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On weak and restrained domination in trees

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Abstract

In a graph $G=(V,E)$ a vertex is said to dominate itself and all its neighbours. A weak dominating set is a set $S \subseteq V$ where for every vertex u not in S there is a vertex v of S adjacent to u with $d_G(v) \leqslant d_G(u)$. A restrained dominating set is a set $S \subseteq V$ where every vertex in $V-S$ is adjacent to a vertex in S as well as another vertex in $V-S$. The weak domination number $\gamma_w(G)$ (resp. restrained domination number $\gamma_r(G)$) is the minimum cardinality of a weak dominating set (resp. restrained dominating set). We determine sharp bounds for the weak and restrained domination numbers of a tree in terms of the domination number, the order, number of leaves and support vertices. More precisely, we show that if T is a tree of order $n \geq 3$ with ℓ leaves and s support vertices, then $\gamma_w(T), \gamma_r(T) \geqslant \lceil (n+2+\ell-s)/3 \rceil$, and $\gamma_w(T), \gamma_r(T) \geq \gamma(T)+\ell-s \geq \lceil (n+2+2\ell-3s)/3 \rceil$ improving those of Hattingh and Rautenbach. We also show that $\gamma_w(T) \leqslant \lfloor (n+2\ell+2s-3)/3 \rfloor$ and $\gamma_r(T) \leqslant \lfloor (n+2\ell+s+1)/3 \rfloor$.

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1. Introduction

In a graph $G=(V,E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v)=\{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v]=N(v) \cup \{v\}$. For a set S of V, the *open neighborhood* is $N(S)=\cup_{v \in S} N(v)$, the *closed neighborhood* is $N[S]=N(S) \cup S$ and $G[S]$ is the *subgraph* induced by the vertices of S. The *degree* of a vertex v denoted by $d_G(v)$ is the size of its open neighborhood. A vertex of degree one is called a *pendent vertex* or a *leaf* and its neighbor is called a *support* vertex. If v is a support vertex of a tree T then L_v will denote the set of the leaves attached at v. A support vertex v is called *strong* if $|L_v|>1$.

A set $S \subseteq V$ is a *dominating set* if for each vertex $v \in V-S$, $N(v) \cap S \neq \emptyset$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set.

In this paper, we are interested in two variations of domination called weak and restrained domination. A set $S \subseteq V$ is a *weak dominating set* (WDS) if every vertex $v \in V-S$ is adjacent to a vertex $u \in D$ where $d_G(v) \geqslant d_G(u)$. The *weak domination number* $\gamma_w(G)$ is the minimum cardinality of a weak dominating set. A dominating set $S \subseteq V$ is a *restrained dominating set* (RDS) if the subgraph induced by the vertices of $V-S$ has no isolated vertices. The *restrained* *domination number* $\gamma_r(G)$ is the minimum cardinality of a restrained dominating set. An *end-dominating set* of G is a dominating set that contains all leaves of G. The *end-domination number* $\gamma_e(G)$ is the minimum cardinality of an end-dominating set. The concept of weak domination was introduced by Sampathkumar and Pushpa Latha [7], while the concept of restrained domination was introduced by Telle and Proskurowski [8], albeit as a vertex partitioning problem. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [4], [5]. We make a couple of straightforward observations.

Observation 1. *If v is a leaf of a graph G, then v is in every weak and restrained dominating set.*

Observation 2. *If G is a connected graph G then,*

1) *If $n \geqslant 3$ then $\gamma_e(G) \leqslant \gamma_w(G)$.*
2) *$\gamma_e(G) \leqslant \gamma_r(G)$.*

In [3], Hattingh and Rautenbach have given a lower bound for the weak and restrained domination numbers in trees.

Theorem 3. *(Hattingh and Rautenbach [3]) If T is a tree of order $n \geqslant 1$, then*

a) *$\gamma_w(T) \geqslant \lceil (n+2)/3 \rceil$ for $n \neq 2$.*
b) *$\gamma_r(T) \geqslant \lceil (n+2)/3 \rceil$.*

In this paper we present different lower and upper bounds of the weak and restrained domination numbers of a tree in terms of the domination number, the order, the number of leaves and support vertices that improve those of Theorem 3.

2. Lower bounds

We begin by giving a lower bound on the end-domination number for every tree.

Theorem 4. *If T is a tree with ℓ leaves and s support vertices, then $\gamma_e(T) \geqslant \lceil (n+2+\ell-s)/3 \rceil$.*

Proof. We proceed by induction on the order n. Clearly the result holds if $diam(T) \in \{0,1\}$. If $diam(T) = 2$ then T is a star $K_{1,n-1}$ where $\gamma_e(T) = n-1 \geqslant \lceil (n+2+\ell-s)/3 \rceil = \lceil 2n/3 \rceil$, and hence the result is valid, establishing the base case.

Assume that every tree T' of order $n' < n$ with ℓ' leaves and s' support vertices satisfies $\gamma_e(T') \geqslant \lceil (n'+2+\ell'-s')/3 \rceil$. Let T be a tree of order n with ℓ leaves and s support vertices.

If any support vertex, say x, of T is strong, then let T' be the tree obtained from T by removing a leaf adjacent to x. Then $\gamma_e(T') = \gamma_e(T) - 1, \ell' = \ell - 1$ and $s' = s$. Applying the inductive hypothesis to T', we obtain the desired result. Henceforth, we can assume that every support vertex of T is adjacent to exactly one leaf.

We now root T at a vertex r of maximum eccentricity $diam(T) \geqslant 3$. Let v be a support vertex at maximum distance from r, u its parent in the rooted tree and S a $\gamma_e(T)$-set. Denote by T_v the subtree induced by a vertex v and its descendants in the rooted tree T. We distinguish between two cases.

Case 1. $d_T(u) \geqslant 3$. Then either u is a support vertex of T or u has a child besides v as a support vertex. Let $T' = T - T_v$. Clearly, $n' = n-2, \ell' = \ell - 1$ and $s' = s-1$. Then S contains the leaf neighbor of v, say v' and without loss of generality $v \notin S$ else we can substitute it by u in S. Thus, $S - \{v'\}$ is an end-dominating set of T', implying that $\gamma_e(T') \leqslant \gamma_e(T) - 1$. Applying the inductive hypothesis to T', it follows that

$$\gamma_e(T) \geqslant \gamma_e(T') + 1 \geqslant \lceil (n' + 2 + \ell' - s')/3 \rceil + 1 = \lceil (n + 3 + \ell - s)/3 \rceil .$$

Case 2. $d_T(u) = 2$. Since $diam(T) \geqslant 3$, let w be the parent of u in the rooted tree. Let $T' = T - T_u$. Then $n' = n-3$. The result is valid if $T = P_4$ or P_5, so let $n' \geq 3$. We can assume that $v' \in S, v \notin S$, $u \notin S$ (else replace v or u by w in S) and $w \in S$. Thus $S - \{v'\}$ is an end-dominating set of T', implying that $\gamma_e(T') \leqslant \gamma_e(T) - 1$. Now if $d_T(w) \geqslant 3$ then $\ell' = \ell - 1$ and $s' = s - 1$. If $d_T(w) = 2$, that is w is a leaf in T', then $\ell' = \ell$ and $s' \leqslant s$. In any case, it follows by induction on T' that:

$$\gamma_e(T) \geqslant \gamma_e(T') + 1 \geqslant \lceil (n' + 2 + \ell' - s')/3 \rceil + 1 = \lceil (n + 2 + \ell - s)/3 \rceil .$$

This achieves the proof. □

According to Observation 2 and Theorem 4, we have the following corollary:

Corollary 5. *If T is a tree of order $n \geqslant 1$, then*

a) $\gamma_w(T) \geqslant \lceil (n + 2 + \ell - s)/3 \rceil$ *for* $n \neq 2$.

b) $\gamma_r(T) \geqslant \lceil (n + 2 + \ell - s)/3 \rceil$.

Clearly since $\ell - s \geqslant 0$ for every tree, Corollary 5 improves Theorem 3.

Proposition 6. *If T is a tree of order at least three with ℓ leaves and s support vertices, then* $\gamma_e(T) \geq \gamma(T) + \ell - s$.

Proof. Let $L(T)$ and $S(T)$ denote the set of leaves and support vertices of T, respectively. Then since every $\gamma_e(T)$-set D contains $L(T)$, $(D - L(T)) \cup S(T)$ is a dominating set of T. Hence $\gamma(T) \leq |D| - \ell + s$. □

Recall that the *connected domination number* of a graph G denoted by $\gamma_c(G)$ is the minimum cardinality of a dominating set whose induced subgraph is connected. In [1], Duchet and Meyniel proved that every connected graph G, satisfies $\gamma(G) \geq (\gamma_c(G) + 2)/3$. Since $\gamma_c(T) = n - \ell$ for every tree T, it follows that $\gamma(T) \geq (n + 2 - \ell)/3$. Note that extremal trees have been characterized by Lemañska [6]. Using this lower bound and Proposition 6, we have:

Corollary 7. *If T is a tree of order at least three with ℓ leaves and s support vertices, then* $\gamma_e(T) \geq \lceil (n + 2\ell - 3s + 2)/3 \rceil$.

By Observation 2, Proposition 6 and Corollary 7, we obtain a second lower bound on $\gamma_w(T)$ and $\gamma_r(T)$ which improves Corollary 5 for $\ell > 2s$.

Corollary 8. *If T is a tree of order $n \geqslant 3$, then*

a) $\gamma_w(T) \geq \gamma(T) + \ell - s \geq \lceil (n + 2 + 2\ell - 3s)/3 \rceil$, *and*

b) $\gamma_r(T) \geq \gamma(T) + \ell - s \geq \lceil (n + 2 + 2\ell - 3s)/3 \rceil$.

All inequalities in Corollary 8 are attained. For example, let T be the tree obtained from a path P_3 by attaching a center vertex of a star $K_{1,3}$ at each leaf of the path. Then $n = 11, \ell = 6, s = 2$, $\gamma(T) = 3$ and $\gamma_w(T) = \gamma_r(T) = 7$.

3. Upper bounds

We give in this section an upper bound for the weak and restrained domination numbers of a nontrivial tree.

Theorem 9. *If T is a nontrivial tree with n vertices, ℓ leaves and s support vertices, then $\gamma_w(T) \leqslant \lfloor (n + 2\ell + 2s - 3)/3 \rfloor$, and this bound is sharp.*

Proof. We proceed by induction on $n \geqslant 2$. Clearly the result is valid if $T = P_2$. If T is a star $K_{1,p} (p \geqslant 2)$, then $\gamma_w(T) = p = \lfloor (n + 2\ell + 2s - 3)/3 \rfloor$, and hence the result holds. Assume that every tree T' of order $2 \leqslant n' < n$ satisfies $\gamma_w(T') \leqslant \lfloor (n' + 2\ell' + 2s' - 3)/3 \rfloor$. Let T be a tree of order n.

Root T at a vertex r of maximum eccentricity $diam(T) \geq 3$. Let v be a support vertex at maximum distance $diam(T) - 1$ from r and u the parent of v. We consider the following two cases:

Case 1. $d_T(u) \geqslant 3$, that is u is either a support vertex or has a child besides v as a support vertex. Let $T' = T - T_v$. Then $n' = n - |L_v| - 1 \geq 3$, $\ell' = \ell - |L_v|$ and $s' = s - 1$. Let S' be any $\gamma_w(T')$-set. Then S' can be extended to a WDS of T by adding the set of leaves L_v with a possibility to replace u (if $u \in S'$) by its parent since its degree will be increased in T. Such a substitution is possible since the goal of u in S' is either to weakly dominate itself or its parent. So $\gamma_w(T) \leqslant \gamma_w(T') + |L_v|$. Now by induction on T' we obtain:

$$\gamma_w(T) \leqslant \lfloor (n' + 2\ell' + 2s' - 3)/3 \rfloor + |L_v| = \lfloor (n + 2\ell + 2s - 6)/3 \rfloor$$

Case 2. $d_T(u) = 2$. Since $diam(T) \geq 3$, let w be the parent of u. If $d_T(w) = 1$, then T is a double star $S_{|L_v|,1}$ and the result holds. Assume that $d_T(w) \geq 3$. Seeing Case 1, we suppose that every subtree rooted at a child of w is a star. If $d_T(w) \geq 4$ or $d_T(w) = 3$ and every leaf of $V(T_w) - L_v$ is at distance one or two from w, then let $T' = T - T_u$ and S' be any $\gamma_w(T')$-set. If $w \notin S'$, then S' can be extended to a WDS of T by adding the set $L_v \cup \{u\}$. Assume now that $w \in S'$. Then w weakly dominates itself or its parent and so S' can be extended to a WDS of T (possibly by replacing w by its parent) by adding the set $L_v \cup \{u\}$. In any case, $\gamma_w(T) \leqslant \gamma_w(T') + |L_v| + 1$. By induction on T', and since $n' = n - |L_v| - 2$, $s' = s - 1$ and $\ell' = \ell - |L_v|$, we obtain:

$$\begin{aligned} \gamma_w(T) &\leqslant \gamma_w(T') + |L_v| + 1 \leqslant \lfloor (n' + 2\ell' + 2s' - 3)/3 \rfloor + |L_v| + 1 \\ &= \lfloor (n + 2\ell + 2s - 4)/3 \rfloor . \end{aligned}$$

Now we examine the case, $d_T(w) = 3$ and every leaf in T_w is at distance three from w. Let y be the second support vertex at distance two from w in T_w. Clearly seeing Case 1, the parent of

y, say z has degree two. Let $T' = T - (T_v \cup T_y)$. Then $n' = n - |L_v| - |L_y| - 2$, $s' \leq s-1$ and $\ell' = \ell - |L_v| - |L_y| + 2$. Since every $\gamma_w(T')$-set contains u and z, such a set can be extended to a WDS of T by adding the set $L_v \cup L_y$, and so $\gamma_w(T) \leqslant \gamma_w(T') + |L_v| + |L_y|$. By induction on T', we have:

$$\begin{aligned}\gamma_w(T) &\leqslant \gamma_w(T') + |L_v| + |L_y| \leqslant \lfloor (n' + 2\ell' + 2s' - 3)/3 \rfloor + |L_v| + |L_y| \\ &= \lfloor (n + 2\ell + 2s - 3) \rfloor /3.\end{aligned}$$

Finally, assume that $d_T(w) = 2$ and let $T' = T - T_u$. Then $n' = n - |L_v| - 2$. It is a routine matter to check the result if $n' = 1$ or 2. Thus assume that $n' \geq 3$. Since every $\gamma_w(T')$-set S' contains w, such a set can be extended to a WDS of T by adding the set L_v. Hence $\gamma_w(T) \leqslant \gamma_w(T') + |L_v|$. Applying the induction on T', and since $\ell' = \ell - |L_v| + 1$ and $s' \leq s$ we have:

$$\begin{aligned}\gamma_w(T) &\leqslant \gamma_w(T') + |L_v| \leqslant \lfloor (n' + 2\ell' + 2s' - 3)/3 \rfloor + |L_v| \\ &= \lfloor (n + 2\ell + 2s - 3)/3 \rfloor .\end{aligned}$$

The bound is attained for stars and paths $P_{3k+2} (k \geq 1)$. □

Theorem 10. *If T is a nontrivial tree with n vertices, ℓ leaves and s support vertices, then $\gamma_r(T) \leqslant \lfloor (n + 2\ell + s + 1)/3 \rfloor$, and this bound is sharp.*

Proof. Again, we proceed by induction on $n \geqslant 2$. It is a routine matter to check the result if $diam(T) \in \{1, 2, 3\}$. Assume that every tree T' of order $2 \leqslant n' < n$ satisfies $\gamma_r(T') \leqslant \lfloor (n' + 2\ell' + s' + 1)/3 \rfloor$, and let T be a tree of order n.

If T contains a strong support vertex, then let T' be the tree obtained from T by removing a leaf adjacent to a strong support vertex. Then $\gamma_r(T') = \gamma_r(T) - 1, n' = n - 1, \ell' = \ell - 1$ and $s' = s$. The result follows by induction on T'. Hence we assume that T contains no strong support vertex.

Root T at a vertex r of maximum eccentricity $diam(T) \geq 4$. Let v be a support vertex at maximum distance $diam(T) - 1$ from r, u the parent of v and w the parent of u in the rooted tree. We distinguish between cases:

Case 1. $d_T(u) \geqslant 3$. Let $T' = T - T_u$. Then $n' = n - 2d_T(u) + 2 - i$, where $i = 1$ if u is not a support vertex and $i = 0$ else. Since the result holds for $0 \leq n' \leq 2$, we assume that $n' \geq 3$. Let S' be any $\gamma_r(T')$-set. If $w \notin S$, then S' can be extended to a RDS of T by adding the leaves of T_u, and possibly v if u is not a support vertex. Hence $\gamma_r(T) \leq \gamma_r(T') + d_T(u) - 1 + i$. Since $w \notin S'$, the degree of w in T must be equal to at least three. So $\ell' = \ell - (d_T(u) - 1)$ and $s' = s - (d_T(u) - 1)$. By induction on T',

$$\begin{aligned}\gamma_r(T) &\leq \gamma_r(T') + d_T(u) - 1 + i \leq \lfloor (n' + 2\ell' + s' + 1)/3 \rfloor + d_T(u) - 1 + i \\ &= \lfloor (n + 2\ell + s + 1 + 2 + 2i - 2d_T(u))/3 \rfloor .\end{aligned}$$

Thus $\gamma_r(T) \leq \lfloor (n + 2\ell + s + 1)/3 \rfloor$ since $d_T(u) \geqslant 3$.

Now if $w \in S'$, then S' can be extended to a RDS of T by adding the leaves of T_u, and hence $\gamma_r(T) \leq \gamma_r(T') + d_T(u) - 1$. Likewise, since $\ell' \leq \ell - (d_T(u) - 1) + 1$ and $s' \leq s - (d_T(u) - 1) + 1$, the result follows by induction on T', and we obtain $\gamma_r(T) \leq \lfloor (n + 2\ell + s + 1 + 5 - 2d_T(u) - i)/3 \rfloor \leq \lfloor (n + 2\ell + s + 1)/3 \rfloor$

Case 2. $d_T(u) = 2$. Let $T' = T - T_u$. Clearly the result holds if $n' = (n - 3) \in \{1, 2\}$. So assume that $n' \geq 3$. Let S' be any $\gamma_r(T')$-set. If $w \in S'$ then S' is extended to a RDS of T by adding the leaf adjacent to v. Hence $\gamma_r(T) \leq \gamma_r(T') + 1$. By induction on T', and since $\ell' \leq \ell$ and $s' \leq s$, we obtain $\gamma_r(T) \leqslant \gamma_r(T') + 1 \leq \lfloor (n + 2\ell + s + 1)/3 \rfloor$.

If $w \notin S'$ then S' is extended to a RDS of T by adding v and its leaf. So $\gamma_r(T) \leq \gamma_r(T') + 2$. Since $w \notin S'$, $\deg_T(w) \geq 3$, and so $\ell' = \ell - 1$ and $s' = s - 1$. Now by induction on T', we have $\gamma_r(T) \leq \gamma_r(T') + 2 \leq \lfloor (n + 2\ell + s + 1)/3 \rfloor$.

The bound is attained for stars and the path P_6. □

Recall that the *independent domination number* $i(G)$ of a graph G is the minimum cardinality of a set S of vertices that is both dominating and independent. The following result is due to Favaron [2]:

Theorem 11. *(Favaron [2]) If T is a tree of order $n \geq 3$ with ℓ leaves, then $i(T) \leq \lfloor (n + \ell)/3 \rfloor$.*

Using Theorem 11, one can see that $\gamma(T) \leq \lfloor (n + s)/3 \rfloor$ holds for every tree T of order $n \geq 3$ with s support vertices.

According to Corollary 8-(b) and Theorem 10, we have the following corollary which also strengthens the above upper bound on $\gamma(T)$ for $\ell \geq 3s + 3$.

Corollary 12. *If T is a tree of order $n \geqslant 3$, then*

$$\gamma(T) \leqslant \min\{\lfloor (n + s)/3 \rfloor, \lfloor (n - \ell + 4s + 1)/3 \rfloor\}.$$

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