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AN ALTERNATIVE DEFINITION OF THE k-IRREDUNDANCE

ODILE FAVARON
L.R.I., URM 8623
Bât. 490, Université de Paris-Sud
91405-Orsay cedex, France
E-mail: of@lri.fr

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Abstract

In accordance with the k-domination and the k-independence introduced by Fink and Jacobson in 1985, Jacobson, Peters and Rall defined in 1990 the concept of k-irredundance. We propose here a slightly different definition avoiding some inconveniences and keeping the main properties of the former one.

Keywords: Domination, Independence, Irredundance

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1. Introduction

For notation and graph theory terminology, especially for domination problems, we in general follow [12]. In a graph $G = (V(G), E(G))$ of order $n(G)$, the *neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. If S is a subset of vertices, its neighborhood is $N_G(S) = \cup_{v \in S} N_G(v)$. The closed neighbohoods of v and S are $N_G[v] = N_G(v) \cup \{v\}$ and $N_G[S] = N_G(S) \cup S$. The *degree* of a vertex v of G, denoted by $d_G(v)$, is the order of its neighborhood. The maximum degree of G is $\Delta(G) = \max\{d_G(v); v \in V(G)\}$. The maximum degree of the subgraph induced in G by $S \subseteq V(G)$ is denoted $\Delta(S)$. When no confusion can arrise we usually write V, E, n, $d(v)$, $N(v)$, ... for $V(G)$, $E(G)$, $n(G)$, $d_G(v)$, $N_G(v)$, ... Given a positive integer k and a subset $S \subseteq V$, the k-*neighborhood* of S is the set $N_k(S)$ of the vertices of G adjacent to at least k vertices of S and its closed k-*neighborhood* is $N_k[S] = N_k(S) \cup S$. For any parameter μ associated to a graph property $\mathcal{P}$, we refer to a set of vertices with Property $\mathcal{P}$ and cardinality $\mu(G)$ as a $\mu(G)$-set or μ-set. An *independent set* S is a set of vertices of G inducing a subgraph with no edge. Equivalently, $\Delta(S) = 0$. A *dominating set* S is a set of vertices such that every vertex in $V \setminus S$ has at least one neighbor in S. Equivalently, $N[S] = V$. In [10, 11] Fink and Jacobson defined a generalization of the concepts of independence and domination. For an integer $k \geq 1$, a set S of V is k-*independent* if $\Delta(S) < k$ and k-*dominating* if every vertex in $V \setminus S$ has at least k neighbors in S. Clearly every subset of a k-independent set is k-independent and every superset of a k-dominating set is k-dominating. Therefore the maximality of a k-independent set and the minimality of a k-dominating set can be defined by addition or deletion of a single vertex. A k-independent set S is maximal in G if for every $x \in V \setminus S$, $S \cup \{x\}$ is not k-independent and a k-dominating set S of G is minimal if for every $x \in S$, $S \setminus \{x\}$ is not k-dominating. The *lower k-independence number* $i_k(G)$ is the minimum cardinality of a maximal k-independent set in G and the *k-independence number* $\beta_k(G)$ is the maximum cardinality of a k-independent set. Similarly, the *k-dominating number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G and the *k-upper dominating number* $\Gamma_k(G)$ is the maximum cardinality of a minimal k-dominating set of G. Clearly $i_k(G) \leq \beta_k(G)$ and $\gamma_k(G) \leq \Gamma_k(G)$. We also notice that the 1-independent sets and the 1-dominating sets are respectively the classical independent sets and dominating sets, so $i_1(G) = i(G)$, $\beta_1(G) = \beta(G)$, $\gamma_1(G) = \gamma(G)$ and $\Gamma_1(G) = \Gamma(G)$.

For $k = 1$, it is well known that an independent set is maximal if and only if it is also dominating and that a maximal independent set is a minimal dominating set, thus implying $\gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G)$ for every graph G. This property cannot be completely generalized because for $k \geq 2$, a maximal k-independent set is not necessarily k-dominating. However there always exist sets which are both k-independent and k-dominating [8], and these sets are maximal k-independent and minimal k-dominating [9]. Hence we still have $\gamma_k(G) \leq \beta_k(G)$ and $i_k(G) \leq \Gamma_k(G)$ for every k. But γ_k may be larger than i_k and Γ_k may be smaller than β_k. Finally, since every k-independent set is $(k+1)$-independent and every $(k+1)$-dominating set of G is k-dominating, the sequences (β_k) and (γ_k) are weakly increasing. Hence

$$\beta(G) = \beta_1(G) \leq \beta_2(G) \leq \beta_3(G) \leq \cdots \leq \beta_\Delta(G) < \beta_{\Delta+1}(G) = n,$$

$$\gamma(G) = \gamma_1(G) \leq \gamma_2(G) \leq \gamma_3(G) \leq \cdots \leq \gamma_\Delta(G) < \gamma_{\Delta+1}(G) = n.$$

As often, the behavior of the minmax or maxmin parameters is less regular and the sequences (i_k) and (Γ_k) are not necessarily monotone. More details and results on k-domination and k-independence can be found in [1, 2, 3, 4, 6, 8, 9, 10, 11, 12, 14].

The concept of irredundance was introduced by Cockayne, Hedetniemi and Miller in 1978 [7] to extend to not necessarily dominating sets the property which makes a dominating set S minimal. Different equivalent formulations can be given for this property.

- $\mathcal{P}_1$: $\forall x \in S$, either x or a vertex x' of $V \setminus S$ which is dominated by S is not dominated by $S \setminus \{x\}$.
- $\mathcal{P}_1'$: $\forall x \in S$, either x or a neighbor x' of x in $V \setminus S$ is not dominated by $S \setminus \{x\}$.
- $\mathcal{P}_1''$: $\forall x \in S$, $N[S] \setminus N[S \setminus \{x\}] \neq \emptyset$.

A subset S of vertices is *irredundant* if it satisfies any of the equivalent properties $\mathcal{P}_1$, $\mathcal{P}_1'$ or $\mathcal{P}_1''$. Any subset of an irredundant set is irredundant and an irredundant set S is maximal if for every $x \in V \setminus S$, $S \cup \{x\}$ is not irredundant. The minimum and maximum cardinalities of a maximal irredundant set are respectively $\mathrm{ir}(G)$ and $\mathrm{IR}(G)$. It is also well-known that a dominating set is minimal if and only if it is irredundant and that a minimal dominating set is maximal irredundant, which leads to the complete inequality chain for every graph G

$$\mathrm{ir}(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq \mathrm{IR}(G). \tag{1}$$

Properties $\mathcal{P}_1$, $\mathcal{P}_1'$, $\mathcal{P}_1''$ related to a subset S of V can be generalized to the case $k \geq 2$ as follows

- $\mathcal{P}_k$: $\forall x \in S$, either x or a vertex x' of $V \setminus S$ which is k-dominated by S is not k-dominated by $S \setminus \{x\}$.
- $\mathcal{P}_k'$: $\forall x \in S$, either x or a neighbor x' of x in $V \setminus S$ is not k-dominated by $S \setminus \{x\}$.
- $\mathcal{P}_k''$: $\forall x \in S$, $N_k[S] \setminus N_k[S \setminus \{x\}] \neq \emptyset$.

Properties $\mathcal{P}_k$ and $\mathcal{P}''_k$ are equivalent and mean that $\forall x \in S$, either x has fewer than k neighbors in S or x has a neighbor x' in $V \setminus S$ adjacent to exactly k vertices of S. Property $\mathcal{P}'_k$, which means that $\forall x \in S$, either x has fewer than k neighbors in S or x has a neighbor x' in $V \setminus S$ adjacent to at most k vertices of S, is weaker than $\mathcal{P}_k$ when $k > 1$. But restricted to k-dominating sets S, the three properties are equivalent since x' has at least k neighbors in S. Hence a k-dominating set is minimal if and only if it satisfies any of them.

In 1990, Jacobson, Peters and Rall [13] extended the concept of irredundance to k-irredundance and defined a k*-irredundant set* as a subset $S \subseteq V$ satisfying Property $\mathcal{P}_k$, or equivalently Property $\mathcal{P}''_k$. A k-irredundant set S is *maximal* if no superset of S is k-irredundant, and the minimum and maximum cardinalities of a maximal k-irredundant set of G are denoted by $\text{ir}_k(G)$ and $\text{IR}_k(G)$. Jacobson et al. established nice properties on these new parameters, in particular they proved that for every k and every graph G,

$$\beta_k(G) \leq \text{IR}_k(G) \quad \text{and} \quad \text{ir}_k(G) \leq \gamma_k(G) \leq \Gamma_k(G) \leq \text{IR}_k(G). \tag{2}$$

However this definition presents two inconveniences pointed out in [13] with interesting examples. First, a subset of a k-irredundant set is not necessarily k-irredundant and consequently a single vertex extension is not sufficient for determining maximality of k-irredundant sets. Second, a k-irredundant set is not necessarily $(k+1)$-irredundant and consequently the sequence of the max parameter $\text{IR}_k(G)$ is not necessarily monotone.

Our purpose is to propose another definition of the k-irredundance keeping the properties (2) satisfied by the first definition but avoiding the two mentioned inconveniences.

2. Another definition of the k-irredundance

To avoid confusion we use throughout the paper an accentuation "prime" to denote all what concerns the new definition.

Definition 1. *A subset S of vertices of a graph G is k-irredundant' if it satisfies $\mathcal{P}'_k$, in other terms if $\forall x \in S$, either x has fewer than k neighbors in S or x has a neighbor x' in $V \setminus S$ adjacent to at most k vertices of S.*

First we observe that the property for a subset of vertices to be k-irredundant' is hereditary.

Proposition 2. *Every subset of a k-irredundant' set S is k-irredundant'.*

Proof. It is sufficient to prove that $\forall y \in S$, $S \setminus \{y\}$ is k-irredundant'. Let $x \in S \setminus \{y\}$. If x has at least k neighbors in $S \setminus \{y\}$, it has at least k neighbors in S and by $\mathcal{P}'_k$ applied to S, x has a neighbor x' in $V \setminus S$ adjacent to at most k vertices of S. Therefore x' has at most k neighbors in $S \setminus \{y\}$ and $S \setminus \{y\}$ satisfies $\mathcal{P}'_k$. □

From Proposition 2 we can define a *maximal k-irredundant' set* of G as a k-irredundant' set S such that $\forall x \in V \setminus S$, $S \cup \{x\}$ is not k-irredundant'. Let $\text{ir}'_k(G)$ and $\text{IR}'_k(G)$ be respectively the minimum and maximum cardinality of a maximal k-irredundant' set of G. For $k = 1$, $\mathcal{P}'_k$ is the same as $\mathcal{P}'_1$ and a (maximal) 1-irredundant' set is a usual (maximal) irredundant set. Therefore $\text{ir}'_1(G) = \text{ir}(G)$ and $\text{IR}'_1(G) = \text{IR}(G)$.

The second property of the k-irredundance' is a direct consequence of the definition of $\mathcal{P}'_k$.

Proposition 3. *Every k-irredundant' set of G is $(k+1)$-irredundant'.* □

Proposition 3 implies that as the sequences $(\beta_k(G))$ and $(\gamma_k(G))$, the sequence (IR'$_k(G)$) is weakly increasing in every graph. Since V is a $(\Delta+1)$-irredundant' set but not a Δ-irredundant' set, we have

$$\mathrm{IR}(G) = \mathrm{IR}'_1(G) \leq \mathrm{IR}'_2(G) \leq \mathrm{IR}'_3(G) \leq \cdots \leq \mathrm{IR}'_\Delta(G) < \mathrm{IR}'_{\Delta+1}(G) = n .$$

Therefore the two inconveniences of the first definition of the k-irredundance mentioned in Section 1 do not exist for the second definition. We now check that the inequalities (2) still hold with ir'_k and IR'_k.

Proposition 4. *Every k-independent set is k-irredundant' and thus $\beta_k(G) \leq \mathrm{IR}'_k(G)$ for every graph and every value of k. Moreover $\beta_\Delta(G) = \mathrm{IR}'_\Delta(G)$.*

Proof. That every k-independent set is k-irredundant' is clear from the definition of $\mathcal{P}'_k$. Let now S be any Δ-irredundant' set of G. If S contains a vertex x with more than $\Delta-1$ neighbors in S, then x has a neighbor x' in $V \setminus S$ adjacent to at most Δ vertices in S. This means that x has degree more than Δ, a contradiction. Hence every vertex of S has degree at most $\Delta-1$ in S and S is Δ-independent. Therefore $\mathrm{IR}'_\Delta(G) \leq \beta_\Delta(G)$, thus implying $\beta_\Delta(G) = \mathrm{IR}'_\Delta(G)$. □

Proposition 5. *A k-dominating set of G is minimal if and only if it is k-irredundant' and a k-dominating k-irredundant' set is a maximal k-irredundant' set of G.*

Proof. We saw in Section 1 that a k-dominating set is minimal if and only if it satisfies $\mathcal{P}'_k$, that is if and only if it is k-irredundant'. Let S be a k-dominating k-irredundant' set and let $x \in V \setminus S$. Since S is k-dominating, x has at least k neighbors in $S' = S \cup \{x\}$ and every neighbor x' of x in $V \setminus S'$ has at least k neighbors in S and thus more than k neighbors in S'. Therefore S' is not k-irredundant' and S is a maximal k-irredundant' set of G. □

Corollary 6. *Every graph G satisfies $\mathrm{ir}'_k(G) \leq \gamma_k(G) \leq \Gamma_k(G) \leq \mathrm{IR}'_k(G)$ for every $k \geq 1$.*

Proof. The inequality chain is an obvious consequence of Proposition 5 since every $\gamma_k(G)$-set or $\Gamma_k(G)$-set is a maximal k-irredundant' set. □

However $\mathrm{ir}'_k(G)$ can be larger or smaller than $i_k(G)$. For instance, the graph H constructed from $k \geq 3$ disjoint cliques A_i of order k by choosing one vertex x_i in each clique A_i and adding the $k(k-1)/2$ edges joining the k vertices x_i satisfies $i_k(H) = k$ and $\mathrm{ir}'_k(H) = k^2 - k > i_k(H)$. The graph J_2 constructed from a clique of vertex set $\{x_1, x_2, x_3, x_4\}$ and six independant sets $A_{i,j}$, $1 \leq i < j \leq 4$, of order $p \geq 3$ by adding for every pair $\{i,j\}$ all the edges joining x_i and x_j to every vertex of $A_{i,j}$ satisfies $i_2(J_2) = p + 2$ and $\mathrm{ir}'_2(J_2) = 4 < i_2(J_2)$. For any k this construction can be generalized to a graph J_k satisfying $i_k(J_k) > \mathrm{ir}'_k(J_k)$.

We give now some properties of IR'. Proposition 7 improves the inequality $\gamma(G) + \mathrm{IR}(G) \leq n$ valid in every graph without isolated vertex [5].

Proposition 7. *Every graph G with order n and minimum degree $\delta \geq 1$ satisfies $\gamma(G) + \mathrm{IR}'_\delta(G) \leq n$.*

Proof. Let S be any $\mathrm{IR}'_\delta(G)$-set. Every vertex $x \in S$ has at least one neighbor in $V \setminus S$, by the definition of δ if x has less than δ neighbors in S, and by the definition of the δ-irredundance' otherwise. Hence $V \setminus S$ is a dominating set of G and $\gamma(G) \leq n - |S| = n - \mathrm{IR}'_\delta(G)$. □

Since $\gamma(G) \geq n/(\Delta+1)$ holds in every graph, Proposition 7 admits the following

Corollary 8. *Every graph with order n, minimum degree $\delta \geq 1$ and maximum degree Δ satisfies* $\mathrm{IR}'_\delta(G) \leq n\Delta/(\Delta+1)$. □

The bounds in Proposition 7 and Corollary 8 are sharp. For instance in a prism $K_p \square K_2$, $n = 2p$, $\delta = \Delta = p$, $\gamma = 2$ and since the IR'_δ-sets are the prisms $K_{p-1}\square K_2$, $\mathrm{IR}'_\delta = 2p-2 = n-\gamma$. In a clique K_n, $\mathrm{IR}'_{n-1} = n-1 = n-\gamma = n\Delta/(\Delta+1)$. In a cycle C_{6p}, $\mathrm{IR}'_2 = 4p = n-\gamma = n\Delta/(\Delta+1)$.

The last proposition relates the new k-irredundance' parameter to the former one.

Proposition 9. $\mathrm{IR}'_k(G) \geq \max\{\mathrm{IR}_j(G) \mid 1 \leq j \leq k\}$ *for every graph G and every value of k.*

Proof. Every $\mathrm{IR}_k(G)$-set S satisfies $\mathcal{P}_k$ by definition and thus also $\mathcal{P}'_k$. Hence S is k-irredundant' and $\mathrm{IR}_k(G) \leq \mathrm{IR}'_k(G)$. Therefore $\mathrm{IR}_j(G) \leq \mathrm{IR}'_j(G)$ for all $1 \leq j \leq k$ and since the sequence (IR'_k) is weakly increasing, $\mathrm{IR}'_j(G) \leq \mathrm{IR}'_k(G)$. Whence $\mathrm{IR}'_k(G) \geq \max\{\mathrm{IR}_j(G) \mid 1 \leq j \leq k\}$. □

It was proved in Corollary 3.7 of [13] that in a graph of minimum degree $\delta \geq 1$, $\gamma(G) \leq \min\{n - \mathrm{IR}_k(G) \mid k = 1, 2, \cdots, \delta\}$. This result is a direct consequence of Propositions 7 and 9.

References

[1] Y. Caro and Y. Roditty, A note on the k-domination number of a graph, *Internat. J. Math. & Math. Sci.,* **13** (1990), 205-206.

[2] Y. Caro and Z. Tuza, Improved lower bounds on k-independence, *J. Graph Theory,* **15** (1991), 99-107.

[3] G. Chen and M. S. Jacobson, On a relationship between 2-dominating and 5-dominating sets in graphs, *J. Combin. Math. Combin. Comput.,* **39** (2001), 139-145.

[4] B. Chen and S. Zhou, Upper bounds for f-domination number of graphs, *Discrete Math.,* **185** (1998), 239-243.

[5] E. J. Cockayne, O. Favaron, C. Payan, A. G. Thomason, Contributions to the theory of domination, independence and irredundance in graphs, *Discrete Math.,* **33** (1981), 249-258.

[6] E. J. Cockayne, B. Gamble and B. Shepherd, An upper bound for the k-domination number of a graph,*J. Graph Theory,* **9** (1985), 533-534.

[7] E. J. Cockayne, S. T. Hedetniemi and D. J. Miller, Properties of hereditary hypergraphs and middle graphs,*Canad. Math. Bull.,* **21** (1978), 461-468.

[8] O. Favaron, k-domination and k-independence in graphs, *Ars Combin.,* **25C** (1988), 159-167.

[9] O. Favaron, On a conjecture of Fink and Jacobson concerning k-domination and k-dependence, *J. Combin. Theory Seri. B,* **39**(1) (1985), 101-102.

[10] J. F. Fink and M. S. Jacobson, n-domination in graphs, in : Graph Theory with Applications to Algorithms and Computer (Kalamazoo, Mich., 1984), eds. Alavi, Chartrand, Lesniak, Lick and Wall, Wiley, New-York, (1985), 283-300.

[11] J. F. Fink and M. S. Jacobson, n-domination, n-dependence and forbidden subgraphs, in : Graph Theory with Applications to Algorithms and Computer (Kalamazoo, Mich., 1984), eds. Alavi, Chartrand, Lesniak, Lick and Wall, Wiley, New York, (1985), 301-311.

[12] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.

[13] M. S. Jacobson, K. Peters and D. F. Rall, On n-irredundance and n-domination, *Ars Combin.,* **29B** (1990), 151-160.

[14] C. Stracke and L. Volkmann, A new domination conception, *J. Graph Theory,* **17** (1993), 315-323.