AKCE J. Graphs. Combin., 2, No. 2 (2005), pp. 87-98

On Point-Set Domination in Graphs II: Minimum PSD-Sets

B. D. Acharya
Department of Science and Technology
Government of India
Technology Bhavan, New Mehrauli Road
New Delhi - 110 016
INDIA
e-mail: *bdacharya@yahoo.com*
and
Purnima Gupta
Department of Mathematics
Sri Venkateswara College
(University of Delhi)
New Delhi - 110 021
INDIA
e-mail: *purnimachandni@rediffmail.com*

Communicated by: S. Arumugam
Received 13 July 2005; accepted 24 October 2005

Abstract

In this paper, we amend the original proof of a result due to E. Sampathkumar and L. Pushpa Latha [6] as also its variation given in [4] and establish new fundamental results on minimum point-set dominating(psd-)sets in separable graphs.

Keywords: Point-set domination, Point-set domination number.

2000 Mathematics Subject Classification: 05C69

1. Introduction

Unless defined or mentioned otherwise, we refer the reader to F. Harary [3] for standard terminology and notation in graph theory.

E. Sampathkumar and L. Pushpa Latha [6] define a set D of vertices in a connected graph $G=(V,E)$ to be a point-set dominating (or, in short, psd-) set of G if for every subset $S \subseteq V-D$ there exists a vertex $v \in D$ such that the subgraph $\langle S \cup \{v\} \rangle$ induced by the set $S \cup \{v\}$ is connected. The set of all psd-sets in G will be denoted $\mathscr{D}_{ps}(G)$.

Remark 1. *Clearly every psd-set of G must be a dominating set of G in the standard sense (cf. O. Ore [5], Ch. 13), but not conversely. We shall denote by $\mathscr{D}(G)$ the set of all dominating sets of G.*

Remark 2. *It may be noted that it is not necessary to impose connectedness of G in the above definition due to the following result.*

Lemma 1. *Let $G=(V,E)$ be any graph and D be any psd-set of G. Then, $\langle V-D\rangle$ is a proper subgraph of a component H of G.*

Proof. Suppose there exist vertices u and v belonging to two different components of G. Since D is a psd-set of G, there must exist $w \in D$ such that $\langle\{u,v,w\}\rangle$ is connected, contrary to the assumption. This implies $V-D \subseteq V(H)$ for some component H of G. Further, since D is a psd-set of G, $D\cap V(H) \in \mathscr{D}_{ps}(H)$. Hence, $D \cap V(H) \neq \phi$ which implies that $\langle V-D\rangle$ is a proper subgraph of H. □

Remark 3. *If H is a component of G then a psd-set of H, together with the vertices of all its other components, forms a psd-set of G.*

For a finite graph G, the minimum cardinality of a psd-set of G is called the point-set domination number of G and is denoted by $\gamma_p(G)$. Further, a psd-set of G with exactly $\gamma_p(G)$ elements is called a γ_p-set (or, a minimum psd-set) of G or, a $\gamma_p(G)$-set. We shall use similar terminology for other parameters of G defined using the notion of cardinality of the subsets of $V(G)\cup E(G)$. We let $\mathscr{D}^0_{ps}(G)$ denote the set of all $\gamma_p(G)$-sets.

Theorem 1. *Let G be a finite graph of order n, and $\mathscr{C}_G$ denote the set of its components. Then*

$$\gamma_p(G) = n - \max_{H\in\mathscr{C}_G} \{|V(H)| - \gamma_p(H)\} \tag{1}$$

Proof. Let D be a γ_p-set of G. By Lemma 1, it follows that there exists $H \in \mathscr{C}_G$ such that $V-D \subset V(H)$. Clearly, $D\cap V(H) \in \mathscr{D}_{ps}(H)$ and since

$$|D| = |D\cap V(H)| + n - |V(H)| \tag{2}$$

we have,

$$\gamma_p(G) \geq n - |V(H)| + \gamma_p(H) \geq n - \max_{H'\in\mathscr{C}_G} \{|V(H')| - \gamma_p(H')\}. \tag{3}$$

On the other hand, by Remark 3, we get

$$\gamma_p(G) \leq n - |V(H)| + \gamma_p(H), \text{ for all } \mathrm{H} \in \mathscr{C}_\mathrm{G} \tag{4}$$

which yields

$$\gamma_p(G) \leq n - \max_{H'\in\mathscr{C}_G} \{|V(H')| - \gamma_p(H')\}. \tag{5}$$

Inequalities (3) and (5) together imply (1). □

Remark 4. *It is easily seen from Theorem 1 that if G is a finite connected graph of order n, then $\gamma_p(G)$ stands well defined as in [6]. Hence, without loss generality, we may assume that G is connected which we shall do hitherforth. Also, G will be assumed to be finite whenever we are dealing with the parameter $\gamma_p(G)$.*

The following two interesting observations made in [6] are indeed easy consequences of the definition of psd-sets.

Proposition 1. [6] *Let* D *be a psd-set of a graph* G *and* $d_G(u,v)$ *denote the distance between the vertices* u *and* v *in* G *. Then*

$$d_G(u,v) \leq 2 \quad \textit{for all} \quad u,v \in V-D. \tag{6}$$

Also, if $\Delta := \Delta(G) = max.\{d(u) : u \in V(G)\}$ *, then for any graph* G *of order* n *one always has*

$$\gamma_p(G) \leq n - \Delta(G). \tag{7}$$

In [1], we pointed out a gap in the proof of an incidentally correct result of Sampathkumar and Pushpa Latha [6] and attempted to rectify the same by proposing an amendment to that 'proof'. However, there was a gap in our proof too, which somehow got repeated in an article by Sampathkumar in [4]. A finally corrected proof of the result pointing out the error and leading to new fundamental results on minimum psd-sets in separable graphs appears in the doctoral dissertation of the second author of the present paper [2], which we reproduce here for making it available to a wider readership. While doing so, we are led to the classification of minimum psd-sets of a connected separable graph.

2. Amendment to a result due to Sampathkumar and Pushpa Latha

The following result was "proved" in [6] (*cf.* Proposition 3) as well as in [4]. As mentioned, it will be pointed out that its proof needs to be corrected.

Proposition 2. [6] *Let* $G = (V,E)$ *be a graph of finite order* n *having cutvertices. Then*

$$\gamma_p(G) = min\{n-\Delta, n-k\} \tag{8}$$

where $k =^{max}_{B\in\mathscr{B}_G} \{|V(B)| - \gamma_p(B)\}$ *, and* $\mathscr{B}_G$ *denotes the set of all blocks of* G *.*

Starting with a γ_p -set D of a finite separable graph G , the proof of this interesting result has been divided into two parts, viz.,

Case 1. Not all vertices of $\langle V-D\rangle$ belong to the same block of G . In this case, it has been shown that $\gamma_p(G) = n - \Delta$.

Case 2. All the vertices of $\langle V-D\rangle$ belong to the same block B of G . In this case, it has been shown that $\gamma_p(G) = n - k$.

The proof in Case 2 begins by the remark, 'In this case, clearly $\gamma_p(G) = n-(|V(B)|-\gamma_p(B)) \geq n-k$. ' We observed in [1] that while the inequality part of this statement is obviously true, the equality part is not true in general and illustrated with the following examples.

For the graph $K_t^+, t \geq 3$, obtained by the adjunction of a pendant vertex at each vertex of the complete graph K_t of order t , and for the independent γ_p -set D consisting of all its pendant vertices, we see that $\langle V-D\rangle = B = K_t$ is the unique nontrivial block of K_t^+ and still $\gamma_p(K_t^+) = t = 2t-t = n-\Delta < n - \{|V(B)| - \gamma_p(B)\} \leq n-k$.

To give an example of a finite separable graph G which has a $\gamma_p(G)$ -set D and a block B such that $V-D \subset V(B)$, yet not satisfying the equality $|D| = \gamma_p(G) = n-(|V(B)|-\gamma_p(B))$, consider the 'dot-composed graph' $G = C_4 \bullet K_2$ which is obtained by the identification of a vertex of C_4

with one of K_2 (*cf.* Harary [3], p.23). Let $B = C_4 = (u_1, u_2, u_3, u_4, u_1)$ and $V(K_2) = \{u_1, v_1\}$. Then $D = \{u_3, v_1\}$ is a $\gamma_p(G)$-set such that $V - D \subset V(B)$ and $|D| = \gamma_p(G) = n - |V(B)| + |V(B) \cap D|$ where $|V(B) \cap D| < \gamma_p(B)$ and, therefore, $\gamma_p(G) < n - (|V(B)| - \gamma_p(B))$.

In both the cases, we find $\gamma_p(G) < n-(|V(B)|-\gamma_p(B))$ and hence the inequality $\gamma_p(G) \geq n-k$ may not hold in general, which was being claimed in Case 2 of the proof in question. Thus, the proof of Case 2 needs amendment. Hence, we shall determine the value of $\gamma_p(G)$ when there exist $\gamma_p(G)$-sets D satisfying $V - D \subseteq V(B)$ for some block B of G and

$$\gamma_p(G) < n - (|V(B)| - \gamma_p(B)) \tag{9}$$

Note that if D is a $\gamma_p(G)$-set and B is a block of G such that $V - D \subseteq V(B)$ then

$$|D| = n - |V(B)| + |V(B) \cap D| \tag{10}$$

so that

$$|V(B) \cap D| \leq \gamma_p(B), \tag{11}$$

for, otherwise $(V(G) - V(B)) \bigcup \{some \gamma_p(B) - set\} = D' \in \mathscr{D}_{ps}(G)$ and $|D'| < |D|$ contrary to the fact that D is a $\gamma_p(G)$-set. Further, graphs in our discussion above are examples in which all the γ_p-sets D satisfy $V - D \subset V(B)$ for some $B \in \mathscr{B}_G$ and strict inequality holds in (11). It is in such cases we get $\gamma_p(G) < n - (|V(B)| - \gamma_p(B))$ and hence the arguments given under Case 2 fail to hold. Therefore, we shall distinguish the cases when strict inequality and equality in (11) occur.

Let $G = (V, E)$ be any graph with cutvertices, $N(u) = \{v \in V(G) : uv \in E(G)\}$ denote the set of *neighbours* of $u \in V(G)$ and let $N[u] = \{u\} \cup N(u)$ denote the *closed neighbourhood* of u. Then, for any $D \in \mathscr{D}_{ps}(G)$ and $B \in \mathscr{B}_G$ such that $V - D \subseteq V(B)$, we define the set $P(B, D) = \{u \in V - D : N(u) \cap (D - V(B)) \neq \phi$ and $N(u) \cap (V(B) \cap D) = \phi\}$.

Lemma 2. *Let $G = (V, E)$ be any graph with cutvertices. $D \in \mathscr{D}_{ps}(G)$ and $B \in \mathscr{B}_G$ be such that $V - D \subseteq V(B)$. Then every vertex of $P(B, D)$ is adjacent to every other vertex of $V - D$.*

Proof. Suppose not. Then there exists $u \in P(B, D)$ which is not adjacent to some other vertex $v \in V - D$. Since D is a psd-set of G, Theorem 1 implies that there must exist $w \in D$ such that (u, w, v) is a uv-path in G. Since B is a block it then follows that $w \in B$. But then $w \in N(u) \cap (B \cap D)$, a contradiction to the definition of $P(B, D)$. □

Remark 5. *Under the hypotheses of Lemma 2, we see that $P(B, D) \neq \phi \Rightarrow \langle P(B, D) \rangle$ is complete.*

Lemma 3. *Let $G = (V, E)$ be any graph with cutvertices, $D \in \mathscr{D}_{ps}(G)$ and $B \in \mathscr{B}_G$ be such that $V - D \subseteq V(B)$. Then,*

$$V(B) \cap D \neq \phi \Rightarrow V(B) \cap D \in \mathscr{D}_{ps}(\langle V(B) - P(B, D) \rangle). \tag{12}$$

Proof. Let S be a subset of $(V - D) - P(B, D)$. Let W be a maximal independent subset of S. If $|W| = 1$, let $W = \{w\}$. Then, $w \in (V - D) - P(B, D)$ whence there exists $d \in V(B) \cap D$ such that $wd \in E$. This implies $\langle S \cup \{d\} \rangle$ is connected. Next, if $|W| > 1$ then $D \in \mathscr{D}_{ps}(G)$ and $W \subseteq V - D$ implying thereby existance of $d \in D$ such that $\langle W \cup \{d\} \rangle$ is connected. So, d is adjacent to more than one vertex of W and hence of B whence $d \in V(B)$. Thus, $d \in V(B) \cap D$ is such that $\langle W \cup \{d\} \rangle$, and therefore $\langle S \cup \{d\} \rangle$, is connected and hence the proof is complete. □

Remark 6. *Note from (12) that if* $P(B,D) = \phi$ *then* $V(B) \cap D \in \mathscr{D}_{ps}(B)$ *. This yields,* $\gamma_p(B) \leq |V(B) \cap D|$ *. This, together with (11), gives* $\gamma_p(B) = |V(B) \cap D|$ *. Thus, in fact, we have*

$$P(B,D) = \phi \Rightarrow \gamma_p(B) = |V(B) \cap D|. \tag{13}$$

Remark 7. *Further, (12) gives*

$$|V(B) \cap D| < \gamma_p(B) \Rightarrow P(B,D) \neq \phi. \tag{14}$$

We can now proceed to amend the proof of Case 2 of Proposition 2; in fact, it gets resolved into two subcases, viz.,(i) $P(B,D) \neq \phi$ and (ii) $P(B,D) = \phi$; where $D \in \mathscr{D}^0_{ps}(G)$.

Subcase i. In this case, for any $u \in P(B,D)$ we have $N(u) \cap D \neq \phi$ and by Lemma 2 u is adjacent to every other vertex of $V - D$. Therefore, $\Delta \geq d(u) \geq |V - D|$ for any $u \in P(B,D)$ and hence we get $\Delta \geq n - \gamma_p(G)$ so that $\gamma_p(G) \geq n - \Delta$. Combining this with the general observation (7), we get $\gamma_p(G) = n - \Delta$ in this case.

Subcase ii. In this case, $|V(B) \cap D| = \gamma_p(B)$ by (13). Using this in (10), we get $\gamma_p(G) = n - |V(B)| + \gamma_p(B)$ and hence

$$\gamma_p(G) = n - (|V(B)| - \gamma_p(B)) \geq n - k. \tag{15}$$

On the other hand, as noted in [6], since any set S of vertices in G consisting of the vertices in all but one block F of G together with a psd-set T of F constitutes a psd-set of G, we must have

$$\gamma_p(G) \leq |S| = n - (|V(F)| - |T|). \tag{16}$$

The inequality (16) must hold, in particular, when $|T| = \gamma_p(F)$. Thus,

$$\gamma_p(G) \leq n - (|V(F)| - \gamma_p(F)) \quad \text{for all} \quad F \in \mathscr{B}_G. \tag{17}$$

In particular, therefore, (17) must hold for any block F for which $|V(F)| - \gamma_p(F) = k(G)$ and hence we get

$$\gamma_p(G) \leq n - k \tag{18}$$

which, together with (15) yields $\gamma_p(G) = n - k$, completing the proof in this case. □

Remark 8. $B \in \mathscr{B}_G$, $D \in \mathscr{D}^0_{ps}(G), V - D \subseteq V(B)$ *and* $P(B,D) = \phi \Rightarrow \gamma_p(G) = n - |V(B)| + \gamma_p(B)$ *where* $|V(B)| - \gamma_p(B) = k$ *.*

Remark 9. *If* D *is a* γ_p*-set which is such that* $V - D \subseteq V(B)$ *for some* $B \in \mathscr{B}_G$*, then* $P(B,D) \neq \phi \Rightarrow |N(u) \cap D| = 1$ *for every* $u \in P(B,D)$ *and every vertex of* $P(B,D)$ *has degree* $\Delta(G)$ *.*

Remark 10. *Let* D *be a* $\gamma_p(G)$*-set such that* $V - D$ *contains vertices of different blocks. Then* $V - D = N(w)$ *for some vertex* w *of degree* $\Delta(G)$ *.*

Remark 11. $\Delta(G) \geq k(G) \Rightarrow V(G) - N(w)$ *is a* $\gamma_p(G)$*-set for every vertex* w *of degree* $\Delta(G)$ *.*

3. Classification of psd-sets in a separable graph

For any separable graph $G=(V,E)$ of order n, let $\mathscr{D}_{ps}(G:X)=\{D\in\mathscr{D}_{ps}(G):$ there exists $B\in\mathscr{B}_G$ with $V-D\subset V(B)\}$, $\mathscr{D}_{ps}(G,Y)=\{D\in\mathscr{D}_{ps}(G):$ there exists $B\in\mathscr{B}_G$ with $V-D=V(B)\}$ and $\mathscr{D}_{ps}(G:Z)=\mathscr{D}_{ps}(G)-(\mathscr{D}_{ps}(G:X)\bigcup\mathscr{D}_{ps}(G,Y))=\{D\in\mathscr{D}_{ps}(G):V-D$ contains vertices of different blocks$\}$. For simplicity, we call a psd-set a Type-X or Type-Y or Type-Z psd-set according to whether D is in $\mathscr{D}_{ps}(G,X),\mathscr{D}_{ps}(G,Y)$ or $\mathscr{D}_{ps}(G,Z)$. We shall put $\mathscr{D}^0_{ps}(G,X)=\mathscr{D}^0_{ps}(G)\bigcap\mathscr{D}_{ps}(G,X),\mathscr{D}^0_{ps}(G,Y)=\mathscr{D}^0_{ps}(G)\bigcap\ \mathscr{D}_{ps}(G,Y)$ and $\mathscr{D}^0_{ps}(G,Z)=\mathscr{D}^0_{ps}(G)\bigcap\mathscr{D}_{ps}(G,Z)$. Clearly,

$$\mathscr{D}_{ps}(G)=\mathscr{D}_{ps}(G,X)\bigcup\mathscr{D}_{ps}(G,Y)\bigcup\mathscr{D}_{ps}(G,Z)$$

and

$$\mathscr{D}^0_{ps}(G)=\mathscr{D}^0_{ps}(G,X)\bigcup\mathscr{D}^0_{ps}(G,Y)\bigcup\mathscr{D}^0_{ps}(G,Z).$$

It can be readily seen that the sets in these decompositions are mutually pairwise disjoint.

Theorem 2. *Let $G=(V,E)$ be any connected graph of order n having cutvertices. Then $\mathscr{D}^0_{ps}(G:Z)=\phi$ if and only if one of the following conditions holds:*

(*i*) $\Delta(G)<k(G)$.

(*ii*) $\Delta(G)=k(G)$ *and every vertex of degree Δ has all its neighbours contained in a single block of G.*

where $k(G):=k$ is as defined in Proposition 2.

Proof. *Necessity.*

Let $\mathscr{D}^0_{ps}(G)\subseteq\mathscr{D}_{ps}(G:X)\bigcup\mathscr{D}_{ps}(G:Y)$ and $D\in\mathscr{D}^0_{ps}(G)$. Then there exists $B\in\mathscr{B}_G$ such that $V-D\subseteq V(B)$.

If $P(B,D)\neq\phi$ then we recall that $\gamma_p(G)=n-\Delta$ and every vertex of $P(B,D)$ has degree Δ. Hence, $u\in P(B,D)\Rightarrow|N(u)\cap V(B)|=\Delta-1$ which means that $|N[u]\cap V(B)|=\Delta$. Further, $\gamma_p(G)=n-\Delta$ and $d(u)=\Delta\Rightarrow V(G)-N(u)\in\mathscr{D}^0_{ps}(G)$. By assumption, there exists $B_1\in\mathscr{B}_G$ such that $N(u)\subseteq V(B_1)$ and since $d(u)\geq 2$ we have $N[u]\subseteq V(B_1)$. Thus, it follows that B and B_1 have at least Δ vertices in common. Since any two blocks have at most one vertex in common, it follows that $\Delta\leq 1$, contrary to the hypothesis. Therefore, $P(B,D)=\phi$. This implies $\gamma_p(G)=n-k$ whence $\Delta\leq k$. Hence, either $\Delta<k$, which is condition (i), or $\Delta=k$ whence $V(G)-N(u)$ is a $\gamma_p(G)$-set for any vertex u of degree Δ. By hypothesis, we hence get $N(u)\subseteq V(B)$ for some $B\in\mathscr{B}_G$ which is condition (ii).

Sufficiency.

First, suppose $\Delta<k$ so that $\gamma_p(G)=n-k$ by Proposition 2. Let D be a $\gamma_p(G)$-set such that $V-D$ contains vertices of different blocks, so that $\gamma_p(G)=n-\Delta$. Therefore, $\gamma_p(G)=n-k=n-\Delta$, which implies $\Delta\geq k$, contrary to the assumption.

Next, let G satisfy condition (ii) and D be a $\gamma_p(G)$-set with $V-D$ having vertices of different blocks. Then, by Remark 10, $V-D=N(w)$ for some vertex w of degree Δ. Therefore, $N(w)$ contains vertices of different blocks which contradicts condition (ii).

Thus, the proof is complete.

□

Remark 12. *Theorem 2 can be equivalently stated as follows :* $\mathscr{D}^0_{ps}(G:Z) \neq \phi$ *if and only if one of the following conditions is satisfied:*

(*i*) $\Delta(G) > k(G)$, *and*

(*ii*) $\Delta(G) = k(G)$ *and there exists a vertex* u *of degree* $\Delta(G)$ *such that* $N(u)$ *is not contained in any single block of* G.

Further, $\mathscr{D}^0_{ps}(G:X)$ may be divided into two classes, viz., $\mathscr{D}^0_{ps}(G:X_1) = \{D \in \mathscr{D}^0_{ps}(G:X) : V - D \subset N(B)$ for some $B \in \mathscr{B}_G$ and $P(B,D) = \phi\}$ and $\mathscr{D}^0_{ps}(G:X_2) = \mathscr{D}^0_{ps}(G:X) - \mathscr{D}^0_{ps}(G:X_1)$.

Theorem 3. $\mathscr{D}^0_{ps}(G:X_1) \neq \phi$ *if and only if* $\Delta(G) \leq k(G)$.

Proof. Necessity.

Let $D \in \mathscr{D}^0_{ps}(G:X_1)$. Then, there exists $B \in \mathscr{B}_G$ such that $V - D \subset V(B)$ and $P(B,D) = \phi$. Hence, $V(B) \cap D$ is a $\gamma_p(B)$-set (by Remark 6) and $\gamma_p(G) = n - |V(B)| + \gamma_p(B)$. Since $\gamma_p(G) = min\{n-\Delta, n-k\}$ we obtain $|V(B)| - \gamma_p(B) = k(G)$ and hence $\gamma_p(G) = n-k$. This implies $n-k \leq n-\Delta$ whence $\Delta(G) \leq k(G)$.

Sufficiency.

Let $\Delta(G) \leq k(G)$. Then, $\gamma_p(G) = n-k$. Let B be a block such that $|V(B)| - \gamma_p(B) = k$. Then, $D = (V(G) - V(B)) \bigcup \{$ a $\gamma_p(B)$-set $\}$ is a psd-set for G such that $|D| = n - |V(B)| + \gamma_p(B) = n-k$. Hence, D is a $\gamma_p(G)-$ set such that $V - D \subset V(B)$ for some block B of G and $P(B,D) = \phi$. Hence, $D \in \mathscr{D}^0_{ps}(G:X_1)$. □

Theorem 4. $\mathscr{D}^0_{ps}(G:Y) \neq \phi$ *if and only if*

(*i*) $\Delta(G) \geq k(G)$, *and*

(*ii*) G *has a block* B *which is a clique of order* $\Delta(G)$ *and* $\langle N[V(B)]\rangle = B^+$ *where for any set* A *of vertices of* G, $N(A)$ *denotes the set of neighbours of vertices in* A *and* $N[A] = A \bigcup N(A)$.

Proof. Let $D \in \mathscr{D}^0_{ps}(G:Y)$. Then, by definition, there must exist $B \in \mathscr{B}_G$ such that $V - D = V(B) = P(B,D)$. By Remark 5, the last equality implies that B is a complete block of G and, by Remark 9, $|N(x) \cap D| = 1$ for every $x \in V(B)$. This implies $\langle N[V(B)]\rangle = B^+$ and $d(x) = |V-D|$ for all $x \in V-D$. Since $|D| \leq n - \Delta \Rightarrow d(x) = |V-D| \geq \Delta$ we get $d(x) = \Delta = |V-D| = |V(B)|$ for every $x \in V-D$. Thus, $d(x) = \Delta(G)$ for all $x \in V(B)$. This also implies that $|V(B)| = \Delta(G)$ and $\langle N[V(B)]\rangle = B^+$, so establishing condition (ii). Next, since $n - \Delta = \gamma_p(G) = min.\{n-\Delta, n-k\}$ condition (i) follows.

For the converse, assume the validity of the conditions of the theorem. Then, (i) $\Rightarrow \gamma_p(G) = n - \Delta$ and hence (ii) $\Rightarrow D = V(G) - V(B)$ is a psd-set of order $n-\Delta$. This yields $D \in \mathscr{D}^0_{ps}(G:Y)$. □

If $\mathscr{D}^0_{ps}(G:Y) \neq \phi$ *how large could be* $\Delta(G) - k(G)$. The following result answers this question.

Theorem 5. $\mathscr{D}^0_{ps}(G:Y) \neq \phi \Rightarrow \Delta(G) \leq k(G)+1$.

Proof. $\mathscr{D}^0_{ps}(G:Y) \neq \phi \Rightarrow$ there exists $B \in \mathscr{B}_G$ such that B is a clique of order $\Delta \Rightarrow |V(B)| - \gamma_p(B) = \Delta - 1 \Rightarrow k(G) = max.\{|V(B')| - \gamma_p(B') : B' \in \mathscr{B}_G\} \geq \Delta - 1$. □

Thus, we have

$$\mathscr{D}^0_{ps}(G:Y) \neq \phi \Rightarrow \Delta(G) \in \{k(G), k(G)+1\}. \tag{19}$$

The next natural question is to identify the blocks that give rise to $\gamma_p(G)-$ *sets* $D \in \mathscr{D}^0_{ps}(G:Y)$. This, however, is answered in the proof of Theorem 5 from which it is clear that they are nothing but the clique blocks of order Δ in G which satisfy $\langle N[B] \rangle = B^+$.

Theorem 6. $\mathscr{D}^0_{ps}(G:X_2) \neq \phi$ *if and only if the following conditions are satisfied:*

(*i*) $\Delta(G) \geq k(G)$

(*ii*) $V(G)$ *can be partitioned into four nonempty subsets* V_1, V_2, V_3 *and* V_4 *such that*

- *(a)* $V_1 \neq \phi$.
- *(b)* $|V_1 \bigcup V_2| = \Delta(G)$.
- *(c)* $\langle V_1 \cup V_2 \bigcup V_3 \rangle \in \mathscr{B}_G$.
- *(d)* V_3 *is a psd-set of* $\langle V_2 \bigcup V_3 \rangle$.
- *(e)* $\langle V_1 \rangle$ *is complete.* $N(x) \cap V_2 = V_2$, $N(x) \cap V_3 = \phi$ *and* $N(x) \cap V_4 \neq \phi$ *for each* $x \in V_1$.

Proof. *Necessity.*

Let G have a γ_p -set D such that $V - D \subset V(B)$ and $P(B, D) \neq \phi$ for some $B \in \mathscr{B}_G$. Then, as shown in the previous section, we must have $\gamma_p(G) = n - \Delta$ so that (i) is satisfied. Hence, let $V_1 = P(B, D)$. $V_2 = (V - D) - V_1, V_3 = V(B) - (V - D) = V(B) \cap D$ and $V_4 = V(G) - V(B)$.

Clearly then $V_1 = P(B,D) \neq \phi$; since $V - D \subset V(B) \Rightarrow V(B) \cap D \neq \phi \Rightarrow V_1 = P(B,D) \subset V - D$ whence $V_2 \neq \phi; V_3 \neq \phi$ as already seen; and since G is not a block, $V_4 \neq \phi$. It is easy to see from the foregoing arguments that the decomposition $\{V_1, V_2, V_3, V_4\}$ of $V(G)$ so defined satisfies the stated conditions.

Sufficiency.

Suppose that $V(G)$ can be partitioned into four subsets as stated in the theorem. Then, firstly by condition (i) and Proposition 2 it follows that $\gamma_p(G) = n - \Delta$.

Next, let $D = V_3 \bigcup V_4$ so that $V - D = V_1 \bigcup V_2 \subset V(B)$ where $B = \langle V_1 \bigcup V_2 \bigcup V_3 \rangle$. Then $P(B, D) = V_1 \neq \phi$ by condition (ii)(a) and $|V - D| = |V_1 \bigcup V_2| = \Delta(G)$ by condition (ii)(b). By (ii)(c) we have $V - D \subset V(B)$. By (ii)(d) and (ii)(e) it follows that D is a $\gamma_p(G)$ -set. Thus, we get $D \in \mathscr{D}^0_{ps}(G:X_2)$ and the proof is complete. □

If $\mathscr{D}^0_{ps}(G:X_2) \neq \phi$, *how large could be* $\Delta(G) - k(G)$. The following result answers this question.

Theorem 7. $\mathscr{D}^0_{ps}(G:X_2) \neq \phi \Rightarrow \Delta(G) \leq k(G)+1.$

Proof. Let $D \in \mathscr{D}^0_{ps}(G : X_2)$. Then, by definition of $\mathscr{D}^0_{ps}(G : X_2)$ there must exist $B \in \mathscr{B}_G$ such that $V - D \subset V(B)$ and $P(B, D) \neq \phi$. Also, $V(B) \cap D$ is a psd-set of $B - P(B, D)$. By the definition of $P(B, D)$, one can easily see that $(V(B) \cap D) \bigcup \{u\}$, for any $u \in P(B, D)$, is a psd-set for B so that

$$\gamma_p(B) \leq |V(B) \cap D| + 1. \tag{20}$$

Also,

$$|V(B) \cap D| \leq \gamma_p(B) \tag{21}$$

for, otherwise, $(V(G) - V(B)) \bigcup \{$ a $\gamma_p(B) - set\}$ would be a psd-set of G having less than $|D|$ vertices contrary to the fact that D is a $\gamma_p(G)$-set. By (20) and (21) we get

$$\gamma_p(B) - 1 \leq |V(B) \cap D| \leq \gamma_p(B). \tag{22}$$

Now, $|D| = n - |V(B)| + |V(B) \cap D| \geq n - |V(B)| + \gamma_p(B) - 1$; that is $n - \Delta \geq n - |V(B)| + \gamma_p(B) - 1$ or, equivalently, $\Delta(G) \leq |V(B)| - \gamma_p(B) + 1 \leq k(G) + 1$. □

Thus, we have

$$\mathscr{D}^0_{ps}(G : X_2) \neq \phi \Rightarrow \Delta(G) \in \{k(G), k(G) + 1\}. \tag{23}$$

Corollary 7.1. *If D is a $\gamma_p(G)$-set such that $V - D \subset V(B)$ and $P(B, D) \neq \phi$ for some $B \in \mathscr{B}_G$, then $|V(B) \cap D| = \gamma_p(B)$ or $|V(B) \cap D| = \gamma_p(B) - 1$.*

The next natural question is to identify the blocks that give rise to $\gamma_p(G)$-sets $D \in \mathscr{D}^0_{ps}(G : X_2)$.

Theorem 8. *If $\Delta(G) = k(G) + 1$ and if D is a $\gamma_p(G)$-set such that $V - D \subset V(B)$ and $P(B, D) \neq \phi$, then $|V(B)| - \gamma_p(B) = k(G)$ and $|V(B) \cap D| = \gamma_p(B) - 1$.*

Proof. Suppose $|V(B)| - \gamma_p(B) < k(G)$. Then, $|D| = n - |V(B)| + |V(B) \cap D| \geq n - |V(B)| + \gamma_p(B) - 1 > n - k(G) - 1$ and hence $|D| \geq n - k(G)$. However, since $\Delta(G) > k(G)$ we must have $|D| = \gamma_p(G) = n - \Delta(G)$. Therefore, we get $n - \Delta \geq n - k$ whence $\Delta \leq k$ contrary to the hypothesis. Thus, we must have $|V(B)| - \gamma_p(B) = k(G)$. Next, by virtue of (22) in the proof of Theorem 7, $|V(B) \cap D| \in \{\gamma_p(B), \gamma_p(B) - 1\}$. If $|V(B) \cap D| = \gamma_p(B)$ we can arrive at the same contradiction. Hence, the result follows. □

On lines analogous to the proof of Theorem 8, one may easily prove the following.

Theorem 9. *Let G be a graph such that $\Delta(G) = k(G)$. If D is a $\gamma_p(G)$-set and $B \in \mathscr{B}_G$ such that $V - D \subset V(B)$ and $P(B, D) \neq \phi$ then $|V(B)| - \gamma_p(B) = k(G)$ or $|V(B)| - \gamma_p(B) = k(G) - 1$.*

Proof. Suppose $|V(B)| - \gamma_p(B) < k(G)$. Then $|D| = n - |V(B)| + |V(B) \cap D| \geq n - |V(B)| + \gamma_p(B) - 1 > n - (k - 1) - 1$ so that $n - \Delta > n - k$, or equivalently $\Delta < k$, a contradiction to our assumption. Thus, the result follows. □

We can show further that the following implications hold under the hypotheses of Theorem 9.

$$|V(B)| - \gamma_p(B) = k(G) \Rightarrow |V(B) \cap D| = \gamma_p(B), \tag{24}$$

$$|V(B)| - \gamma_p(B) = k(G) - 1 \Rightarrow |V(B) \cap D| = \gamma_p(B) - 1. \tag{25}$$

Examples of both the types are shown in Figure-1.

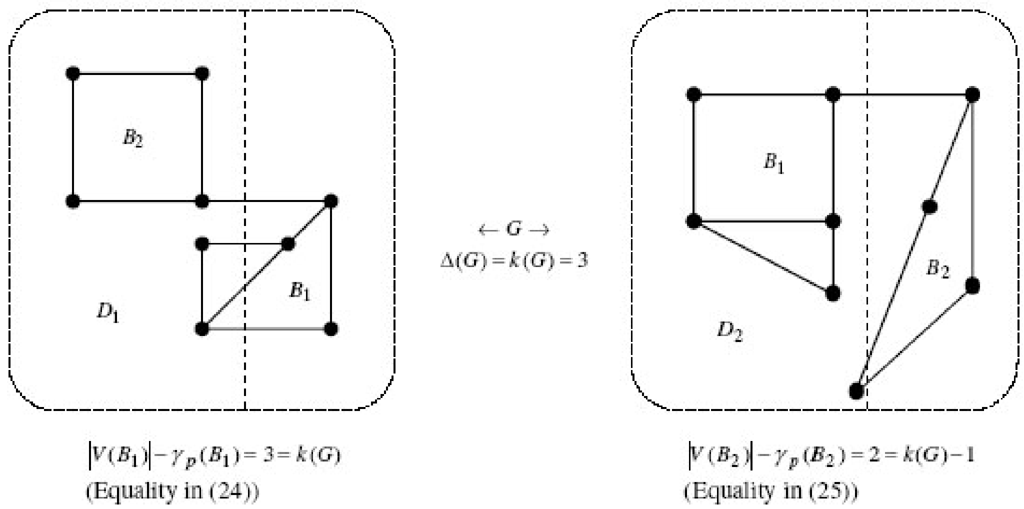


Figure 1.

Next interest is to know about the blocks B that are involved with the members $D \in \mathscr{D}^0_{ps}(G : X_1)$; that is, such that $V - D \subset V(B)$ and $P(B, D) = \phi$. In this case, obviously, $V(B) \cap D$ is a $\gamma_p(B)$ -set and $n - k(G) \geq \gamma_p(G) = n - |V(B)| + \gamma_p(B) \geq n - k(G)$ so that $|V(B)| - \gamma_p(B) = k(G)$. Notice that $|V(B) \cap D| = \gamma_p(B)$ in this case.

Thus, in every case, we have shown that if $D \in \mathscr{D}^0_{ps}(G : X)$ then $\gamma_p(G) \in \{n - |V(B)| + \gamma_p(B), n - |V(B)| + \gamma_p(B) - 1\}$, answering the question how much less could $\gamma_p(G)$ be from the quantity $n - |V(B)| + \gamma_p(B)$, raised while considering the inequality (15).

Further, one might be interested to know whether the same block B of G could be involved in deciding the membership of a psd-set D_1 in $\mathscr{D}^0_{ps}(G : X_1)$ and a psd-set D_2 in $\mathscr{D}^0_{ps}(G : X_2)$. The answer is in the affirmative (obviously, if $\Delta(G) = k(G)$), as illustrated in Figure -2.

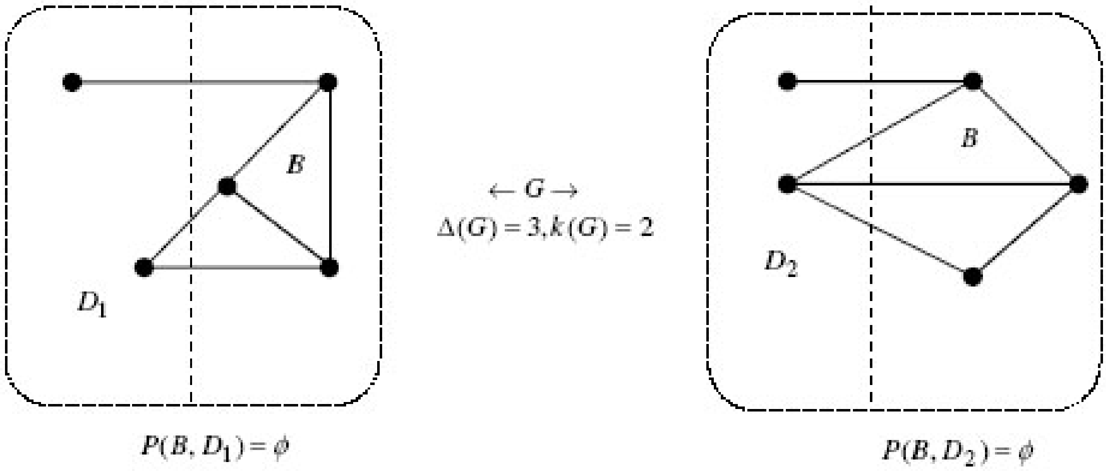
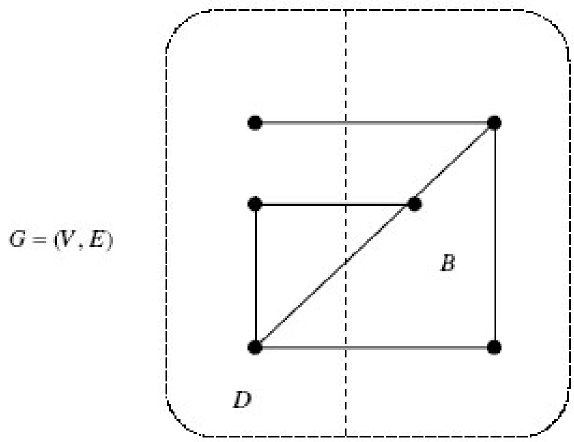


Figure 2.

Remark 13. *If D is a $\gamma_p(G)$-set such that $V - D \subset V(B)$ and $P(B, D) \neq \phi$ for some $B \in \mathscr{B}_G$ (i.e., $D \in \mathscr{D}^0_{ps}(G : X_2)$), then $V(B) \cap D$ is a psd-set for $\langle B - P(B, D)\rangle$ but need not be a $\gamma_p(\langle B - P(B, D)\rangle)$-set as illustrated by the example in Figure - 3.*



$V - D \subset V(B)$
$P(B, D) \neq \phi$
$V(B) \cap D \in \mathscr{D}_{ps}(\langle V(B) - P(B, D)\rangle)$
but $V(B) \cap D$ is not a $\gamma_p(\langle V(B) - P(B, D)\rangle)$-set
Figure 3.

References

[1] B.D. Acharya and P. Gupta, On point-set domination in graphs, In: *Proceedings of National Seminar on Recent Developments in Mathematics*, Karnatak Universiy, Dharwad, (1996), 106-108.

[2] P. Gupta, Certain extensions of the notion of domination in graphs, Ph.D. Thesis, University of Delhi, September, 1998.

[3] F. Harary,*Graph Theory*, Addison-Wesley, Reading, Mass., 1972.

[4] T. Haynes, S.T. Hedetniemi and P.J. Slater, *Domination in Graphs - Advanced topics*, Marcel Dekker, New York, 1998.

[5] O. Ore, *Theory of Graphs*, American Mathematical Society Colloq. Publi., No. 38, 1962.

[6] E. Sampathkumar and L. Pushpa Latha, Point-set domination number of a graph, *Indian J. Pure Appl. Math.,* **24**(4)(1993), 225-229.