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7-Shredders in 7-Connected Graphs

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Abstract

For a graph G, a subset S of $V(G)$ is called a shredder if $G-S$ consists of three or more components. We show that if G is a 7-connected graph of order at least 41, then the number of shredders of cardinality 7 of G is less than or equal to $(2|V(G)|-8)/3$.

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1. Introduction

In this paper, we consider only finite, undirected, simple graphs with no loops and no multiple edges. Let $G=(V(G),E(G))$ be a graph. For $x \in V(G)$, we let $N_G(x)$ denote the set of vertices adjacent to x in G. For $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph induced by S in G, and $G-S$ denotes the subgraph obtained from G by deleting all vertices in S together with the edges incident with them; thus $G-S=\langle V(G)-V(S)\rangle$.

As is introduced by Cheriyan and Thurimella in [1], a subset S of $V(G)$ is called a *shredder* if $G-S$ consists of three or more components. A shredder of cardinality k is referred to as a k-shredder. In [2; Theorem 1], it is proved that if $k \geq 5$ and G is a k-connected graph, then the number of k-shredders of G is less than $2|V(G)|/3$, and it is shown that for each fixed $k \geq 5$, the coefficient $2/3$ in the upper bound is best possible. For $k=5$, it is shown in [3; Theorem 3] that if G is a 5-connected graph of order at least 135, then the number of 5-shredders of G is less than or equal to $(2|V(G)|-10)/3$, and for $k=6$, it is shown in [5] that if G is a 6-connected graph of order at least 325, then the number of 6-shredders of G is less than or equal to $(2|V(G)|-9)/3$. It is also shown that both of these two bounds are attained by infinitely many graphs (for results concerning the case where $1 \leq k \leq 4$, the reader is referred to [4] and [2; Theorem 2]). In this paper, we prove:

Theorem. *Let G be a 7-connected graph of order at least 41. Then the number of 7-shredders of G is less than or equal to*

$$(2|V(G)|-8)/3.$$

We here construct an infinite family of graphs G which attain the bound $(2|V(G)|-8)/3$ in the Theorem. Let $m \geq 8$. Define an auxiliary graph H_m of order m by letting

$$\begin{aligned} V(H_m) &= \{v_i | 1 \leq i \leq m\}, \\ E(H_m) &= \{v_i v_{i+3} | 1 \leq i \leq m-3\} \\ &\cup \{v_1v_2,\ v_1v_3,\ v_2v_3,\ v_{m-2}v_{m-1},\ v_{m-2}v_m,\ v_{m-1}v_m\}. \end{aligned}$$

We define a graph G_m of order $3m-5$ by adding $m-5$ vertices to the so-called lexicographic product of H_m and the null graph of order 2. More precisely, we let

$$\begin{aligned} V(G_m) &= \{x_{i,j} | 1 \leq i \leq m,\ 1 \leq j \leq 2\} \cup \{\alpha_i | 4 \leq i \leq m-3\} \cup \{a\}, \\ E(G_m) &= \{x_{i,j}x_{i+3,k} | 1 \leq i \leq m-3,\ 1 \leq j,k \leq 2\} \\ &\cup \{x_{i-1,j}\alpha_i,\ x_{i,j}\alpha_i,\ x_{i+1,j}\alpha_i | 4 \leq i \leq m-3,\ 1 \leq j \leq 2\} \\ &\cup \{x_{i,j}a,\ x_{k,j}a | 1 \leq i \leq 2,\ m-1 \leq k \leq m,\ 1 \leq j \leq 2\} \\ &\cup \{x_{4,j}a,\ x_{m-3,j}a | 1 \leq j \leq 2\} \\ &\cup \{\alpha_i a | 4 \leq i \leq m-3\} \\ &\cup \{x_{1,j}x_{2,k},\ x_{1,j}x_{3,k},\ x_{2,j}x_{3,k} | 1 \leq j,k \leq 2\} \\ &\cup \{x_{m-2,j}x_{m-1,k},\ x_{m-2,j}x_{m,k},\ x_{m-1,j}x_{m,k} | 1 \leq j,k \leq 2\}. \end{aligned}$$

Then, as we shall see below, G_m is 7-connected, and has $2m-6$ 7-shredders

$\{x_{i,1}, x_{i,2}, x_{i+1,1}, x_{i+1,2}, x_{i+2,1}, x_{i+2,2}, a\}$ $(2 \leq i \leq m-3)$,

$\{x_{i-3,1}, x_{i-3,2}, x_{i+3,1}, x_{i+3,2}, \alpha_{i-1}, \alpha_i, \alpha_{i+1}\}$ $(5 \leq i \leq m-4)$,

$\{x_{1,1}, x_{1,2}, x_{3,1}, x_{3,2}, x_{5,1}, x_{5,2}, a\}$,

$\{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{6,1}, x_{6,2}, \alpha_4\}$,

$\{x_{1,1}, x_{1,2}, x_{7,1}, x_{7,2}, \alpha_4, \alpha_5, a\}$,

$\{x_{m-6,1}, x_{m-6,2}, x_{m,1}, x_{m,2}, \alpha_{m-4}, \alpha_{m-3}, a\}$,

$\{x_{m-5,1}, x_{m-5,2}, x_{m-1,1}, x_{m-1,2}, x_{m,1}, x_{m,2}, \alpha_{m-3}\}$,

$\{x_{m-4,1}, x_{m-4,2}, x_{m-2,1}, x_{m-2,2}, x_{m,1}, x_{m,2}, a\}$.

Thus the number of 7-shredders of G_m is $2m-6 = (2(3m-5)-8)/3 = (2|V(G_m)|-8)/3$.

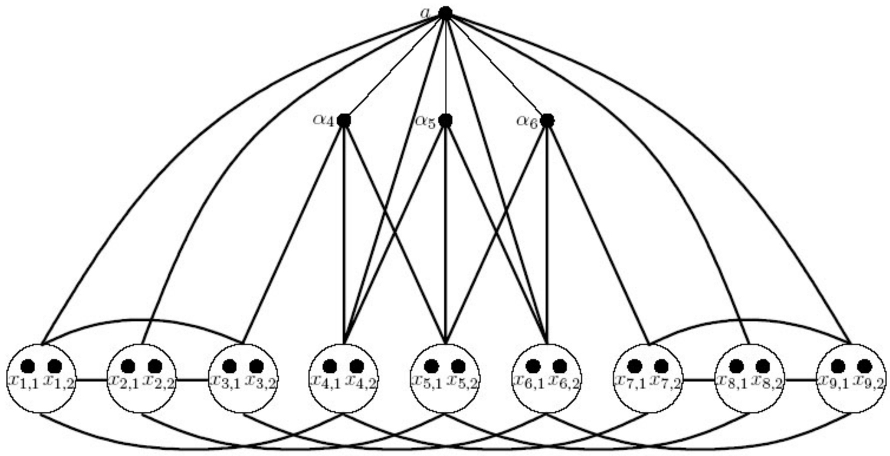

Figure 1. $m = 9$

For completeness, we include the proof of the assertion that G is 7-connected. In the proof, we make use of the following claim, which can easily be verified.

Claim 1.1. *Let G be a connected graph, and let $S \subseteq V(G)$ be a cutset with minimum cardinality. Let u, v be two vertices of G such that $N_G(u) = N_G(v)$. Then we have $\{u, v\} \subseteq S$ or $\{u, v\} \cap S = \emptyset$.*

The following claim immediately follows from the definition of H_m.

Claim 1.2. *H_m is 2-connected. Further if T is a cutset of H_m with $|T| = 2$, then $T = \{v_k, v_{k+3l}\}$ for some k, l with $l \geq 2$ and $1 \leq k < k + 3l \leq m$.*

Now let $G = G_m$, and set $A = \{\alpha_i | 4 \leq i \leq m - 3\}, X_i = \{x_{i,1}, x_{i,2}\}\ (1 \leq i \leq m), B = \cup_{1 \leq i \leq m} X_i$. The following claim follows from Claims 1.1 and 1.2.

Claim 1.3. *$\langle B \rangle$ is 4-connected. Further if R is a cutset of $\langle B \rangle$ with $|R| = 4$, then $R = X_k \cup X_{k+3l}$ for some k, l with $l \geq 2$ and $1 \leq k < k + 3l \leq m$.*

Let $S \subseteq V(G)$ be a cutset of G with minimum cardinality and, by way of contradiction, suppose that $|S| \leq 6$.

Claim 1.4. $(\{a\} \cup A) \cap S \neq \emptyset$.

Proof. Suppose $(\{a\} \cup A) \cap S = \emptyset$. Then since $\langle \{a\} \cup A \rangle$ is connected and $N_G(x) \cap (\{a\} \cup A) \neq \emptyset$ for each $x \in B$ by the definition of G, $G - S$ is connected. This contradicts the assumption that S is a cutset of G.

Claim 1.5. *$\langle B-S\rangle$ is disconnected.*

Proof. Suppose that $\langle B-S\rangle$ is connected. Since $|S|\leq 6$, it follows from Claim 1.4 that $|B\cap S|\leq 5$. On the other hand, $|N_G(\alpha)\cap B|\geq 6$ for each $\alpha\in\{a\}\cup A$ by the definition of G. Hence $N_G(\alpha)\cap(B-S)\neq\emptyset$ for each $\alpha\in\{a\}\cup A$, which means that $G-S$ is connected, a contradiction.

By Claims 1.1, 1.3, 1.4 and 1.5, we easily obtain the following claim.

Claim 1.6.

(i) *$|S\cap B|=4$, and $S\cap B=X_k\cup X_{k+3l}$ for some k,l with $l\geq 2$ and $1\leq k<k+3l\leq m$.*

(ii) *$1\leq|S\cap(\{a\}\cup A)|\leq 2$.*

Let k,l be as in Claim 1.6, and let $B_1=\cup_{1\leq i\leq l-1}X_{k+3i}$ and $B_2=B-S-B_1$. Then the following claim follows from the definition of G.

Claim 1.7.

(i) *$\langle B_1\rangle$ is connected or $B_1=X_{k+3}$.*

(ii) *$\langle B_2\rangle$ is connected.*

Claim 1.8. *There exists $\beta\in(\{a\}\cup A)-S$ such that $\langle(B-S)\cup\{\beta\}\rangle$ is connected.*

Proof. Take $x\in B_1$. Then $x\in X_i$ for some i with $4\leq i\leq m-3$. By the definition of G, this implies $|N_G(x)\cap(\{a\}\cup A)|=3$, and hence $N_G(x)\cap((\{a\}\cup A)-S)\neq\emptyset$ by Claim 1.6 (ii). Let $\beta\in N_G(x)\cap((\{a\}\cup A)-S)$. Note that we have $B_1\subseteq N_G(\beta)$ in the case where $B_1=X_{k+3}$. Now if $\beta=a$, then $N_G(\beta)\supseteq X_1\cup X_2$; if $\beta\in A$, then $N_G(\beta)\cap(X_{i-1}\cup X_{i+1})\neq\emptyset$. Since we have $B_2\cap(X_1\cup X_2)\neq\emptyset$ and $B_2\supseteq X_{i-1}\cup X_{i+1}$, this implies $N_G(\beta)\cap B_2\neq\emptyset$. Hence $\langle(B-S)\cup\{\beta\}\rangle$ is connected by Claim 1.7.

Now since $N_G(\alpha)\cap(B-S)\neq\emptyset$ for each $\alpha\in(\{a\}\cup A)-S$, it follows from Claim 1.8 that $G-S$ is connected. This contradicts the assumption that S is a cutset of G, completing the proof of the assertion that G is 7-connected.

2. Preliminary Result

Throughout the rest of this paper, let G be a 7-connected graph, and let $\mathscr{S}$ denote the set of 7-shredders of G. For each $S\in\mathscr{S}$, we define $\mathscr{K}(S)$, $\mathscr{L}(S)$ and $L(S)$ as follows. Let $S\in\mathscr{S}$. We let $\mathscr{K}(S)$ denote the set of componetns of $G-S$. Write $\mathscr{K}(S)=\{H_1,\cdots,H_s\}$ $(s=|\mathscr{K}(S)|)$. We may assume $|V(H_1)|\geq|V(H_2)|\geq\cdots\geq|V(H_s)|$ (any such labeling will do). Under this notation, we let $\mathscr{L}(S)=\mathscr{K}(S)-\{H_1\}$ and $L(S)=\cup_{2\leq i\leq s}V(H_i)$; thus $L(S)=\cup_{C\in\mathscr{L}(S)}V(C)$. Now let $\mathscr{L}=\cup_{S\in\mathscr{S}}\mathscr{L}(S)$. A member F of $\mathscr{L}$ is said to *saturated* if there exists a subset $\mathscr{C}$ of $\mathscr{L}-\{F\}$ such that $V(F)=\cup_{C\in\mathscr{C}}V(C)$.

Let $S, T \in \mathscr{S}$ with $S \neq T$. We say that S *meshes* with T if S intersects with at least two member of $\mathscr{K}(T)$. It is easy to see that if S meshes with T, then T intersects with all members of $\mathscr{K}(S)$, and hence T meshes with S and S intersects with all members of $\mathscr{K}(T)$ (see [1; Lemma 4.3 (1)]).

The following two lemmas are proved in [4; Lemmas 2.1 and 3.1] (see also [2; Lemmas 3.2 and 3.4]).

Lemma 2.1. *Let $S, T \in \mathscr{S}$ with $S \neq T$, and suppose that S does not mesh with T. Then one of the following holds:*

(i) $L(S) \cap L(T) = \emptyset$, $(L(S) \cup L(T)) \cap (S \cup T) = \emptyset$, *and no edge of G joins a vertex in $L(S)$ and a vertex in $L(T)$;*

(ii) *there exists $C \in \mathscr{L}(S)$ such that $V(C) \supseteq L(T)$ (so $L(S) \supseteq L(T)$); or*

(iii) *there exists $D \in \mathscr{L}(T)$ such that $V(D) \supseteq L(S)$ (so $L(T) \supseteq L(S)$).*

Lemma 2.2. *Let $S, T \in \mathscr{S}$ with $S \neq T$, and suppose that S meshes with T. Then the following hold.*

(i) $S \supseteq L(T)$ *or* $T \supseteq L(S)$.

(ii) $L(S) \cap L(T) = \emptyset$.

The following lemma is proved in [2; Lemma 3.6].

Lemma 2.3. *Let $F \in \mathscr{L}$, and suppose that F is saturated. Then $|V(F)| \geq 4$.*

The following lemmas are proved in [3; Lemmas 2.9 through 2.12].

Lemma 2.4. *Let $S \in \mathscr{S}$, and let $p = |\mathscr{L}(S)|$. Set $\mathscr{T} = \{T \in \mathscr{S} | L(T) \subseteq L(S)\}$. Then $|\mathscr{T}| \leq (2|L(S)| - 2p + 3)/3 \leq (2|L(S)| - 1)/3$.*

Lemma 2.5. *Let $X \subseteq V(G)$. Set $\mathscr{T} = \{T \in \mathscr{S} | L(T) \subseteq X\}$ and $\mathscr{L}_0 = \cup_{T \in \mathscr{T}} \mathscr{L}(T)$, and suppose that no component in $\mathscr{L}_0$ is saturated. Then $|\mathscr{T}| \leq |X|/2$.*

Lemma 2.6. *Let $S, T \in \mathscr{S}$, and suppose that S meshes with T and $L(S) \not\subseteq T$. Then $L(T) \subseteq S$ and $|L(T)| \leq 3$.*

Lemma 2.7. *Suppose that $|V(G)| \geq 15$. Let $S, T \in \mathscr{S}$, and suppose that S meshes with T, $L(S) \subseteq T$ and $L(T) \subseteq S$. Then $|L(S)| + |L(T)| \leq 7$.*

The following lemma follows from Lemmas 2.6 and 2.7.

Lemma 2.8. *Suppose that $|V(G)| \geq 15$. Let $S, T \in \mathscr{S}$, and suppose that S meshes with T and $|L(S)| \geq 4$. Then $L(T) \subseteq S$ and $|L(T)| \leq 3$.*

As an immediate corollary of Lemma 2.8, we obtain the following lemma.

Lemma 2.9. *Suppose that* $|V(G)| \geq 15$ *. Let* $S, T \in \mathscr{S}$ *with* $S \neq T$ *, and suppose that* $|L(S)|, |L(T)| \geq 4$ *. Then* S *does not mesh with* T *.*

3. Proof of the Theorem

We continue with the notation of the preceeding section, and prove the Theorem. Thus let $|V(G)| \geq 41$ and , by way of contradiction, suppose that

$$|\mathscr{S}| \geq (2|V(G)| - 7)/3. \tag{3.1}$$

We define an order relation $\leq$ in $\mathscr{S}$ as follows:

$$S \leq T \Longleftrightarrow L(S) \subseteq L(T) \ (S, T \in \mathscr{S}).$$

Let $S_1, \cdots, S_m$ be the maximal members of $\mathscr{S}$ with respect to the order relation $\leq$. We may assume $|L(S_1)| \geq \cdots \geq |L(S_m)|$. Let $p_i = |\mathscr{L}(S_i)|$ for each i , and let $W = V(G) - (L(S_1) \cup \cdots \cup L(S_m)\ )$. Arguing as in [3; Claims 3.2 through 3.4], we obtain the following three claims. We include sketches of their proofs for the convenience of the reader.

Claim 3.1.

(i) $m + 2|W| \leq 7$.

(ii) $2p_1 + (m-1) + 2|W| \leq 10$.

Sketch of Proof. By (3.1) and Lemma 2.4, $(2|V(G)| - 7)/3 \leq \sum_{1 \leq i \leq m} \ (2|L(S_i)| - 2p_i + 3)/3$, and hence $2(p_1 + \cdots + p_m) - 3m + 2|W| \leq 7$. Since $p_i \geq 2$ for all i , both (i) and (ii) follow from this.

Claim 3.2. $|L(S_1)| \geq 4$.

Sketch of Proof. If $|L(S_1)| \leq 3$, then by Claim 3.1 (i), $|V(G)| \leq 3m + |W| \leq 21$, which contradicts the assumption that $|V(G)| \geq 41$.

Claim 3.3. $m \geq 2$ *and* $|L(S_2)| \geq 4$.

Sketch of Proof. Suppose that $m = 1$ or $|L(S_2)| \leq 3$. Then by Claim 3.1 (ii), $|V(G) - L(S_1)| \leq 3(m-1) + |W| \leq 30 - 6p_1$, and hence $|V(G) - (S_1 \cup L(S_1))| \leq 23 - 6p_1$, which implies $|L(S_1)| \leq p_1(23 - 6p_1)$. Consequently $|V(G)| \leq p_1(23 - 6p_1) + 30 - 6p_1 \leq 40$ because $p_1 \geq 2$, which contradicts the assumption that $|V(G)| \geq 41$.

By Lemma 2.9, Cliam 3.2 and Claim 3.3 imply that S_1 does not mesh with S_2 . Since $L(S_1) \cap L(S_2) = \emptyset$ by the maximality of $L(S_1)$ and $L(S_2)$, $L(S_1) \cap S_2 = L(S_2) \cap S_1 = \emptyset$ by Lemma 2.1.

Write $\mathscr{K}(S_1) - \mathscr{L}(S_1) = \{C_1\}$ and $\mathscr{K}(S_2) - \mathscr{L}(S_2) = \{C_2\}$; thus $C_1 = G - S_1 - L(S_1)$ and $C_2 = G - S_2 - L(S_2)$. We define $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_{1,1}$, $\mathscr{T}_{1,2}$, $\mathscr{T}_{1,3}$, $\mathscr{T}_{2,1}$, $\mathscr{T}_{2,2}$, $\mathscr{T}_{2,3}$ as follows:

$$\begin{aligned}
\mathscr{T}_1 &= \{T \in \mathscr{S} | L(T) \cap (S_1 \cup S_2) = \emptyset\}, \\
\mathscr{T}_2 &= \{T \in \mathscr{S} | L(T) \subseteq S_1 \cup S_2\}, \\
\mathscr{T}_{1,1} &= \{T \in \mathscr{S} | L(T) \subseteq L(S_1)\}, \\
\mathscr{T}_{1,2} &= \{T \in \mathscr{S} | L(T) \subseteq L(S_2)\}, \\
\mathscr{T}_{1,3} &= \{T \in \mathscr{S} | L(T) \subseteq V(C_1) \cap V(C_2)\}, \\
\mathscr{T}_{2,1} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_1 - S_2\}, \\
\mathscr{T}_{2,2} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_2 - S_1\}, \\
\mathscr{T}_{2,3} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_1 \cap S_2\}.
\end{aligned}$$

In view of the maximality of $L(S_1)$ and $L(S_2)$ and Claims 3.2 and 3.3, it follows from Lemmas 2.1 and 2.8 that $\mathscr{T}_1$ is the set of those members of $\mathscr{S}$ which mesh with neither S_1 nor S_2 , and $\mathscr{T}_2$ is the set of those members of $\mathscr{S}$ which mesh with S_1 or S_2 . Thus $\mathscr{S} = \mathscr{T}_1 \cup \mathscr{T}_2$ (disjoint union). Further by Lemma 2.1, $\mathscr{T}_1 = \mathscr{T}_{1,1} \cup \mathscr{T}_{1,2} \cup \mathscr{T}_{1,3}$ (disjoint union) and , by Lemma 2.8, $\mathscr{T}_2 = \mathscr{T}_{2,1} \cup \mathscr{T}_{2,2} \cup \mathscr{T}_{2,3}$ (disjoint union).

The following two claims immediately follow from Lemma 2.4 (see also [3; Claim 3.6]).

Claim 3.4. $|\mathscr{T}_{1,i}| \leq (2|L(S_i)| - 1)/3$ $(i = 1, 2)$.

Claim 3.5. $|\mathscr{T}_{1,3}| \leq 2|V(C_1) \cap V(C_2)|/3$.

Since $|L(T)| \leq 3$ for each $T \in \mathscr{T}_2$ by Lemma 2.8, the following claim follows from Lemmas 2.3 and 2.5 (see also [3; Claim 3.8]).

Claim 3.6.

(i) $|\mathscr{T}_{2,1}| \leq |S_1 - S_2|/2$.

(ii) $|\mathscr{T}_{2,2}| \leq |S_2 - S_1|/2$.

(iii) $|\mathscr{T}_{2,3}| \leq |S_1 \cap S_2|/2$.

Now it follows from Claims 3.4, 3.5 and 3.6 that

$$
\begin{aligned}
|\mathscr{S}| &= |\mathscr{T}_1| + |\mathscr{T}_2| \\
&= |\mathscr{T}_{1,1}| + |\mathscr{T}_{1,2}| + |\mathscr{T}_{1,3}| + |\mathscr{T}_{2,1}| + |\mathscr{T}_{2,2}| + |\mathscr{T}_{2,3}| \\
&\leq (2|L(S_1| - 1)/3 + (2|L(S_2)| - 1)/3 + 2|V(C_1) \cap V(C_2)|/3 \\
&\quad + \lfloor |S_1 - S_2|/2 \rfloor + \lfloor |S_2 - S_1|/2 \rfloor + \lfloor |S_1 \cap S_2|/2 \rfloor \\
&= (2(|L(S_1)| + |L(S_2)| + |V(C_1) \cap V(C_2)|) - 2)/3 \\
&\quad + 2\lfloor (7 - |S_1 \cap S_2|)/2 \rfloor + \lfloor |S_1 \cap S_2|/2 \rfloor \\
&= (2(|V(G)| - |S_1 \cup S_2|) - 2)/3 + 2\lfloor (7 - |S_1 \cap S_2|)/2 \rfloor + \lfloor |S_1 \cap S_2|/2 \rfloor \\
&= (2|V(G)| + 2|S_1 \cap S_2| - 30)/3 + 2\lfloor (7 - |S_1 \cap S_2|)/2 \rfloor + \lfloor |S_1 \cap S_2|/2 \rfloor .
\end{aligned}
$$

Since $0 \leq |S_1 \cap S_2| \leq 6$, this implies that $|\mathscr{S}| \leq (2|V(G)| - 8)/3$, which contradicts (3.1). This completes the proof of the Theorem.

References

[1] J.Cheriyan and R.Thurimella, Fast algorithms for k-shredders and k-node connectivity augumentation, Proc. 28th ACM STOC, 1996, pp. 37-46.

[2] Y.Egawa, k-Shredders in k-connected graphs, preprint.

[3] Y.Egawa and Y.Okadome, 5-Shredders in 5-connected graphs, preprint.

[4] T.Jordán, On the number of shredders, *J. Graph Theory,* **31**(1999), 195-200.

[5] M.Tsugaki, 6-Shredders in 6-connected graphs, *SUT J. Math.,* **39**(2003), 211-224.