ON WEAK AND RESTRAINED DOMINATION IN TREES

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Abstract

In a graph \( G = (V, E) \) a vertex is said to dominate itself and all its neighbours. A weak dominating set is a set \( S \subseteq V \) where for every vertex \( u \) not in \( S \) there is a vertex \( v \) of \( S \) adjacent to \( u \) with \( d_G(v) \leq d_G(u) \). A restrained dominating set is a set \( S \subseteq V \) where every vertex in \( V - S \) is adjacent to a vertex in \( S \) as well as another vertex in \( V - S \). The weak domination number \( \gamma_w(G) \) (resp. restrained domination number \( \gamma_r(G) \)) is the minimum cardinality of a weak dominating set (resp. restrained dominating set). We determine sharp bounds for the weak and restrained domination numbers of a tree in terms of the domination number, the order, number of leaves and support vertices. More precisely, we show that if \( T \) is a tree of order \( n \geq 3 \) with \( \ell \) leaves and \( s \) support vertices, then \( \gamma_w(T), \gamma_r(T) \geq [(n + 2 + \ell - s)/3] \), and \( \gamma_w(T), \gamma_r(T) \geq \gamma(T) + \ell - s \geq [(n + 2 + 2\ell - 3s)/3] \) improving those of Hattingh and Rautenbach. We also show that \( \gamma_w(T) \leq [(n + 2\ell + 2s - 3)/3] \) and \( \gamma_r(T) \leq [(n + 2\ell + s + 1)/3] \).

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1. Introduction

In a graph \( G = (V, E) \), the open neighborhood of a vertex \( v \in V \) is \( N(v) = \{u \in V \mid uv \in E\} \) and the closed neighborhood is \( N[v] = N(v) \cup \{v\} \). For a set \( S \subseteq V \), the open neighborhood is \( N(S) = \bigcup_{v \in S} N(v) \), the closed neighborhood is \( N[S] = N(S) \cup S \) and \( G[S] \) is the subgraph induced by the vertices of \( S \). The degree of a vertex \( v \) denoted by \( d_G(v) \) is the size of its open neighborhood. A vertex of degree one is called a pendent vertex or a leaf and its neighbor is called a support vertex. If \( v \) is a support vertex of a tree \( T \) then \( L_v \) will denote the set of the leaves attached at \( v \). A support vertex \( v \) is called strong if \( |L_v| > 1 \).

A set \( S \subseteq V \) is a dominating set if for each vertex \( v \in V - S \), \( N(v) \cap S \neq \emptyset \). The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set.

In this paper, we are interested in two variations of domination called weak and restrained domination. A set \( S \subseteq V \) is a weak dominating set (WDS) if every vertex \( v \in V - S \) is adjacent to a vertex \( u \in D \) where \( d_G(v) \geq d_G(u) \). The weak domination number \( \gamma_w(G) \) is the minimum cardinality of a weak dominating set. A dominating set \( S \subseteq V \) is a restrained dominating set (RDS) if the subgraph induced by the vertices of \( V - S \) has no isolated vertices. The restrained
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The domination number \( \gamma_r(G) \) is the minimum cardinality of a restrained dominating set. An end-dominating set of \( G \) is a dominating set that contains all leaves of \( G \). The end-domination number \( \gamma_e(G) \) is the minimum cardinality of an end-dominating set. The concept of weak domination was introduced by Sampathkumar and Pushpa Latha [7], while the concept of restrained domination was introduced by Telle and Proskurowski [8], albeit as a vertex partitioning problem. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [4], [5]. We make a couple of straightforward observations.

**Observation 1.** If \( v \) is a leaf of a graph \( G \), then \( v \) is in every weak and restrained dominating set.

**Observation 2.** If \( G \) is a connected graph then,

1) If \( n \geq 3 \) then \( \gamma_e(G) \leq \gamma_w(G) \).
2) \( \gamma_e(G) \leq \gamma_r(G) \).

In [3], Hattingh and Rautenbach have given a lower bound for the weak and restrained domination numbers in trees.

**Theorem 3.** (Hattingh and Rautenbach [3]) If \( T \) is a tree of order \( n \geq 1 \), then

a) \( \gamma_w(T) \geq \lceil (n+2)/3 \rceil \) for \( n \neq 2 \).

b) \( \gamma_r(T) \geq \lceil (n+2)/3 \rceil \).

In this paper we present different lower and upper bounds of the weak and restrained domination numbers of a tree in terms of the domination number, the order, the number of leaves and support vertices that improve those of Theorem 3.

2. Lower bounds

We begin by giving a lower bound on the end-domination number for every tree.

**Theorem 4.** If \( T \) is a tree with \( \ell \) leaves and \( s \) support vertices, then \( \gamma_e(T) \geq \lceil (n+2+\ell-s)/3 \rceil \).

**Proof.** We proceed by induction on the order \( n \). Clearly the result holds if \( diam(T) \in \{0,1\} \). If \( diam(T) = 2 \) then \( T \) is a star \( K_{1,n-1} \) where \( \gamma_e(T) = n-1 \geq \lceil (n+2+\ell-s)/3 \rceil = \lceil 2n/3 \rceil \), and hence the result is valid, establishing the base case.

Assume that every tree \( T' \) of order \( n' < n \) with \( \ell' \) leaves and \( s' \) support vertices satisfies \( \gamma_e(T') \geq \lceil (n'+2+\ell'-s')/3 \rceil \). Let \( T \) be a tree of order \( n \) with \( \ell \) leaves and \( s \) support vertices.

If any support vertex, say \( x \), of \( T \) is strong, then let \( T' \) be the tree obtained from \( T \) by removing a leaf adjacent to \( x \). Then \( \gamma_e(T') = \gamma_e(T) - 1, \ell' = \ell - 1 \) and \( s' = s \). Applying the inductive hypothesis to \( T' \), we obtain the desired result. Henceforth, we can assume that every support vertex of \( T \) is adjacent to exactly one leaf.

We now root \( T \) at a vertex \( r \) of maximum eccentricity \( diam(T) \geq 3 \). Let \( v \) be a support vertex at maximum distance from \( r \), \( u \) its parent in the rooted tree and \( S \) a \( \gamma_e(T) \)-set. Denote
by $T_v$ the subtree induced by a vertex $v$ and its descendants in the rooted tree $T$. We distinguish between two cases.

**Case 1.** $d_T(u) \geq 3$. Then either $u$ is a support vertex of $T$ or $u$ has a child besides $v$ as a support vertex. Let $T' = T - T_v$. Clearly, $n' = n - 2, \ell' = \ell - 1$ and $s' = s - 1$. Then $S$ contains the leaf neighbor of $v$, say $v'$ and without loss of generality $v \notin S$ else we can substitute it by $u$ in $S$. Thus, $S - \{v\}$ is an end-dominating set of $T'$, implying that $\gamma_e(T') \leq \gamma_e(T) - 1$. Applying the inductive hypothesis to $T'$, it follows that

$$\gamma_e(T) \geq \gamma_e(T') + 1 \geq [(n' + 2 + \ell' - s')/3] + 1 = [(n + 3 + \ell - s)/3].$$

**Case 2.** $d_T(u) = 2$. Since $\text{diam}(T) \geq 3$, let $w$ be the parent of $u$ in the rooted tree. Let $T' = T - T_u$. Then $n' = n - 3$. The result is valid if $T = P_3$ or $P_5$, so let $n' \geq 3$. We can assume that $v' \in S, v \notin S, w \notin S$ (else replace $v$ or $u$ by $w$ in $S$) and $w \in S$. Thus $S - \{v'\}$ is an end-dominating set of $T'$, implying that $\gamma_e(T') \leq \gamma_e(T) - 1$. Now if $d_T(w) \geq 3$ then $\ell' = \ell - 1$ and $s' = s - 1$. If $d_T(w) = 2$, that is $w$ is a leaf in $T'$, then $\ell' = \ell$ and $s' \leq s$. In any case, it follows by induction on $T'$ that:

$$\gamma_e(T) \geq \gamma_e(T') + 1 \geq [(n' + 2 + \ell' - s')/3] + 1 = [(n + 2 + \ell - s)/3].$$

This achieves the proof.

According to Observation 2 and Theorem 4, we have the following corollary:

**Corollary 5.** If $T$ is a tree of order $n \geq 1$, then

a) $\gamma_w(T) \geq [(n + 2 + \ell - s)/3]$ for $n \neq 2$.

b) $\gamma_r(T) \geq [(n + 2 + \ell - s)/3]$.

Clearly since $\ell - s \geq 0$ for every tree, Corollary 5 improves Theorem 3.

**Proposition 6.** If $T$ is a tree of order at least three with $\ell$ leaves and $s$ support vertices, then $\gamma_e(T) \geq \gamma(T) + \ell - s$.

**Proof.** Let $L(T)$ and $S(T)$ denote the set of leaves and support vertices of $T$, respectively. Then since every $\gamma_e(T)$-set $D$ contains $L(T)$, $(D - L(T)) \cup S(T)$ is a dominating set of $T$. Hence

$$\gamma(T) \leq |D| - \ell + s. \qed$$

Recall that the connected domination number of a graph $G$ denoted by $\gamma_c(G)$ is the minimum cardinality of a dominating set whose induced subgraph is connected. In [1], Duchet and Meyniel proved that every connected graph $G$, satisfies $\gamma(G) \geq (\gamma_c(G) + 2)/3$. Since $\gamma_c(T) = n - \ell$ for every tree $T$, it follows that $\gamma(T) \geq (n + 2 - \ell)/3$. Note that extremal trees have been characterized by Lemańiska [6]. Using this lower bound and Proposition 6, we have:

**Corollary 7.** If $T$ is a tree of order at least three with $\ell$ leaves and $s$ support vertices, then $\gamma_e(T) \geq [(n + 2\ell - 3s + 2)/3]$.

By Observation 2, Proposition 6 and Corollary 7, we obtain a second lower bound on $\gamma_w(T)$ and $\gamma_r(T)$ which improves Corollary 5 for $\ell > 2s$.\[\text{Mustapha Chellali}\]
Corollary 8. If $T$ is a tree of order $n \geq 3$, then

a) $\gamma_w(T) \geq \gamma(T) + \ell - s - \left\lceil (n + 2 + 2\ell - 3s)/3 \right\rceil$, and

b) $\gamma_r(T) \geq \gamma(T) + \ell - s - \left\lceil (n + 2 + 2\ell - 3s)/3 \right\rceil$.

All inequalities in Corollary 8 are attained. For example, let $T$ be the tree obtained from a path $P_3$ by attaching a center vertex of a star $K_{1,3}$ at each leaf of the path. Then $n = 11, \ell = 6, s = 2$, $\gamma(T) = 3$ and $\gamma_w(T) = \gamma_r(T) = 7$.

3. Upper bounds

We give in this section an upper bound for the weak and restrained domination numbers of a nontrivial tree.

Theorem 9. If $T$ is a nontrivial tree with $n$ vertices, $\ell$ leaves and $s$ support vertices, then $\gamma_w(T) \leq \left\lceil (n + 2\ell + 2s - 3)/3 \right\rceil$, and this bound is sharp.

Proof. We proceed by induction on $n \geq 2$. Clearly the result is valid if $T = P_2$. If $T$ is a star $K_{1,p}(p \geq 2)$, then $\gamma_w(T) = p = \left\lceil (n + 2\ell + 2s - 3)/3 \right\rceil$, and hence the result holds. Assume that every tree $T'$ of order $2 \leq n' < n$ satisfies $\gamma_w(T') \leq \left\lceil (n' + 2\ell' + 2s' - 3)/3 \right\rceil$. Let $T$ be a tree of order $n$.

Root $T$ at a vertex $r$ of maximum eccentricity $\text{diam}(T) \geq 3$. Let $v$ be a support vertex at maximum distance $\text{diam}(T) - 1$ from $r$ and $u$ the parent of $v$. We consider the following two cases:

Case 1. $d_T(u) \geq 3$, that is $u$ is either a support vertex or has a child besides $v$ as a support vertex. Let $T' = T - T_v$. Then $n' = n - |L_v| - 1 \geq 3$, $\ell' = \ell - |L_v|$ and $s' = s - 1$. Let $S'$ be any $\gamma_w(T')$-set. Then $S'$ can be extended to a WDS of $T$ by adding the set of leaves $L_v$ with a possibility to replace $u$ (if $u \in S'$) by its parent since its degree will be increased in $T$. Such a substitution is possible since the goal of $u$ in $S'$ is either to weakly dominate itself or its parent. So $\gamma_w(T) \leq \gamma_w(T') + |L_v|$. Now by induction on $T'$ we obtain:

$$\gamma_w(T) \leq \left\lceil (n' + 2\ell' + 2s' - 3)/3 \right\rceil + |L_v| = \left\lceil (n + 2\ell + 2s - 6)/3 \right\rceil$$

Case 2. $d_T(u) = 2$. Since $\text{diam}(T) \geq 3$, let $w$ be the parent of $u$. If $d_T(w) = 1$, then $T$ is a double star $S_{L_v,1}$ and the result holds. Assume that $d_T(w) \geq 3$. Seeing Case 1, we suppose that every subtree rooted at a child of $w$ is a star. If $d_T(w) \geq 4$ or $d_T(w) = 3$ and every leaf of $V(T_w) - L_v$ is at distance one or two from $w$, then let $T' = T - T_u$ and $S'$ be any $\gamma_w(T')$-set. If $w \notin S'$, then $S'$ can be extended to a WDS of $T$ by adding the set $L_v \cup \{u\}$. Assume now that $w \in S'$. Then $w$ weakly dominates itself or its parent and so $S'$ can be extended to a WDS of $T$ (possibly by replacing $w$ by its parent) by adding the set $L_v \cup \{u\}$. In any case, $\gamma_w(T) \leq \gamma_w(T') + |L_v| + 1$. By induction on $T'$, and since $n' = n - |L_v| - 2$, $s' = s - 1$ and $\ell' = \ell - |L_v|$, we obtain:

$$\gamma_w(T) \leq \gamma_w(T') + |L_v| + 1 \leq \left\lceil (n' + 2\ell' + 2s' - 3)/3 \right\rceil + |L_v| + 1 = \left\lceil (n + 2\ell + 2s - 4)/3 \right\rceil.$$

Now we examine the case, $d_T(w) = 3$ and every leaf in $T_w$ is at distance three from $w$. Let $y$ be the second support vertex at distance two from $w$ in $T_w$. Clearly seeing Case 1, the parent of
Let \( T' = T - (T_v \cup T_y) \). Then \( n' = n - |L_v| - |L_y| - 2 \), \( s' \leq s - 1 \) and \( \ell' = \ell - |L_v| - |L_y| + 2 \). Since every \( \gamma_w(T') \)-set contains \( u \) and \( z \), such a set can be extended to a WDS of \( T \) by adding the set \( L_v \cup L_y \), and so \( \gamma_w(T) \leq \gamma_w(T') + |L_v| + |L_y| \). By induction on \( T' \), we have:

\[
\gamma_w(T) \leq \gamma_w(T') + |L_v| + |L_y| \leq \left\lfloor \frac{(n' + 2\ell' + 2s' - 3)}{3} \right\rfloor + |L_v| + |L_y|
= \left\lfloor \frac{(n + 2\ell + 2s - 3)}{3} \right\rfloor.
\]

Finally, assume that \( d_T(w) = 2 \) and let \( T' = T - T_u \). Then \( n' = n - |L_v| - 2 \). It is a routine matter to check the result if \( n' = 1 \) or \( 2 \). Thus assume that \( n' \geq 3 \). Since every \( \gamma_w(T') \)-set \( S' \) contains \( w \), such a set can be extended to a WDS of \( T \) by adding the set \( L_v \). Hence \( \gamma_w(T) \leq \gamma_w(T') + |L_v| \). Applying the induction on \( T' \), and since \( \ell' = \ell - |L_v| + 1 \) and \( s' \leq s \) we have:

\[
\gamma_w(T) \leq \gamma_w(T') + |L_v| \leq \left\lfloor \frac{(n' + 2\ell' + 2s' - 3)}{3} \right\rfloor + |L_v|
= \left\lfloor \frac{(n + 2\ell + 2s - 3)}{3} \right\rfloor.
\]

The bound is attained for stars and paths \( P_{3k+2}(k \geq 1) \). 

**Theorem 10.** If \( T \) is a nontrivial tree with \( n \) vertices, \( \ell \) leaves and \( s \) support vertices, then \( \gamma_r(T) \leq \left\lfloor \frac{(n + 2\ell + s + 1)}{3} \right\rfloor \), and this bound is sharp.

**Proof.** Again, we proceed by induction on \( n \geq 2 \). It is a routine matter to check the result if \( \text{diam}(T) \in \{1, 2, 3\} \). Assume that every tree \( T' \) of order \( 2 \leq n' < n \) satisfies \( \gamma_r(T') \leq \left\lfloor \frac{(n' + 2\ell' + s' + 1)}{3} \right\rfloor \), and let \( T \) be a tree of order \( n \).

If \( T \) contains a strong support vertex, then let \( T' \) be the tree obtained from \( T \) by removing a leaf adjacent to a strong support vertex. Then \( \gamma_r(T') = \gamma_r(T) - 1, n' = n - 1, \ell' = \ell - 1 \) and \( s' = s \). The result follows by induction on \( T' \). Hence we assume that \( T \) contains no strong support vertex.

Root \( T \) at a vertex \( r \) of maximum eccentricity \( \text{diam}(T) \geq 4 \). Let \( v \) be a support vertex at maximum distance \( \text{diam}(T) - 1 \) from \( r \), \( u \) the parent of \( v \) and \( w \) the parent of \( u \) in the rooted tree. We distinguish between cases:

**Case 1.** \( d_T(u) \geq 3 \). Let \( T' = T - T_u \). Then \( n' = n - 2d_T(u) + 2 - i \), where \( i = 1 \) if \( u \) is not a support vertex and \( i = 0 \) else. Since the result holds for \( 0 \leq n' \leq 2 \), we assume that \( n' \geq 3 \). Let \( S' \) be any \( \gamma_r(T') \)-set. If \( w \notin S \), then \( S' \) can be extended to a RDS of \( T' \) by adding the leaves of \( T_u \), and possibly \( v \) if \( u \) is a support vertex. Hence \( \gamma_r(T) \leq \gamma_r(T') + d_T(u) - 1 + i \). Since \( w \notin S' \), the degree of \( w \) in \( T \) must be equal to at least three. So \( \ell' = \ell - (d_T(u) - 1) \) and \( s' = s - (d_T(u) - 1) \). By induction on \( T' \),

\[
\gamma_r(T) \leq \gamma_r(T') + d_T(u) - 1 + i \leq \left\lfloor \frac{(n' + 2\ell' + s' + 1)}{3} \right\rfloor + d_T(u) - 1 + i
= \left\lfloor \frac{(n + 2\ell + s + 1 + 2 + 2i - 2d_T(u))}{3} \right\rfloor.
\]

Thus \( \gamma_r(T) \leq \left\lfloor \frac{(n + 2\ell + s + 1)}{3} \right\rfloor \) since \( d_T(u) \geq 3 \).

Now if \( w \in S' \), then \( S' \) can be extended to a RDS of \( T \) by adding the leaves of \( T_u \), and hence \( \gamma_r(T) \leq \gamma_r(T') + d_T(u) - 1 \). Likewise, since \( \ell' \leq \ell - (d_T(u) - 1) + 1 \) and \( s' \leq s - (d_T(u) - 1) + 1 \), the result follows by induction on \( T' \), and we obtain \( \gamma_r(T) \leq \left\lfloor \frac{(n + 2\ell + s + 1 + 5 - 2d_T(u) - i)}{3} \right\rfloor \leq \left\lfloor \frac{(n + 2\ell + s + 1)}{3} \right\rfloor \).

**Case 2.** \( d_T(u) = 2 \). Let \( T' = T - T_u \). Clearly the result holds if \( n' = (n - 3) \in \{1, 2\} \). So assume that \( n' \geq 3 \). Let \( S' \) be any \( \gamma_r(T') \)-set. If \( w \in S' \) then \( S' \) is extended to a RDS of \( T \) by
adding the leaf adjacent to $v$. Hence $\gamma_r(T) \leq \gamma_r(T') + 1$. By induction on $T'$, and since $\ell' \leq \ell$ and $s' \leq s$, we obtain $\gamma_r(T) \leq \gamma_r(T') + 1 \leq [(n + 2\ell + s + 1)/3]$.

If $w \not\in S'$ then $S'$ is extended to a RDS of $T$ by adding $v$ and its leaf. So $\gamma_r(T) \leq \gamma_r(T') + 2$. Since $w \not\in S'$, $\deg_T(w) \geq 3$, and so $\ell' = \ell - 1$ and $s' = s - 1$. Now by induction on $T'$, we have $\gamma_r(T) \leq \gamma_r(T') + 2 \leq [(n + 2\ell + s + 1)/3]$.

The bound is attained for stars and the path $P_6$.

Recall that the independent domination number $i(G)$ of a graph $G$ is the minimum cardinality of a set $S$ of vertices that is both dominating and independent. The following result is due to Favaron [2]:

**Theorem 11.** (Favaron [2]) If $T$ is a tree of order $n \geq 3$ with $\ell$ leaves, then $i(T) \leq [(n + \ell)/3]$.

Using Theorem 11, one can see that $\gamma(T) \leq [(n + s)/3]$ holds for every tree $T$ of order $n \geq 3$ with $s$ support vertices.

According to Corollary 8- (b) and Theorem 10, we have the following corollary which also strengthens the above upper bound on $\gamma(T)$ for $\ell \geq 3s + 3$.

**Corollary 12.** If $T$ is a tree of order $n \geq 3$, then

$$\gamma(T) \leq \min\{[(n + s)/3], [(n - \ell + 4s + 1)/3]\}.$$ 

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**References**


