VERTICES BELONGING TO ALL OR TO NO MINIMUM DOUBLE DOMINATING SETS IN TREES

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Abstract
In a graph $G = (V,E)$, a vertex dominates itself and all its neighbors. A double dominating set of $G$ is a dominating set that dominates every vertex of $G$ at least twice. In this paper, we characterize vertices that are in all or in no minimum double dominating sets in trees.

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1. Introduction and preliminary results

For a simple graph $G = (V,E)$, the open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. For a set $S$, we denote by $\langle S \rangle$ the subgraph induced by the vertices of $S$. A set $S \subseteq V$ is a dominating set if for each vertex $v \in V - S$, $N(v) \cap S \neq \emptyset$. For more treatment on domination in graphs, see [7, 8].

A set $S$ is a double-dominating set, abbreviated DDS, if for every vertex $v \in V$, $|N[v] \cap S| \geq 2$, that is, $v$ is in $S$ and has at least one neighbor in $S$ or $v$ is in $V - S$ and has at least two neighbors in $S$. The double-domination number $\gamma_{x2}(G)$ is the minimum cardinality of a double-dominating set of $G$. We call a double-dominating set of cardinality $\gamma_{x2}(G)$ a $\gamma_{x2}(G)$-set. Double domination was introduced by Harary and Haynes [6] and studied for example in ([1, 3, 6])

For a property $P$ of a vertex subset of a graph $G$, let $\mu_P(G)$ denote the minimum (or maximum) cardinality of a set with the property $P$. Many researchers were interested in characterizing the vertices of $G$ that are in all or in no set with the property $P$ and cardinality $\mu_P(G)$. Indeed, Hammer et.al., [5] have characterized those vertices in a graph for independent sets with maximum cardinalities, Mynhardt [9] has characterized the vertices in all or in no minimum dominating set of trees and Cockayne et.al., [4] have characterized the set of vertices contained in all or in no total dominating set of trees (a total dominating set is a dominating set $S$ where $\langle S \rangle$ has no isolated vertex).

In this paper, we investigate vertices belonging to all or to no minimum double-dominating set in a tree. Let us give some definition and notation.

For a tree $T$ we define the sets $A_{x2}(T)$ and $N_{x2}(T)$ by

$A_{x2}(T) = \{v \in V(T) \mid v \text{ is in every } \gamma_{x2}(T)-\text{set}\}$, and

$N_{x2}(T) = \{v \in V(T) \mid v \text{ is in no } \gamma_{x2}(T)-\text{set}\}$. 

The degree of a vertex \( v \), denoted by \( \text{deg}_G(v) \), is the number of vertices adjacent to \( v \) and the diameter of \( G \) is \( \text{diam}(G) = \max\{d(x,y) \mid x, y \in V(G)\} \) where \( d(x,y) \) is the length of the shortest path between \( x \) and \( y \). Specifically, for a vertex \( v \) in a rooted tree \( T \), we let \( \mathcal{C}(v) \) and \( \mathcal{D}(v) \) denote the set of children and descendants, respectively, of \( v \), and we define \( \mathcal{D}[v] = \mathcal{D}(v) \cup \{v\} \). The maximal subtree at \( v \) is the subtree of \( T \) induced by \( \mathcal{D}[v] \), and is denoted by \( T_v \). A leaf is a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We denote the set of leaves of \( T \) by \( L(T) \) and the set of support vertices by \( S(T) \). If \( T \) is a tree \( T \) rooted at a vertex \( v \), then we denote by \( L(v) \) the set of leaves of \( T \) distinct from \( v \), that is, \( L(v) = \mathcal{D}(v) \cap L(T) \). Also, a vertex of degree at least three is called a branch vertex, and we denote by \( B(T) \) the set of such vertices. For a vertex branch \( w \) of \( T_v \), we define \( \mathcal{P}^3(w) \) as the set of leaves \( u \in L(w) \) such that \( d(w,u) \equiv j \pmod{3} \) with \( j = 0,1,2 \), and every vertex of the \( w-u \) path different to \( w \) has degree at most two.

We give below some useful observations.

**Observation 1.** In any graph \( G \), every DDS contains all support and pendent vertices.

**Observation 2.** If \( P_n \) is a path with \( n \geq 2 \), then \( \gamma_{\times 2}(P_n) = 2n/3 + 1 \) if \( n \equiv 0 \pmod{3} \) and \( 2 \lfloor n/3 \rfloor \) otherwise.

**Observation 3.** A path \( P_n \) with \( n \equiv 2 \pmod{3} \) has a unique minimum double dominating set.

The following lemma will be used in the next section.

**Lemma 4.** Let \( T' \) be a tree and \( v \) a vertex of \( V(T') \). If \( T \) is a tree obtained from \( T' \) by adding a path \( P_3 = xyz \) and an edge \( u \), where \( u \) is any leaf of \( T' \) such that \( v \notin N[u] \), then

(a) \( \gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2 \);

(b) \( v \in \mathcal{A}_{\times 2}(T') \) if and only if \( v \in \mathcal{A}_{\times 2}(T) \);

(c) \( v \in \mathcal{N}_{\times 2}(T') \) if and only if \( v \in \mathcal{N}_{\times 2}(T) \).

**Proof.** (a) By Observation 1, \( u \) and its support vertex are in every \( \gamma_{\times 2}(T') \)-set. Such a set can be extended to a DDS of \( T \) by adding \( \{y,z\} \), so \( \gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 2 \). On the other hand, if \( D \) is any \( \gamma_{\times 2}(T) \)-set, then by Observation 1, \( z,y \in D \). Now if \( x \notin D \), then \( D' = D \cap V(T') \) contains \( u \) and is a DDS of \( T' \). If \( x \in D \), then \( |D'| = |D| - 3 \) and \( x \) can be replaced in \( D \) with \( u \) if \( u \notin D \) or with its support vertex in \( T' \), say \( w \), if \( u \in D \). In both cases, the resulting set is a DDS of \( T' \) of cardinality \( |D| - 2 \). Thus \( \gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 2 \), implying that \( \gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2 \).

(b) Suppose that \( v \notin \mathcal{A}_{\times 2}(T') \). Let \( D' \) be a \( \gamma_{\times 2}(T') \)-set that does not contain \( v \). Since \( u \in D' \), \( D' \cup \{y,z\} \) is a \( \gamma_{\times 2}(T) \)-set that does not contain \( v \), and so \( v \notin \mathcal{A}_{\times 2}(T) \). Conversely, assume that \( v \in \mathcal{A}_{\times 2}(T') \) and let \( D \) be any \( \gamma_{\times 2}(T) \)-set with \( D' = D \cap V(T') \). If \( x \notin D \), then \( D' \) is an DDS of \( T' \) with \( |D'| = |D| - 2 \). Hence, \( D' \) is a \( \gamma_{\times 2}(T') \)-set with \( v \in D' \subset D \). If \( x \in D \), then as discussed in (a), \( x \) can be replaced by \( u \) or \( w \). Consequently, the resulting set is a \( \gamma_{\times 2}(T) \)-set that contains \( v \) since \( v \notin u \) and \( w \). Therefore \( v \in \mathcal{A}_{\times 2}(T) \).

(c) Suppose that \( v \notin \mathcal{N}_{\times 2}(T') \). Let \( D' \) be a \( \gamma_{\times 2}(T') \)-set that contains \( v \). Clearly, \( D' \cup \{y,z\} \) is a \( \gamma_{\times 2}(T) \)-set containing \( v \) so \( v \notin \mathcal{N}_{\times 2}(T) \). Conversely, suppose that \( v \in \mathcal{N}_{\times 2}(T') \) and let \( D \) be any \( \gamma_{\times 2}(T) \)-set with \( D' = D \cap V(T') \). If \( x \notin D \), then \( D' \) is a \( \gamma_{\times 2}(T') \)-set. Now if \( x \in D \), then as seen in proof of item (a), \( x \) can be replaced by \( u \) or \( w \). Thus the resulting set minus \( z \) and \( y \) is a \( \gamma_{\times 2}(T') \)-set. Now since \( v \neq u \) and \( v \neq w \), we deduce that \( v \in \mathcal{N}_{\times 2}(T) \).
2. The Pruning of a Tree

Let us first define for a tree $T$ rooted at $v$ the set $W^*(T_v)$ by:

$$W^*(T_v) = \{w^* \in C(v) \mid D(w^*) \cap B(T_v) = \emptyset, |P^2(w^*)| \geq 2 \text{ and } P^0(w^*) \cup P^1(w^*) = \emptyset\}$$

The following straightforward observations will be useful for the next.

Observation 5. Let $T$ be a tree rooted at $v$ with $|W^*(T_v)| \geq 2$, $C(v) - W^*(T_v) \neq \emptyset$, and let $w^* \in W^*(T_v)$. Then $v \in \mathcal{A}_{\geq 2}(T_v)$ (resp. $\mathcal{N}_{\geq 2}(T_v)$) if and only if $v \in \mathcal{A}_{\geq 2}(T'_v)$ (resp. $\mathcal{N}_{\geq 2}(T'_v)$) where $T'$ is the tree obtained from $T_v$ by removing, for every vertex $z \in W^*(T_v) - \{w^*\}$, $z$ and all its descendants, that is $T'_v = T_v - \bigcup_{z \in W^*(T_v) - \{w^*\}} T_z$.

Observation 6. Let $T$ be a tree rooted at $v$ and $w^* \in W^*(T_v)$ with $|P^2(w^*)| \geq 3$. Then $v \in \mathcal{A}_{\geq 2}(T_v)$ (resp. $\mathcal{N}_{\geq 2}(T_v)$) if and only if $v \in \mathcal{A}_{\geq 2}(T'_v)$ (resp. $\mathcal{N}_{\geq 2}(T'_v)$) where $T'_v$ is the tree obtained from $T_v$ with replacing $D[w^*]$ with a $P_5$ of center $w^*$.

In order to characterize the sets $\mathcal{A}_{\geq 2}(T)$ and $\mathcal{N}_{\geq 2}(T)$ for any nontrivial tree $T$, we will use a technique called tree pruning introduced by Mynhardt [9] and used later by Cockayne, Henning and Mynhardt [4].

Let $v$ be a vertex of a nontrivial tree $T$ that is neither a support vertex nor a leaf. Using the process described below, with respect to the root $v$ on every branch vertex, the tree $T_v$ is transformed to another tree $\tilde{T}_v$, called the pruning of $T_v$, in which every vertex $u \notin W^*(T_v) \cup \{v\}$ has degree at most two. As a consequence, the properties of the vertex $v$ to be in $\mathcal{A}_{\geq 2}(T)$ or $\mathcal{N}_{\geq 2}(T)$ will be preserved in $\tilde{T}_v$.

Let $T = T_v$ be a nontrivial tree rooted at a vertex $v$. If every vertex $u \notin W^*(T) \cup \{v\}$ has degree at most two then $\tilde{T}_v = T_v$. Otherwise, let $u$ be a branch vertex at maximum distance from $v$. Then apply the following process:

- If $|P^1(u)| \geq 1$, then delete $D(u)$ and attach a path $P_1$ at $u$.

- If $|P^2(u)| \geq 1$, $|P^0(u)| \geq 1$ and $P^1(u) = \emptyset$, then delete $D(u)$ and attach a path $P_1$ at $u$.

- If $|P^2(u)| \geq 2$ and $P^0(u) \cup P^1(u) = \emptyset$, then
  - If $u \notin C(v)$, then delete $D(u)$ and attach two paths $P_2$ at $u$.
  - If $d(v, u) = 2$ and $p(u) \notin B(T)$, then delete $D(u)$ and attach a path $P_2$ at $u$.
  - If either $d(v, u) \geq 3$ or $d(u, v) = 2$ and $p(u) \in B(T)$, then delete $D[u]$.

- If $|P^0(u)| \geq 2$ and $P^1(u) \cup P^2(u) = \emptyset$, then delete $D(u)$ and attach a path $P_3$ at $u$.

To illustrate this technique, we consider the tree in Figure 1.(a) where $v$, $w$, $u$, $x$, $y$ and $z$ are the branch vertices of $T$. At this step, $w$ is the the branch vertex at maximum distance from $v$, since $|P^1(w)| \geq 1$, we delete $D(w)$ and we attach a path $P_1$ at $w$. Then since $y$ is the branch vertex at maximum distance from $v$ and $|P^1(y)| \geq 1$, we delete $D(y)$ and we attach a $P_1$ at $y$. Now it remains three branch vertices $z$, $u$ and $x$ at distance one from $v$. Let us
consider $z$. Since $z \in C(v)$, $P^1(z) \cup P^0(z) = \emptyset$ and $|P^2(z)| \geq 2$, we delete $D(z)$ and we attach two paths $P_2$ at $z$. Now since $|P^0(x)| \geq 1$, $|P^2(x)| \geq 1$ and $P^1 = \emptyset$, then we delete $D(x)$ and we attach a $P_1$ at $x$. Finally since $|P^0(u)| \geq 2$ and $P^1(u) \cup P^2(u) = \emptyset$, we delete $D(u)$ and we attach $P_3$ at $u$.

Figure 1. The pruning of a tree rooted at $v$.

Lemma 7. Let $T$ be a tree rooted at $v$ and $u \neq v$ a branch vertex at maximum distance from $v$ with $k_1 = |P^1(u)|$, $k_2 = |P^2(u)|$ and $k_3 = |P^0(u)|$. If

1. $k_1 \geq 1$, let $T'$ be the tree obtained from $T$ by deleting $D(u)$ and attaching a $P_1$ to $u$.

2. $k_2 \geq 1$, $k_3 \geq 1$ and $k_1 = 0$, let $T'$ be the tree obtained from $T$ by deleting $D(u)$ and attaching a $P_1$ to $u$.

3. $k_2 \geq 2$, $k_1 + k_3 = 0$ and $u \in C(v)$, let $T'$ be the tree obtained from $T$ by deleting $D(u)$ and attaching two paths $P_2$ to $u$.

4. $k_2 \geq 2$, $k_1 + k_3 = 0$, $d(v,u) = 2$ and $p(u) \notin B(T)$, let $T'$ be the tree obtained from $T$ by deleting $D(u)$ and attaching a $P_2$ to $u$.

5. $k_2 \geq 2$, $k_1 + k_3 = 0$, and either $d(v,u) \geq 3$ or $p(u) \in B(T) - \{v\}$, let $T'$ be the tree obtained from $T$ by deleting $D(u)$.

6. $k_3 \geq 2$ and $k_1 + k_2 = 0$, let $T'$ be the tree obtained from $T$ by deleting $D(u)$ and attaching a $P_3$ to $u$. 

Then for each case we have:
(a) $v \in \mathcal{A}_{x_2}(T)$ if and only if $v \in \mathcal{A}_{x_2}(T')$.
(b) $v \in \mathcal{N}_{x_2}(T)$ if and only if $v \in \mathcal{N}_{x_2}(T')$.

Proof. We first note that Lemma 4 allows us to reduce the tree $T_v$ by replacing every $u - x$ path of $T$ with a $u - x$ path of length $j$, where $j = 3, 1, 2$ if $x \in P_i(u)$, $i = 0, 1, 2$, respectively. So we may assume that every leaf of $T_v$ is at distance at most three from $u$.

Let $a_i$, $t_ju_j$ and $x_ky_kz_k$ be paths of order 1, 2 and 3 respectively, attached to $u$ where $a_i, u_j, z_k \in L(T) \cap D(u)$, for $0 \leq i \leq k_1$, $0 \leq j \leq k_2$ and $0 \leq k \leq k_3$. We consider the following cases:

Case 1. $k_1 \geq 1$.

Let $T' = T - (D(u) - \{a_1\})$. Since every $\gamma_{x_2}(T')$-set can be extended to a DDS of $T$ by adding the set $X = \{a_i\} \cup \{t_ju_j\} \cup \{y_kz_k\}$ where $i \in \{2, \ldots, k_1\}$, $j \in \{1, \ldots, k_2\}$ and $k \in \{1, \ldots, k_3\}$, $\gamma_{x_2}(T) \leq \gamma_{x_2}(T') + (k_1 - 1) + 2k_2 + 2k_3$. On the other hand, if $D$ is a $\gamma_{x_2}(T)$-set with $D' = D \cap T'$ then it is a routine matter to check that $D' = D - X$ is a DDS of $T'$. Thus $\gamma_{x_2}(T') \leq \gamma_{x_2}(T) - (k_1 - 1) - 2k_2 - 2k_3$ implying the equality.

(a) Suppose that $v \in \mathcal{A}_{x_2}(T')$ and let $D$ be a $\gamma_{x_2}(T)$-set. As seen before $D' = D - X$ is a $\gamma_{x_2}(T')$-set. Thus $v \in D' \cap D$ and hence $v \in \mathcal{A}_{x_2}(T)$. Conversely, suppose that $v \in \mathcal{A}_{x_2}(T)$ and let $S'$ be a $\gamma_{x_2}(T')$-set. Then $S = S' \cup X$ is a $\gamma_{x_2}(T)$-set, and so $v \in S$. Now since $v \notin D[u]$, $v \in S'$ and hence $v \in \mathcal{A}_{x_2}(T')$.

(b) Suppose that $v \in \mathcal{N}_{x_2}(T')$ and let $D$ be an arbitrary $\gamma_{x_2}(T)$-set. We have seen that $D' = D - X$ is a $\gamma_{x_2}(T')$-set. So $v \notin D'$ and since $v \notin D[u]$, $v \in \mathcal{N}_{x_2}(T)$. Conversely, suppose that $v \in \mathcal{N}_{x_2}(T)$ and let $S'$ be a $\gamma_{x_2}(T')$-set. Since $S = S' \cup X$ is a $\gamma_{x_2}(T)$-set, $v \notin S$. Since $v \notin D[u]$, $v \notin S'$ and hence $v \in \mathcal{N}_{x_2}(T')$.

The proof of item (b) will be omitted for the next since it is similar to the proof of item (a).

Case 2. $k_2 \geq 1$, $k_3 \geq 1$ and $k_1 = 0$.

Let $T' = T - (D(u) - \{t_1\})$. Then $\gamma_{x_2}(T) \leq \gamma_{x_2}(T') + 2(k_2 - 1) + 2k_3 + 1$ because every $\gamma_{x_2}(T')$-set can be extended to a DDS of $T$ by adding the set $Y = \{u_1, t_j, u_j\} \cup \{y_kz_k\}$ where $j \in \{2, \ldots, k_2\}$ and $k \in \{1, \ldots, k_3\}$. Let $D$ be an arbitrary $\gamma_{x_2}(T)$-set. If $u \notin D$, then $D' = D - Y$ is a DDS of $T'$. If $u \notin D$, then by minimality $k_3 = 1$, and so $x_1 \in D$. It follows that $D' = (D - \{u_1, t_j, u_j\} \cup \{x_1, y_k, z_k\}) \cup \{u\}$ where $2 \leq j \leq k_2$ is also a DDS of $T'$. Both cases yield $|D'| = |D| - 2(k_2 - 1) - 2k_3 - 1$. Thus $\gamma_{x_2}(T') \leq \gamma_{x_2}(T) - 2(k_2 - 1) - 2k_3 - 1$ implying the equality.

(a) Suppose that $v \in \mathcal{A}_{x_2}(T')$ and let $D$ be a $\gamma_{x_2}(T)$-set. We have seen above that $D' = \{u\} \cup (D - \{u_1, t_j, u_j\} \cup \{x_1, y_kz_k\})$ where $2 \leq j \leq k_2$ and $1 \leq k \leq k_3$ is a $\gamma_{x_2}(T')$-set. Thus $v \in D'$ and since $v \notin D[u]$, $v \in D$. It follows that $v \in \mathcal{A}_{x_2}(T)$. Conversely, suppose that $v \in \mathcal{A}_{x_2}(T)$ and let $S'$ be a $\gamma_{x_2}(T')$-set. Then $S = S' \cup \{u_1, t_j, u_j\} \cup \{y_kz_k\}$ where $2 \leq j \leq k_2$ and $1 \leq k \leq k_3$ is a $\gamma_{x_2}(T)$-set, so $v \in S$. Since $v \notin D[u]$, $v \in S'$, and so $v \in \mathcal{A}_{x_2}(T')$.

Case 3. $k_2 \geq 2$, $k_1 + k_3 = 0$ and $u \in C(v)$. Let $T' = T - (D(u) - \{t_1, u_1, t_2, u_2\})$.

This case follows from lemma 4 and observation 6.

Case 4. $k_2 \geq 2$, $k_1 + k_3 = 0$, $d(v, u) = 2$ and $p(u) \notin B(T)$. Let $T' = T - (D(u) - \{t_1, u_1\})$.

Then $\gamma_{x_2}(T) \leq \gamma_{x_2}(T') + 2(k_2 - 1)$ since any $\gamma_{x_2}(T')$-set is extended to a DDS of $T$ by adding the set $X = \{t_j, u_j\}$ where $2 \leq j \leq k_2$. Now let $D$ be an arbitrary $\gamma_{x_2}(T)$-set. If $u \in D$,
then $D - X$ is a DDS of $T'$. If $u \notin D$, then $p(u)$ must be in $D$ and hence $D - X$ is a DDS of $T'$, so $\gamma_{x_2}(T') \leq \gamma_{x_2}(T) - 2(k_2 - 1)$. Thus we have $\gamma_{x_2}(T') = \gamma_{x_2}(T) - 2(k_2 - 1)$.

(a) Suppose that $v \in A_{x_2}(T')$ and let $D$ be an arbitrary $\gamma_{x_2}(T)$-set. We know that $D' = D - X$ is a $\gamma_{x_2}(T')$-set. Then $v \in D' \subset D$ and $v \in A_{x_2}(T)$. Conversely, suppose that $v \in A_{x_2}(T)$ and let $S'$ be any $\gamma_{x_2}(T')$-set. We have seen that $S = S' \cup X$ is a $\gamma_{x_2}(T)$-set, so $v \in S$. Since $v \notin D[u]$, then $v \in S'$ and $v \in A_{x_2}(T')$.

Case 5. $k_2 \geq 2$, $k_1 + k_3 = 0$, and either $d(v, u) \geq 3$ or $p(u) \in B(T) - \{v\}$. Let $T' = T - D[u]$.

Clearly any $\gamma_{x_2}(T')$-set can be extended to a DDS of $T'$ by adding the set $X = \{j, u_j\}$ where $j \in \{1, \ldots, k_2\}$, and so $\gamma_{x_2}(T) \leq \gamma_{x_2}(T') + 2k_2$. Now let $D$ be a $\gamma_{x_2}(T)$-set. If $u \notin D$, then $D' = D - X$ is a DDS of $T'$. Assume now that $u \in D$. If $p(u) \notin D$, then $D'' = (D - (X \cup \{u\})) \cup \{p(u)\}$ is a DDS of $T'$, else $(p(u) \in D)$, then by minimality $D''' = (D - (X \cup \{u\})) \cup \{x\}$ is a DDS of $T'$. Assume that $x \in N(p(u)) - \{v, u\}$, and hence $\gamma_{x_2}(T') \leq \gamma_{x_2}(T) - 2k_2$. It follows that $\gamma_{x_2}(T') = \gamma_{x_2}(T) - 2k_2$.

(a) Suppose that $v \in A_{x_2}(T')$ and let $D$ be any $\gamma_{x_2}(T)$-set. We showed before depending on $D$ that one of $D'$, $D''$ or $D'''$ is a $\gamma_{x_2}(T')$-set. Since $v \notin \{p(u), x\}$, $v$ is in one of $D'$, $D''$ or $D'''$, so $v \in D$ and $v \in A_{x_2}(T)$. Conversely, suppose that $v \in A_{x_2}(T)$ and let $S'$ be any $\gamma_{x_2}(T')$-set. We have seen that $S = S' \cup X$ is a $\gamma_{x_2}(T)$-set, so $v \in S$. Since $v \notin D[u]$, then $v \in S'$ and hence $v \in A_{x_2}(T')$.

Case 6. $k_3 \geq 2$ and $k_1 + k_3 = 0$. Let $T' = T - (D(u) - \{x_1, y_1, z_1\})$.

Let $S'$ be a $\gamma_{x_2}(T')$-set. If $u \in S'$, then $S = S' \cup X$ where $X = \{y_k, z_k\}$ with $k \in \{2, \ldots, k_3\}$ is a DDS of $T$. Else, $x_1 \in S'$ and since $u$ is dominated twice by $x_1$ and $p(u)$, then $S = (S' - \{x_1\}) \cup X \cup \{u\}$ is a DDS of $T$. Thus $\gamma_{x_2}(T) \leq \gamma_{x_2}(T') + 2(k_3 - 1)$. Now let $D$ be any $\gamma_{x_2}(T)$-set. If $u \notin D$ then without loss of generality $x_k \notin D$ for $k \in \{2, \ldots, k_3\}$, and so $D' = D - X$ is a DDS of $T'$. If $u \notin D$, then by the minimality of $D$, $k_3 = 2$ and $\{x_1, x_2\} \subset D$. Thus $D''' = (D - \{x_2, y_2, z_2\}) \cup \{u\}$ is a DDS of $T'$ of cardinality $\gamma_{x_2}(T) - 2(k_3 - 1)$ where $k_3 = 2$. It follows that $\gamma_{x_2}(T') \leq \gamma_{x_2}(T) - 2(k_3 - 1)$ and so $\gamma_{x_2}(T') = \gamma_{x_2}(T) - 2(k_3 - 1)$.

(a) Suppose that $v \in A_{x_2}(T')$ and let $D$ be any $\gamma_{x_2}(T)$-set. Then depending on whether $u$ is contained in $D$ or no, $D'$ or $D''$ respectively is a $\gamma_{x_2}(T')$-set. Since $v \notin D[u]$, $v \in D'$ (or $D''$) and hence $v \in D$. It follows that $v \in A_{x_2}(T)$. Conversely, suppose that $v \in A_{x_2}(T)$ and let $S'$ be a $\gamma_{x_2}(T')$-set. As seen above $S$ is a $\gamma_{x_2}(T)$-set, so $v \in S$. Since $v \notin D[u]$, $v \in S'$, so $v \in A_{x_2}(T')$.

3. Characterizations

The next theorem gives a necessary and sufficient condition for a vertex $v$ of a nontrivial tree $T$ to be in $A_{x_2}(T_v)$ ( resp. in $N_{x_2}(T_v)$).

**Theorem 8.** Let $T$ be a tree rooted at $v$ such that $\deg(u) \leq 2$ for every $u \notin W^*(T) \cup \{v\}$. Then

a) $v \in A_{x_2}(T)$ if and only if at least one of the following conditions is verified:

- $v$ is a support vertex;
- $v$ is a leaf;
- $|P^1(v)| \geq 2$;
- $|P^0(v)| \geq 3$;
- \(|P^1(v)| = 1\) and \(|P^0(v)| \in \{1, 2\};
- \(|P^1(v)| = 1\), \(W^*(T) \neq \emptyset\) and \(P^2(v) \cup P^0(v) = \emptyset;\)
- \(|P^0(v)| = 2\) and \(|P^2(v)| \geq 1.\)

b) \(v \in N_{x,2}(T)\) if and only if \(|P^2(v)| \geq 2\) and \(P^1(v) \cup P^0(v) = \emptyset.\)

Proof. By Observation 1, the theorem is valid if \(v\) is a support vertex or a leaf. So suppose that \(v\) is neither a support vertex nor a leaf. By Lemma 4, the tree \(T_v\) can be reduced to a tree \(T_v^*\) by replacing each \(v - b\) path of \(T\) with a \(v - b\) path of length \(j\) where \(j = 3, 4, 2\) if \(b \in P^1(v)\) and \(i = 0, 1, 2\), respectively. Likewise for every \(w^* \in W^*(T_v)\) we replace every \(w^* - b\) path where \(b \in P^2(w^*)\) with a path of length two. Thus every leaf of \(T_v^*\) is at distance 2, 3 or 4 from \(v\) and hence by Lemma 4, \(v \in A_{x,2}(T_v^*)\) if and only if \(v \in A_{x,2}(T_v)\).

We first show the sufficient condition. Let \(D\) be an arbitrary \(\gamma_{x,2}(T)\)-set.

Case 1. \(|P^1(v)| \geq 2.\)

Let \(u\) and \(x\) be two vertices of \(P^1(v)\) where \(P_u = vu_1u_2u_3u\) and \(P_x = vx_1x_2x_3x\). Then by Observation 1, \(\{u, u_3, x_3, x_3\} \subset D.\) If \(v \notin D,\) then \(\{u_1, u_2, x_1, x_2\} \subset D.\) In this case \(D' = \{v\} \cup D - \{u_2, x_2\}\) is a DDS of \(T\) of cardinality \(|D| - 1,\) a contradiction. Thus \(v \in D\) and so \(v \in A_{x,2}(T).\)

Case 2. \(|P^0(v)| \geq 3.\)

Let \(u, x\) and \(w\) be three vertices of \(P^0(v)\) where \(P_u = vu_1u_2u_3u\) and \(P_x = vx_1x_2x\) and \(P_w = vw_1w_2w.\) Then \(\{u, u_2, x, x_2, w, w_2\} \subset D.\) Now if \(v \notin D,\) then \(\{u_1, x_1, w_1\} \subset D,\) and hence \(D' = \{v\} \cup D - \{x_1, w_1\}\) is a DDS of \(T\) of cardinality \(|D| - 1,\) a contradiction. Thus \(v \in D\) and so \(v \in A_{x,2}(T).\)

Case 3. \(|P^1(v)| = 1\) and \(|P^0(v)| \in \{1, 2\}.\)

Let \(u\) and \(x\) be two vertices of \(P^1(v)\) and \(P^0(v)\) where \(P_u = vu_1u_2u_3u\) and \(P_x = vx_1x_2x.\) Then \(\{u, u_3, x_2\} \subset D.\) Now if \(v \notin D,\) then \(\{u_1, u_2, x_1\} \subset D,\) and hence \(\{v\} \cup D - \{u_2, x_1\}\) is a DDS of \(T\) of cardinality less than \(|D|,\) a contradiction. It follows that \(v \in D\) and so \(v \in A_{x,2}(T).\)

Case 4. \(|P^1(v)| = 1, W^* \neq \emptyset\) and \(P^0(v) \cup P^2(v) = \emptyset.\)

Let \(u \in P^1(v)\) and \(w^* \in W^*(T)\) where \(P_u = vu_1u_2u_3u.\) Clearly \(w^*\) is dominated twice by its children. Now assume that \(v \notin D.\) Then \(V(P_u) \cup D[w^*] \subset D,\) but then \(\{v\} \cup D - \{u_2, w^*\}\) is a DDS of \(T\) of cardinality less than \(|D|,\) a contradiction. It follows that \(v \in D\) and so \(v \in A_{x,2}(T).\)

Case 5. \(|P^0(v)| = 2\) and \(|P^2(v)| \geq 1.\)

Let \(u, x \in P^0(v)\) and \(w \in P^2(v)\) where \(P_u = vu_1u_2u, P_x = vx_1x_2x\) and \(P_w = vw_1w.\) Then \(\{u, u_2, x, x_2, w, w_1\} \subset D.\) Now if \(v \notin D,\) then \(\{u_1, x_1\} \subset D\) which implies that \(\{v\} \cup D - \{x_1, u_1\}\) is a DDS of \(T\) of cardinality \(|D| - 1,\) a contradiction. Thus \(v \in D\) and so \(v \in A_{x,2}(T).\)

Case 6. \(|P^2(v)| \geq 2\) and \(P^1(v) \cup P^0(v) = \emptyset.\)

Note that \(W^*(T)\) may be non empty and every vertex (if any) of \(W^*(T)\) is dominated twice by its children. Then \(D\) is a unique \(\gamma_{x,2}(T)\)-set that does not contain \(v.\) It follows that \(v \in N_{x,2}(T).\)

In addition to the previous cases, we consider the following five cases to prove the necessary condition.
Theorem 9. Let \( v \) be a vertex of the tree \( T \), then

- \( v \in A_{x_2}(T) \) if and only if \( v \in A_{x_2}(T_v) \).
- \( v \in N_{x_2}(T) \) if and only if \( v \in N_{x_2}(T_v) \).
References


