{N}_{\times 2}(T)$.

Case 9. $\left|P^0(v)\right| = 1$, $\left|P^2(v)\right| \geq 1$ and $P^1(v) = \emptyset$.

Let $u \in P^1(v)$ where $P_u = vu_1u_2u$. Then every $\gamma_{\times 2}(T)$-set must contain v or u_1 and clearly $v \notin \mathcal{A}_{\times 2}(T) \cup \mathcal{N}_{\times 2}(T)$.

Case 10. $W^* \neq \emptyset$, $\left|P^0(v) \cup P^2(v)\right| = 1$ and $P^1(v) = \emptyset$.

By Observation 5, assume that $W^*(T) = \{w^*\}$. If $\left|P^0(v)\right| = 1$, then T is formed by a path $P_4 = vx_1x_2x$ and a path P_5 of center w^* joined to v. Then $\{x_1, x_2, x, w^*\} \cup D(w^*)$ and $\{x_1, x_2, x, v\} \cup D(w^*)$ are two $\gamma_{\times 2}(T)$-sets. Thus $v \notin \mathcal{A}_{\times 2}(T) \cup \mathcal{N}_{\times 2}(T)$. Now if $\left|P^2(v)\right| = 1$ then T is formed by a path $P_3 = vx_1x$ and a path P_5 of center w^* joined to v. Then $\{x_1, x, w^*\} \cup D(w^*)$ and $\{x_1, x, v\} \cup D(w^*)$ are two $\gamma_{\times 2}(T)$-sets, and so $v \notin \mathcal{A}_{\times 2}(T) \cup \mathcal{N}_{\times 2}(T)$.

Case 11. $|W^*(T)| \geq 2$ and $P^0(v) \cup P^1(v) \cup P^2(v) = \emptyset$.

Then $C(v) = W^*(T)$. Since every vertex of $W^*(T)$ is double dominated by its children, it remains to dominate v twice. For this we need any two vertices of $W^*(T)$ or any vertex of $W^*(T)$ with v itself. It follows that $v \notin \mathcal{A}_{\times 2}(T) \cup \mathcal{N}_{\times 2}(T)$. □

According to Lemma 7 and Theorem 8, we have our main result:

Theorem 9. *Let v be a vertex of the tree T, then*

- $v \in \mathcal{A}_{\times 2}(T)$ *if and only if* $v \in \mathcal{A}_{\times 2}(\bar{T}_v)$.
- $v \in \mathcal{N}_{\times 2}(T)$ *if and only if* $v \in \mathcal{N}_{\times 2}(\bar{T}_v)$.

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