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Extremal Problems on Detectable Colorings of Connected Graphs With Cycle Rank 2

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Abstract

Let G be a connected graph of order $n \geq 3$ and let $c : E(G) \to \{1, 2, \ldots, k\}$ be a coloring of the edges of G (where adjacent edges may be colored the same). For each vertex v of G, the color code of v with respect to c is the k-tuple $c(v) = (a_1, a_2, \cdots, a_k)$, where a_i is the number of edges incident with v that are colored i ($1 \leq i \leq k$). The coloring c is detectable if distinct vertices have distinct color codes. The detection number $\det(G)$ of G is the minimum positive integer k for which G has a detectable k-coloring. A connected graph of order $n \geq 4$ and size m is said to have cycle rank 2 if $m = n + 1$. For each integer $n \geq 4$, let $D_2(n)$ be the maximum detection number among all connected graphs of order n with cycle rank 2 and $d_2(n)$ the minimum detection number among all connected graphs of order n with cycle rank 2. The numbers $D_2(n)$ and $d_2(n)$ are determined for all integers $n \geq 4$. Furthermore, for integers $k \geq 2$ and $n \geq 4$, there exists a connected graph G of order n having cycle rank 2 and $\det(G) = k$ if and only if $d_2(n) \leq k \leq D_2(n)$.

Keywords: Detectable coloring, Detection number, Cycle rank.

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1. Introduction

We refer to the book [7] for graph theory notation and terminology not described in this paper. A problem in graph theory that has received increased attention during the past 35 years concerns studying methods to distinguish the vertices of a connected graph. One of the earlier methods is due to Sumner [15] and Entringer and Gassman [8]. They studied graphs G for which the equality of the open neighborhoods of every two vertices of G implies that the vertices are the same. In this case, the vertices of G are uniquely determined by their open neighborhoods.

Erwin and Harary [9] introduced the idea of selecting a subset S of the vertex set of a graph G such that the subgroup of $\mathrm{Aut}(G)$ that fixes every vertex of S is the identity group. Albertson and Collins [3] and Harary [11] introduced the notion of coloring the vertices of G in such a way that the subgroup of color-preserving automorphisms of $\mathrm{Aut}(G)$ is the identity group. In these ways, the vertices of a graph G can be distinguished from one another with the aid of certain automorphisms of G.

Another way to distinguish the vertices of a connected graph G from one another was introduced by Harary and Melter [12] and Slater [14]. In this method, an ordered set W of vertices of G, say $W = \{ w_1, w_2, \ldots, w_k \}$, is found and each vertex v is assigned the ordered k-tuple $c_W(v) = (a_1, a_2, \ldots, a_k)$, where $a_i = d(v, w_i)$ is the distance between v and w_i for $1 \leq i \leq k$. The ordered k-tuple $c_W(v)$ is sometimes called the *distance code* of v. If distinct vertices of G have distinct distance codes, then the vertices of G are distinguishable.

Harary and Plantholt [13] introduced yet another way to distinguish the vertices of a graph G by assigning colors to the edges of G in such a way that for every two vertices of G, one of the vertices is incident with an edge assigned one of these colors that the other vertex is not. They referred to the minimum number of colors needed to accomplish this as the *point-distinguishing chromatic index* of G.

There is still another manner in which differences among the vertices of a connected graph G can be detected and that combines a number of the features of the methods mentioned above. Let G be a connected graph of order $n \geq 3$ and let $c : E(G) \to \{1, 2, \ldots, k\}$ be a coloring of the edges of G for some positive integer k (where adjacent edges may be colored the same). The *color code* of a vertex v of G (with respect to c) is the ordered k-tuple

$$c(v) = (a_1, a_2, \cdots, a_k) \quad \text{(or simply, } c(v) = a_1 a_2 \cdots a_k\text{)},$$

where a_i is the number of edges incident with v that are colored i for $1 \leq i \leq k$. Therefore, $\sum_{i=1}^{k} a_i = \deg_G v$. The coloring c is called *detectable* if distinct vertices have distinct color codes; that is, for every two vertices of G, there exists a color such that the number of incident edges with that color is different for these two vertices. The *detection number* $\det(G)$ of G is the minimum positive integer k for which G has a detectable k-coloring. Such a coloring is called a *minimum detectable coloring.* Since every nontrivial graph contains at least two vertices having the same degree, the vertices of a nontrivial connected graph cannot be distinguished by their degrees alone. Therefore, every connected graph of order 3 or more has detection number at least 2. To illustrate these concepts, consider the graph G shown in Figure 1(a).

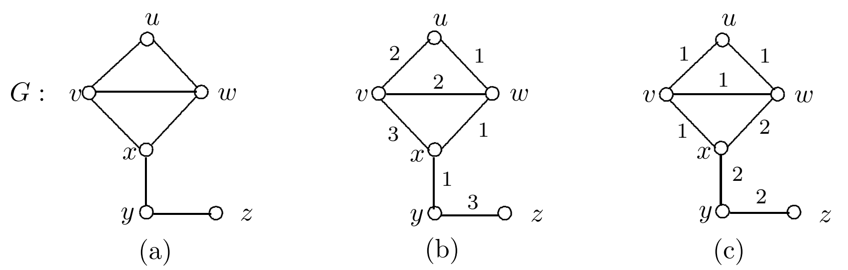


Figure 1: A detectable coloring of a graph

A coloring of the edges of G is shown in Figure 1(b). For this 3-coloring c, the color codes of its vertices are $c(u) = 110$, $c(v) = 021$, $c(w) = 210$, $c(x) = 201$, $c(y) = 101$, $c(z) = 001$. Since the vertices of G have distinct color codes, c is a detectable coloring. Figure 1(c) shows yet another detectable coloring c' of the graph G of Figure 1(a). For this coloring, $c'(u) = 20$, $c'(v) = 30$, $c'(w) = 21$, $c'(x) = 12$, $c'(y) = 02$, $c'(z) = 01$. The coloring c' uses only two colors. Once a detectable 2-coloring for the graph G of Figure 1(c) was obtained, we can immediately conclude that $\det(G) = 2$ as every connected graph of order 3 or more has detection number at least 2.

The concept of detectable colorings was studied in [1, 2, 4, 5, 6, 10]. Among the results obtained previously are the following (see [2, 6, 10]).

Theorem A *If c is a detectable k-coloring of a connected graph G of order at least 3, then G contains at most $\binom{r+k-1}{r}$ vertices of degree r.*

Theorem B *Let G be connected graph of order $n \geq 3$. If H is a connected subgraph of G, then* $\det(G) - \det(H) \leq m(G) - m(H)$.

The detection numbers of complete graphs and complete bipartite graphs have been determined and detectable colorings of connected r-regular graphs and trees have been studied as well (see [2, 4, 5, 6, 10]). The detection number of the cycle C_n of order n was established in [6], which we state next.

Theorem C *Let $n \geq 3$ be an integer and let $\ell = \left\lceil \sqrt{n/2} \right\rceil$. Then*

$$\det(C_n) = \begin{cases} 2\ell & \text{if } 2\ell^2 - \ell + 1 \leq n \leq 2\ell^2 \\ 2\ell - 1 & \text{if } 2(\ell-1)^2 + 1 \leq n \leq 2\ell^2 - \ell. \end{cases}$$

Let G be a connected graph of order n and size m. The number of edges that must be deleted from G to obtain a spanning tree of G is $m - n + 1$. The number $m - n + 1$ is called the *cycle rank* (or *Betti number*) of G. Thus the cycle rank of a tree is 0 and the cycle rank of a unicyclic graph (a connected graph with exactly one cycle) is 1. The cycle rank of a connected graph of order n and size $m = n + 1$ is therefore 2. A graph H is called a *subdivision* of a graph G if H is obtained from G by inserting one or more vertices of degree 2 into one or more edges of G. For this purpose, we also say that a graph is a subdivision of itself. If H is a subdivision of a graph G, then H and G have the same cycle rank. If G is a connected graph of order n whose cycle rank is ψ, then $m = (n-1) + \psi \leq \binom{n}{2}$ and so

$$n \geq \left\lceil \frac{3 + \sqrt{1 + 8\psi}}{2} \right\rceil.$$

In the special case where the cycle rank is 2, it follows that $n \geq 4$. As is the case with trees, there is an formula that gives the number of end-vertices in a connected graph G having cycle rank ψ in terms of ψ and the number of vertices of G having degree 3 or more.

Proposition 1.1. *Let G be a nontrivial connected graph having maximum degree Δ and cycle rank ψ. If n_i is the number of vertices of degree i in G, where $1 \leq i \leq \Delta$, then*

$$n_1 = (2 - 2\psi) + n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta. \tag{1}$$

Proof. Suppose that G has order n and size m. Then $m = (n-1) + \psi$,

$$n = \textstyle\sum_{i=1}^{\Delta} n_i, \text{ and } 2m = \sum_{i=1}^{\Delta} i n_i.$$

Therefore, $2m = 2(n - 1 + \psi) = 2n - 2 + 2\psi$ and so

$$\sum_{i=1}^{\Delta} i n_i = 2 \sum_{i=1}^{\Delta} n_i - 2 + 2\psi. \tag{2}$$

Solving for n_1 in (2), we obtain (1). □

For integers ψ and n, where $\psi \geq 0$ and $n \geq \left\lceil \frac{3+\sqrt{1+8\psi}}{2} \right\rceil$, let $D_\psi(n)$ denote the maximum detection number among all connected graphs of order n with cycle rank ψ and $d_\psi(n)$ the minimum detection number among all connected graphs of order n with cycle rank ψ. That is, if $\mathcal{G}_{\psi,n}$ is the set of all connected graphs of order n with cycle rank ψ, then

$$\begin{aligned} D_\psi(n) &= \max\{\det(G): G \in \mathcal{G}_{\psi,n}\} \\ d_\psi(n) &= \min\{\det(G): G \in \mathcal{G}_{\psi,n}\}. \end{aligned}$$

Thus $D_0(n)$ is the maximum detection number among all trees of order n and $D_1(n)$ is the maximum detection number among all unicyclic graphs of order n; while $d_0(n)$ and $d_1(n)$ are minimum detection numbers among all trees and unicyclic graphs of order n, respectively. These numbers have been determined (see [10]).

Theorem D *Let $n \geq 3$ be an integer. Then $D_0(n) = n - 1$ and $d_0(n) = \left\lceil \frac{-5+\sqrt{8n+41}}{2} \right\rceil$.*

Theorem E *Let $n \geq 3$ be an integer. Then*

$$D_1(n) = \begin{cases} 3 & \text{if } n = 3, 4, 5 \\ n-3 & \text{if } n \geq 6, \end{cases} \quad \text{and } d_1(n) = \left\lceil \frac{-5+\sqrt{8n+25}}{2} \right\rceil.$$

Furthermore, in the case $\psi = 0$ or $\psi = 1$, it was known that for integers $k \geq 2$ and $n \geq 3$, there exists a connected graph G of order n having cycle rank ψ and $\det(G) = k$ if and only if $d_\psi(n) \leq k \leq D_\psi(n)$.

In this work, we investige detectable colorings of connected graphs of order n having cycle rank 2. We study some extremal problems concerning the detection numbers of connected graphs with cycle rank 2, in particular, the problems of determining how large and how small the detection number of such a connected graph of a fixed order can be. Thus, for each integer $n \geq 4$, let $\mathcal{G}_{2,n}$ denote the set of all connected graphs of order n with cycle rank 2, let $D_2(n)$ denote the maximum detection number among all graphs belonging to $\mathcal{G}_{2,n}$, and let $d_2(n)$ denote the minimum detection number among all graphs belonging to $\mathcal{G}_{2,n}$. Since $\mathcal{G}_{2,4} = \{K_4 - e\}$ and $\det(K_4 - e) = 2$, it follows that $D_2(4) = d_2(4) = 2$. All connected graphs of order 5 or 6 with cycle rank 2 are shown in Figure 2 together with a minimum detectable coloring for each. Therefore, $D_2(5) = D_2(6) = 3$ and $d_2(5) = d_2(6) = 2$.

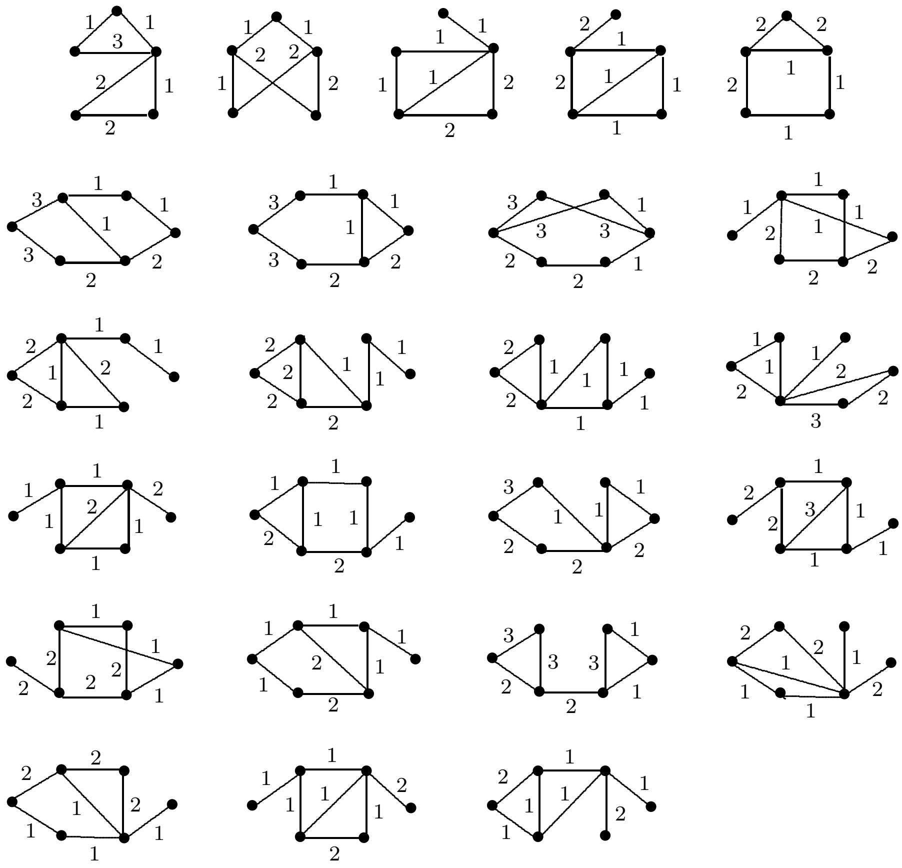


Figure 2: Graphs in $\mathcal{G}_{2,5}$ and $\mathcal{G}_{2,6}$ with minimum detectable colorings

2. The Maximum Detection Number

We begin by determining $D_2(n)$. Unlike the case where trees and unicyclic graphs are involved, it is much more challenging to establish a formula for $D_2(n)$. We have seen that if G is a connected graph of order n having cycle rank 2, then $n \geq 4$. Since $D_2(4) = 2$ and $D_2(5) = D_2(6) = 3$, we may assume that $n \geq 7$. Observe that the graph $G \in \mathcal{G}_{2,n}$ shown in Figure 3 has order n and detection number $n-4$ for $n \geq 7$. Thus $D_2(n) \geq n-4$ for $n \geq 7$. We show, in fact, that $D_2(n) = n-4$ for $n \geq 7$. In order to verify this, we first establish several preliminary results.

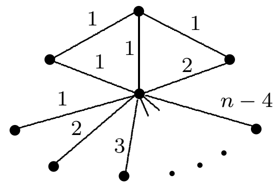
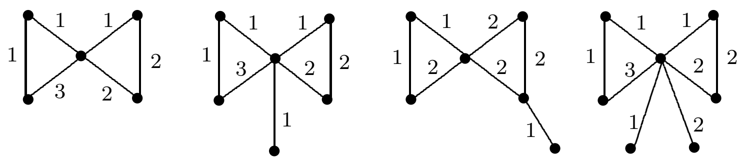


Figure 3: A graph $G \in \mathcal{G}_{2,n}$ with $\det(G) = n-4$

We begin by considering some special classes of graphs in $\mathcal{G}_{2,n}$. Observe that each of the four graphs in Figure 4 has cycle rank 2. Note that the subscript of each of these graphs equals the number of pendant edges that it contains. A minimum detectable coloring for each such graph is also shown in Figure 4. We now describe four classes of connected graphs with cycle rank 2 constructed from the four graphs shown in Figure 4.



B_0 H_1 F_1 J_2

Figure 4: Four graphs with cycle rank 2

(1) Let $\mathcal{B}$ be the set of all graphs that are subdivisions of the graph B_0 in Figure 4. That is, $\mathcal{B}$ consists of all connected graphs obtained from two cycles C and C' , by identifying a vertex in C and a vertex in C' . In particular, $B_0 \in \mathcal{B}$. For each integer $n \geq 5$, let $\mathcal{B}_n$ be the set of graphs of order n in $\mathcal{B}$. So $\mathcal{B}_5 = \{B_0\}$. If $B \in \mathcal{B}_n$, then B has one vertex of degree 4 and $n-1$ vertices of degree 2.

(2) Let $\mathcal{H}$ be the set of all subdivisions of the graph H_1 in Figure 4. That is, $\mathcal{H}$ consists of all connected graphs obtained from a graph $B \in \mathcal{B}$ by adding a new vertex, joining this vertex to the unique vertex of degree 4 in B , and then possibly subdividing the newly added pendant edge. For each integer $n \geq 6$, let $\mathcal{H}_n$ be the set of graphs of order n in $\mathcal{H}$. Thus if $H \in \mathcal{H}_n$, then H has one end-vertex, one vertex of degree 5, and $n-2$ vertices of degree 2.

(3) Let $\mathcal{F}$ be the set of all subdivisions of the graph F_1 in Figure 4. That is, $\mathcal{F}$ consists of all connected graphs obtained from a graph $B \in \mathcal{B}$ by adding a new vertex, joining this vertex to a vertex of degree 2 in B , and then possibly subdividing the newly added pendant edge.

For each integer $n \geq 6$, let $\mathcal{F}_n$ be the set of graphs of order n in $\mathcal{F}$. Thus if $F \in \mathcal{F}_n$, then F has one end-vertex, one vertex of degree 3, one vertex of degree 4, and $n-3$ vertices of degree 2.

(4) Let $\mathcal{J}$ be the set of all subdivisions of the graph J_2 in Figure 4. That is, $\mathcal{J}$ consists of all connected graphs obtained from a graph $B \in \mathcal{B}$ by adding two vertices, joining each of the two new vertices to the unique vertex of degree 4 in B, and then possibly subdividing one or both newly added pendant edges. For each integer $n \geq 7$, let $\mathcal{J}_n$ be the set of graphs of order n in $\mathcal{J}$. Thus if $J \in \mathcal{J}_n$, then H has two end-vertices, one vertex of degree 6, and $n-3$ vertices of degree 2.

In order to establish upper bounds for the detection numbers of graphs belonging to any of the four classes (1)-(4) in terms of their orders, we first present two useful results, which are consequences of Theorems B and C, respectively. For a graph G, let $n(G)$ denote the order of G and $m(G)$ the size of G.

Lemma 2.1. *Let G be a connected graph with cycle rank 2 and let H be a connected subgraph of G.*

(a) *If H has cycle rank 2, then* $\det(G) \leq \det(H) + n(G) - n(H)$.

(b) *If H is unicyclic, then* $\det(G) \leq \det(H) + n(G) - n(H) + 1$.

Proof. We first verify (a). Since G and H both have cycle rank 2, it follows that $m(G) = n(G)+1$ and $m(H) = n(H)+1$. It then follows by Theorem B that

$$\det(G) \leq \det(H) + m(G) - m(H) = \det(H) + n(G) - n(H).$$

The proof of (b) is similar as that of (a) except that $m(H) = n(H)$. □

Corollary 2.2. *Let $n \geq 3$ be an integer. Then $\det(C_n) = 3$ for $3 \leq n \leq 5$ and $\det(C_n) \leq n-3$ for $n \geq 6$.*

Lemma 2.3. *If $B \in \mathcal{B}_n$ where $n \geq 5$, then*

$$\det(B) \quad \leq \quad \begin{cases} n-2 & \textit{if } n = 5 \\ n-3 & \textit{if } n = 6 \\ n-4 & \textit{if } n \geq 7. \end{cases}$$

Proof. Let v be the vertex of degree 4 in B. Suppose that $N(v) = \{w, x, y, z\}$, where w and y belong to one component of $B - v$ and x and z belong to the other component of $B - v$. Let C be the cycle of order $n-1$ obtained from $B - v$ by joining w to x and y to z. Suppose that $\det(C) = k$ and let c be a detectable k-coloring of C. Define a k-coloring $c' : E(B) \to \{1, 2, \ldots, k\}$ of B by

$$c'(e) = \begin{cases} c(wx) & \text{if } e = vw \text{ or } e = vx \\ c(yz) & \text{if } e = vy \text{ or } e = vz \\ c(e) & \text{otherwise.} \end{cases}$$

The color codes of the vertices of B are all those of C together with the color code of v. Since v is the only vertex of degree 4 in B, it follows that c' is a detectable k-coloring of B and so $\det(B) \leq k = \det(C)$. Since $\det(C_4) = \det(C_5) = 3$ and $\det(C) \leq (n-1) - 3 = n-4$ if the order $n-1$ of C is at least 6 (or $n \geq 7$) by Corollary 2.2, we have the desired result. □

Lemma 2.4. *If $H \in \mathcal{H}_n$, where $n \geq 6$, then*

$$\det(H) \quad \leq \quad \begin{cases} n-3 & \textit{if } n = 6 \\ n-4 & \textit{if } n \geq 7. \end{cases}$$

Proof. Let $B \in \mathcal{B}$ be a subgraph of H, let v be the unique vertex of degree 4 in B, and let w be the vertex in $V(H) - V(B)$ that is adjacent to v. Furthermore, let H' be the subgraph of H obtained from B by adding the pendant edge vw to B. Thus $n(H') = n(B) + 1$. Suppose that $\det(B) = k$ and let c be a detectable k-coloring of B. The coloring $c' : E(H') \to \{1, 2, \ldots, k\}$ of H' defined by $c'(e) = 1$ if $e = vw$ and $c'(e) = c(e)$ if $e \neq vw$ is a detectable k-coloring of H'. Thus, $\det(H') \leq k = \det(B)$. By Lemma 2.3,

$$\det(B) \leq \begin{cases} n(B) - 2 & \text{if } n(B) = 5 \\ n(B) - 3 & \text{if } n(B) = 6 \\ n(B) - 4 & \text{if } n(B) \geq 7. \end{cases}$$

Since $\det(H') \leq \det(B)$ and $n(B) = n(H') - 1$, it follows that

$$\det(H') \quad \leq \quad \begin{cases} n(H') - 3 & \text{if } n(H') = 6 \\ n(H') - 4 & \text{if } n(H') = 7 \\ n(H') - 5 & \text{if } n(H') \geq 8. \end{cases}$$

Thus $\det(H') \leq n(H') - 3$ if $n(H') = 6$ and $\det(H') \leq n(H') - 4$ if $n(H') \geq 7$. If $n = 6$, then $H = H'$, which is isomorphic to the graph H_1 of Figure 4, and so $\det(H) = \det(H') = 3 = n - 3$. Thus we may assume that $n \geq 7$. If $n(H') \geq 7$, then by Lemma 2.1,

$$\det(H) \leq \det(H') + n(H) - n(H') \leq n(H') - 4 + n(H) - n(H') = n - 4 .$$

If $n(H') = 6$, then H contains a subgraph H'' that is isomorphic the graph of Figure 5. Since H'' has a (minimum) detectable 3-coloring, as shown in Figure 5, it follows that $\det(H'') = 3 = n(H'') - 4$. Thus $\det(H) \leq n - 4$ by Lemma 2.1. $\square$

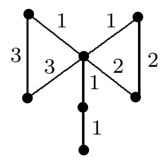


Figure 5: A subgraph of H in the proof of Lemma 2.4

Lemma 2.5. *If $F \in \mathcal{F}_n$ where $n \geq 6$, then* $\det(F) \leq n - 4$.

Proof. Let $B \in \mathcal{B}$ be a subgraph of F. If the order of B is 5, then F contains a subgraph F' that is isomorphic to the graph F_1 in Figure 4. Since $\det(F_1) = 2 = n(F_1) - 4$, it follows by Lemma 2.1 that $\det(F) \leq n - 4$. If the order of B is 6, then F contains a subgraph F' that is isomorphic to one of the graphs of order 7 in Figure 6. In each case, $\det(F') = 3 = n(F') - 4$ and so $\det(F) \leq n - 4$ by Lemma 2.1. If the order of B is 7 or more, then $\det(B) \leq n(B) - 4$ by Lemma 2.3. It then follows by Lemma 2.1 that $\det(F) \leq n - 4$. $\square$

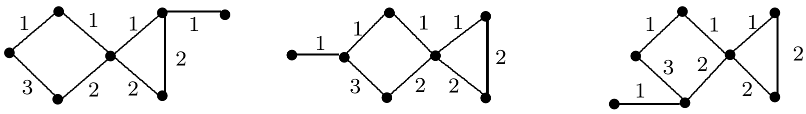


Figure 6: Subgraphs of F in the proof of Lemma 2.5

Lemma 2.6. *If $J \in \mathcal{J}_n$ where $n \geq 7$, then* $\det(J) \leq n-4$.

Proof. Let $B \in \mathcal{B}$ be a subgraph in J. If the order of B is 5, then J contains a subgraph isomorphic to the graph J_2 in Figure 4. Since $\det(J_2) = 3 = n(J_2) - 4$, it follows by Lemma 2.1 that $\det(J) \leq n-4$. If the order of B is 6 or more, then J contains a subgraph $H \in \mathcal{H}$ of order 7 or more. Since $\det(H) \leq n(H) - 4$ by Lemma 2.4, it follows that $\det(J) \leq n-4$ by Lemma 2.1. □

The following lemma will be useful to us.

Lemma 2.7. *Let $G \in \mathcal{G}_{2,n}$ where $n \geq 7$ and let g be the girth of G. If $g \geq 6$, then*

$$\det(G) \quad \leq \quad \begin{cases} n-3 & \text{if } g = 6 \\ n-4 & \text{if } g \geq 7 \end{cases}$$

Proof. Let $C_g : v_1, v_2, \cdots, v_g, v_1$ be a cycle of order g in G. Since $G \neq C_g$, there is a vertex w of G such that $w \notin V(C_g)$ and w is adjacent to a vertex on C_g, say w is adjacent to v_g. Let F be the graph obtained from C_g by adding the edge $v_g w$. Then F is a unicyclic subgraph of order $g+1$ and size $g+1$ in G. Let $C_{g-1} : v_1, v_2, \cdots, v_{g-1}, v_1$ be a cycle of order $g-1$. Suppose that $\det(C_{g-1}) = k$ and let c be a detectable k-coloring of C_{g-1}. The k-coloring $c' : E(F) \to \{1, 2, \ldots k\}$ of F defined by

$$c'(e) = \begin{cases} c(v_{g-1}v_1) & \text{if } e = v_g w, v_g v_1, v_g v_{g-1} \\ c(e) & \text{otherwise} \end{cases}$$

is detectable. Thus, $\det(F) \leq \det(C_{g-1})$. If $g = 6$, then $\det(C_{g-1}) = \det(C_5) = 3$. Thus $\det(F) \leq 3$ and so

$$\det(G) \leq \det(F) + m(G) - m(F) \leq 3 + (n+1) - 7 = n-3$$

by Theorem B. If $g \geq 7$ (and so $g-1 \geq 6$), then $\det(C_{g-1}) \leq (g-1) - 3 = g-4$ by Corollary 2.2. Thus $\det(F) \leq g-4$ and so

$$\det(G) \leq \det(F) + m(G) - m(F) \leq (g-4) + (n+1) - (g+1) = n-4$$

by Lemma 2.1. □

We are now prepared to establish the following.

Theorem 2.8. *If* $G \in \mathcal{G}_{2,n}$ *where* $n \geq 7$ *, then* $\det(G) \leq n - 4$.

Proof. Since G contains at least two cycles, G has a subgraph F that is isomorphic to one of three types of graphs in Figure 7:

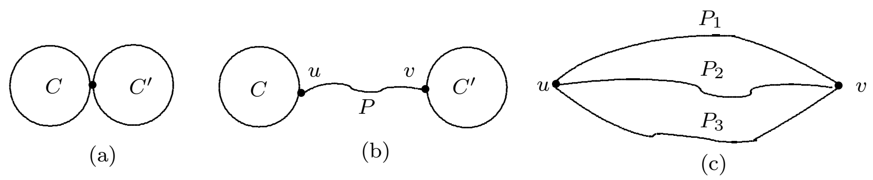


Figure 7: Three possible types of subgraphs

(1) a graph obtained from two cycles C and C' , by identifying a vertex in C and a vertex in C' , that is, a graph in $\mathcal{B}$ as shown in Figure 7(a),

(2) a graph obtained from two disjoint cycles C and C' and a path P of length 1 or more by identifying an end-vertex u of P with a vertex of C and identifying the other end-vertex v of P with a vertex of C' as shown in Figure 7(b),

(3) a subdivision of $K_4 - e$, that is, a graph consisting of three internally disjoint $u - v$ paths P_i $(1 \leq i \leq 3)$, as shown in Figure 7(c), where at least two paths P_i $(1 \leq i \leq 3)$ have length 2 or more.

We now consider these three cases.

Case 1 **.** F is isomorphic to some graph $B \in \mathcal{B}$.

If the order of F is 7 or more, then $\det(F) \leq n(F) - 4$ by Lemma 2.3. It then follows by Lemma 2.1 that $\det(G) \leq n - 4$. If the order of F is 6, then G contains a subgraph F' of order 7 or more such that $F' \in \mathcal{H}$ or $F' \in \mathcal{F}$. Thus $\det(F') \leq n(F') - 4$ by Lemma 2.4 or by Lemma 2.5. Hence $\det(G) \leq n - 4$ by Lemma 2.1.

We now assume that the order of F is 5. If G contains a subgraph F' of order 7 or more that belongs to one of the classes $\mathcal{H}$, $\mathcal{F}$ or $\mathcal{J}$, then $\det(F') \leq n(F') - 4$ by Propositions 2.4,2.5, or 2.6 and so $\det(G) \leq n - 4$ by Lemma 2.1. Otherwise, G contains a subgraph F' that is isomorphic to one of the graphs of order 7 in Figure 8, where a minimum detectable 2-coloring is given for each graph. In each case, $\det(F') = 2 = n(F') - 5$. Thus $\det(G) \leq n - 5 < n - 4$ by Lemma 2.1.

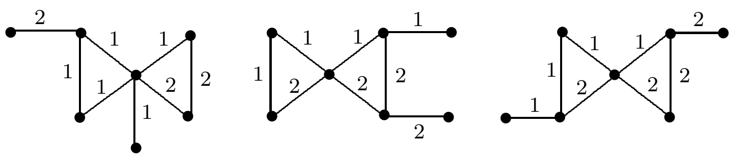


Figure 8: Subgraphs of G in Case 1

Case 2. F is isomorphic to a graph belonging to the class of graphs described in (2) .

If the girth g of G is 7 or more, then $\det(G) \leq n-4$ by Lemma 2.7. On the other hand, if $3 \leq g \leq 6$, then G contains a subgraph F that is isomorphic to one of the graphs shown in Figure 9, where a minimum detectable coloring is given for each such graph. If F is isomorphic to one of the unicyclic graphs in Figure 9, then $\det(F) = n(F)-5$; while if F is isomorphic to one of the graphs with cycle rank 2 in Figure 9, then $\det(F) \leq n(F)-4$. In either case, $\det(G) \leq n-4$ by Lemma 2.1.

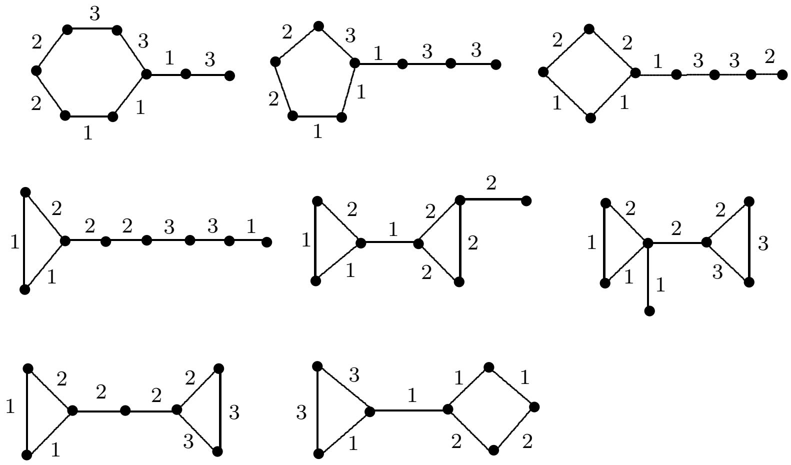


Figure 9: Possible subgraphs of G in Case 2

Case 3. F is isomorphic to a graph belonging to the class of graphs described in (3).

If the girth g of G is 7 or more, then $\det(G) \leq n-4$ by Lemma 2.7. If the girth g of G is 6 or 5, then G contains a subgraph F that is isomorphic to one of the graphs shown in Figure 10, where a minimum detectable 3-coloring is given for each such graph. If F is isomorphic to one of the two unicyclic graphs in Figure 10, then $\det(F) = n(F)-5$; while if F is isomorphic to the graph with cycle rank 2 in Figure 10, then $\det(F) = n(F)-4$. In either case, $\det(G) \leq n-4$ by Lemma 2.1.

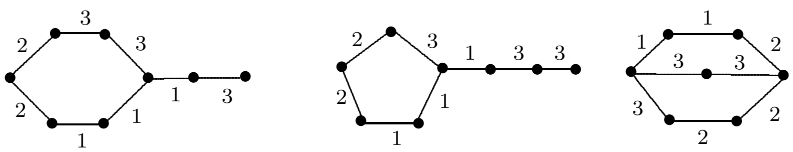


Figure 10: Possible subgraphs of girth 6 or 5

If the girth g of G is 4, then G contains a subgraph F that is isomorphic to one of the graphs shown in Figure 11, where a minimum detectable 2-coloring is given for each such graph. If F is isomorphic to one of the unicyclic graphs in Figure 11, then $\det(F) = n(F)-5$; while if F is isomorphic to one of the graphs with cycle rank 2 in Figure 11, then $\det(F) = n(F)-4$. In either case, $\det(G) \leq n-4$ by Lemma 2.1.

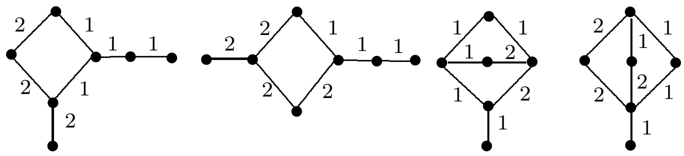


Figure 11: Possible subgraphs of girth 4

If the girth g of G is 3, then G contains a subgraph F that is isomorphic to one of the graphs shown in Figure 12, where a minimum detectable coloring is given for each such graph. If F is isomorphic to the unicyclic graph in Figure 12, then $\det(F) = n(F) - 5$; while if F is isomorphic to one of the graphs with cycle rank 2 in Figure 12, then $\det(F) \leq n(F) - 4$. In either case, $\det(G) \leq n - 4$ by Lemma 2.1. □

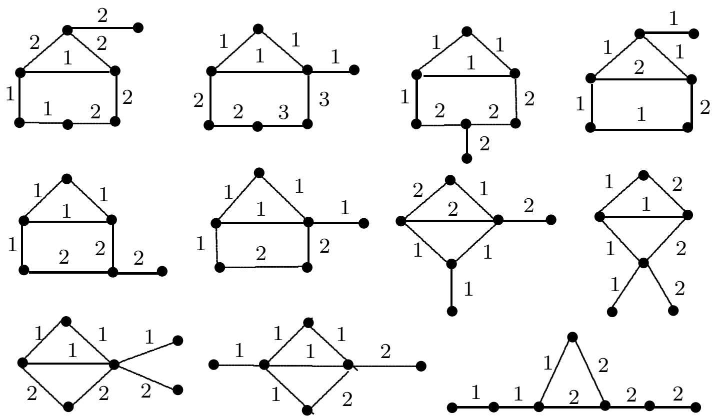


Figure 12: Possible subgraphs of girth 3

We have seen that $D_2(4) = 2$ and $D_2(5) = D_2(6) = 3$. Furthermore, there is a graph $G \in \mathcal{G}_{2,n}$ ($n \geq 7$) having detection number $n-4$. These observations together with Theorem 2.8 yield the following.

Corollary 2.9. *Let $n \geq 4$. Then*

$$D_2(n) = \begin{cases} 2 & \text{if } n = 4 \\ 3 & \text{if } n = 5, 6 \\ n-4 & \text{if } n \geq 7. \end{cases}$$

3. The Minimum Detection Number

In this section we determine the minimum detection number among all connected graphs of order n having cycle rank 2. In order to do this, we first present a useful result that is a consequence Proposition 1.1.

Corollary 3.1. *Let G be a nontrivial connected graph having maximum degree Δ and cycle rank 2 . If n_i is the number of vertices of degree i in G , where $1 \leq i \leq \Delta$, then*

$$n_1 + 2 = n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta.$$

Furthermore, we make an observation. According to Theorem A, if $G \in \mathcal{G}_{2,n}$ and $\det(G) = k$, then G has at most k end-vertices and at most $\frac{k^2+k}{2}$ vertices of degree 2. It then follows by Corollary 3.1 that G has at most $k+2$ vertices of degree 3 or more. Consequently,

$$n \le k + \tfrac{k^2+k}{2} + (k+2) = \tfrac{k^2+5k+4}{2}.$$

We are now prepared to present the following.

Theorem 3.2. *Let $n \ge 6$ be an integer. If $k \ge 2$ is the unique integer such that*

$$\frac{(k-1)^2+5(k-1)+4}{2} + 1 \le n \le \frac{k^2+5k+4}{2}$$

then $d_2(n) = k$.

Proof. First, we show that if $n \ge \frac{(k-1)^2+5(k-1)+4}{2} + 1 = \frac{k^2+3k+2}{2}$, then $d_2(n) \ge k$. Assume, to the contrary, that there exists a connected graph G of order $n \ge \frac{k^2+3k+2}{2}$ having cycle rank 2 such that $\det(G) \le k-1$. By Theorem A, G has at most $k-1$ end-vertices and at most $\frac{k(k-1)}{2}$ vertices of degree 2. Since G has at most $k-1$ end-vertices, it follows from Corollary 3.1 that G has at most $k+1$ vertices of degree 3 or more. Thus

$$n \le (k-1) + \tfrac{k^2-k}{2} + (k+1) = \tfrac{k^2+3k}{2},$$

which is a contradiction. Thus, as claimed, $d_2(n) \ge k$.

Next, we show for each integer n with $\frac{k^2+3k+2}{2} \le n \le \frac{k^2+5k+4}{2}$ that $d_2(n) \le k$ by constructing a graph $G \in \mathcal{G}_{2,n}$ of order n having detection number k. For $6 \le n \le 9$, it follows that $k = 2$. Figure 13 shows four graphs of order n for $n = 6, 7, 8, 9$, respectively, having cycle rank 2 and detection number 2. Thus the theorem holds for $6 \le n \le 9$.

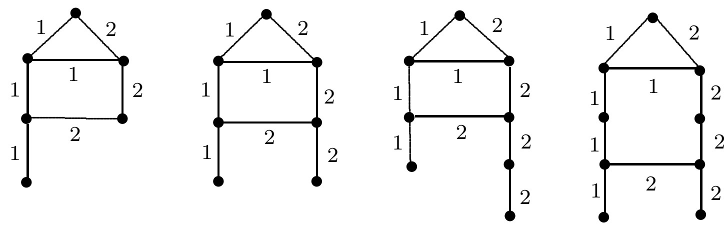


Figure 13: Graphs of order n with detection number 2 for $6 \le n \le 9$

We may now assume that $n \ge 10$. Hence there exists a unique integer $k \ge 3$ such that

$$\tfrac{(k-1)^2+5(k-1)+4}{2} + 1 \le n \le \tfrac{k^2+5k+4}{2}.$$

Therefore, $\frac{(k-1)^2+5(k-1)}{2} + 1 \le n-2 \le \frac{k^2+5k}{2}$. It then follows by the proof of Theorem E that there is a unicyclic graph of order $n-2$ with detection number k. With the aid of the proof of Theorem E, we construct a graph of order n, cycle rank 2, and detection number k in the following three steps:

(1) We construct a unicyclic graph H_k of order $\frac{k^2+5k}{2}$ having detection number k.

(2) From H_k we construct a unicyclic graph H of order $n-2$ having detection number k.

(3) From H we construct a graph G of order n having cycle rank 2 and detection number k.

We now consider two cases, according to whether k is odd or k is even.

Case 1. k is odd. Then $k = 2\ell - 1$ for some integer $\ell \geq 2$.

We now construct a unicyclic graph H_k of order $\frac{k^2+5k}{2}$ having detection number k. Let $C_{2\ell^2-\ell} : v_1, v_2, \ldots, v_{2\ell^2-\ell}, v_1$ be a cycle of length $2\ell^2 - \ell$ and, for $1 \leq i \leq k$, let Q_i be a copy of K_2 with $V(Q_i) = \{u_{i,1}, u_{i,2}\}$. Then the graph H_k is obtained from $C_{2\ell^2-\ell}$ and the graphs Q_i ($1 \leq i \leq k$) by adding the edges $v_{2i}u_{i,1}$ ($1 \leq i \leq k$). Observe that H_k is a unicyclic graph of order $(2\ell^2 - \ell) + 2(2\ell - 1) = \frac{k^2+5k}{2}$. By the proof of Theorem E, H_k has detection number k. We now construct a unicyclic graph H of order $n-2$ having detection number k. Let $n - 2 = \frac{k^2+5k}{2} - p$, where $0 \leq p \leq k+1$. If $p = 0$, then $H = H_k$ has the desired properties. We next consider two subcases, according to whether $1 \leq p \leq k$ or $p = k+1$.

Subcase 1.1. $1 \leq p \leq k$. Let H be the unicyclic graph of order $\frac{k^2+5k}{2} - p$ obtained from H_k by suppressing the vertices $u_{i,1}$ so that the edges $v_{2i}u_{i,1}$ and $u_{i,1}u_{i,2}$ become the single edge $v_{2i}u_{i,2}$ where $1 \leq i \leq p$.

Subcase 1.2. $p = k+1$. Thus $n - 2 = \frac{k^2+3k-2}{2}$. In this case, consider the unicyclic graph of order $\frac{k^2+3k-2}{2} + 1 = \frac{k^2+3k}{2}$ described in Subcase 1.1 (that is, when $p = k$ in Subcase 1.1). We delete the edge $v_{2k}v_{2k-1}$, identify the vertices v_{2k} and v_{2k-1}, and label this new vertex by v. This gives us a unicyclic graph H of order $\frac{k^2+3k-2}{2}$.

Therefore, we have constructed a graph H of order $n-2$ and size $n-2$ having exactly k end-vertices. Furthermore, it was shown in the proof of Theorem E that $\det(H) = k$.

Next, let G be the graph obtained from H by (1) adding the vertices $u_{1,3}$ and $u_{2,3}$ and (2) joining $u_{1,3}$ to $u_{1,2}$, joining $u_{2,3}$ to $u_{2,2}$ and joining $u_{1,2}$ to $u_{2,2}$. Then G is a graph of order n and size $n+1$, that is, $G \in \mathcal{G}_{2,n}$. By the proof of Theorem E, there exists a detectable k-coloring c of H such that $c(v_2u_{1,2}) = 1$, $c(v_4u_{2,2}) = 2$, and $(200\ldots01)$ and $(0200\ldots01)$ are not the color codes of any vertices of degree 3 in H. Thus, we define a k-coloring $c^* : E(G) \to \{1, 2, \ldots, k\}$ of G by

$$c^*(e) = \begin{cases} 1 & \text{if } e = u_{1,2}u_{1,3} \\ 2 & \text{if } e = u_{2,2}u_{2,3} \\ k & \text{if } e = u_{1,2}u_{2,2} \\ c(e) & \text{otherwise.} \end{cases}$$

Observe that the color codes of the vertices of H are those of G except that

(i) the color code of $u_{1,2}$ in H is now the color code of $u_{1,3}$ in G,

(ii) the color code of $u_{2,2}$ in H is now the color code of $u_{2,3}$ in G, and

(iii) the color codes of $u_{1,2}$ and $u_{2,2}$ in G are $(200\ldots01)$ and $(0200\ldots01)$, respectively.

Hence c^* is a detectable k-coloring for G and so $\det(G) \leq k$. The graph G is shown in Figure 14 for $n = 21$ and $k = 5$. In this case, the graph H_5 has order 25, the graph H has order $n-2=19$, and H is constructed from H_5 in Subcase 1.2 for $p=k+1=6$. The graph G is then obtained from H by adding the two vertices $u_{1,3}$ and $u_{2,3}$ and the three edges $u_{1,2}u_{1,3}$, $u_{2,2}u_{2,3}$, and $u_{1,3}u_{2,3}$.

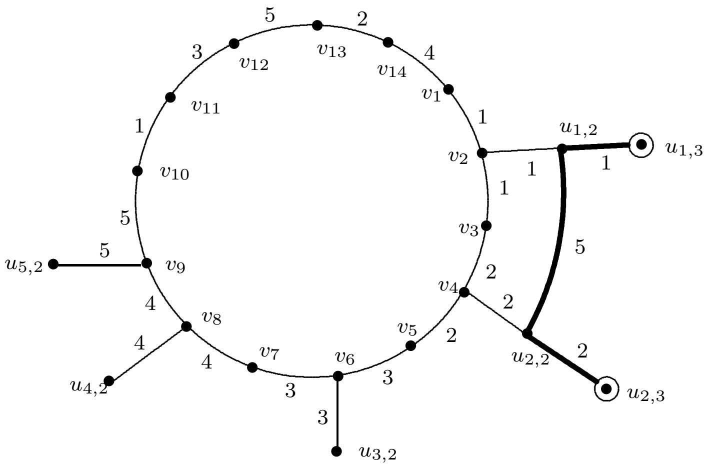


Figure 14: A graph $G \in \mathcal{W}_{21}$ with a detectable 5-coloring in Case 1

Case 2. k is even. Then $k=2\ell$ for some positive integer $\ell \geq 2$.

As with Case 1, we first construct a graph H_k of order $\frac{k^2+5k}{2}$ having detection number k. Let $C_{2\ell^2}: v_1, v_2, \ldots, v_{2\ell^2}, v_1$ be a cycle of length of $2\ell^2$. For $1 \leq i \leq \ell$, let Q_i be a copy of K_2 with $V(Q_i)=\{u_{i,1}, u_{i,2}\}$ and for $\ell+1 \leq i \leq 2\ell$, let $Q_i: u_{i,1}, u_{i,2}, u_{i,3}$ be a copy of a path of length 2. Then the graph H_k is obtained from $C_{2\ell^2}$ and the graphs Q_i ($1 \leq i \leq k$) by adding the edges $v_{2i}u_{i,1}$ ($1 \leq i \leq k$). The graph H_k is a unicyclic graph of order $2\ell^2+2\ell+3\ell=\frac{k^2+5k}{2}$ and detection number k. We next construct a unicyclic graph H of order $n-2$ having detection number k. Let $n-2=\frac{k^2+5k}{2}-p$, where $0 \leq p \leq k+1$. If $p=0$, then $H=H_k$ has the desired properties. We now consider three subcases, according to whether $1 \leq p \leq \ell$, $\ell+1 \leq p \leq k$, or $p=k+1$.

Subcase 2.1. $1 \leq p \leq \ell$. Let H be the unicyclic graph of order $\frac{k^2+5k}{2}-p$ obtained from H_k by suppressing the vertices $u_{i,1}$ for $1 \leq i \leq p$ so that the edges $v_{2i}u_{i,1}$ and $u_{i,1}u_{i,2}$ become the single edge $v_{2i}u_{i,2}$.

Subcase 2.2. $\ell+1 \leq p \leq k$. Let H be the unicyclic graph of order $\frac{k^2+5k}{2}-p$ obtained from the unicyclic graph of order $\frac{k^2+5k}{2}-\frac{k}{2}=\frac{k^2+4k}{2}$ described in Subcase 2.1 (that is, when $p=\ell$ in Subcase 2.1) by suppressing the vertices $u_{i,2}$ so that the edges $u_{i,1}u_{i,2}$ and $u_{i,2}u_{i,3}$ become the single edge $u_{i,1}u_{i,3}$ where $\ell+1 \leq i \leq p$.

Subcase 2.3. $p=k+1$. Hence $n-2=\frac{k^2+3k-2}{2}$. Consider the unicyclic graph of order $\frac{k^2+3k-2}{2}+1=\frac{k^2+3k}{2}$ described in Subcase 2.2 (that is, when $p=k$ in Subcase 2.2). We delete the edge $v_{2k}v_{2k-1}$, identify the vertices v_{2k} and v_{2k-1}, and label this new vertex by v. This gives us a unicyclic graph H of order $\frac{k^2+3k-2}{2}$.

As in Case 1, the desired graph $G \in \mathcal{G}_{2,n}$ is then obtained from H by (1) adding the vertices $u_{1,3}$ and $u_{2,3}$ and (2) joining $u_{1,3}$ to $u_{1,2}$, joining $u_{2,3}$ to $u_{2,2}$ and joining $u_{1,2}$ to $u_{2,2}$. An argument similar to the one in Case 1 shows that $\det(G) \le k$. Figure 15 shows such a graph for $n = 33$ and $k = 6$. In this case, the graph H_6 has order 33, the graph H has order $n - 2 = 31$, and H is constructed from H_6 in Subcase 2.1 for $p = 2$. The graph G then is obtained from H by adding the two vertices $u_{1,3}$ and $u_{2,3}$ and the three edges $u_{1,2}u_{1,3}$, $u_{2,2}u_{2,3}$, and $u_{1,3}u_{2,3}$. □

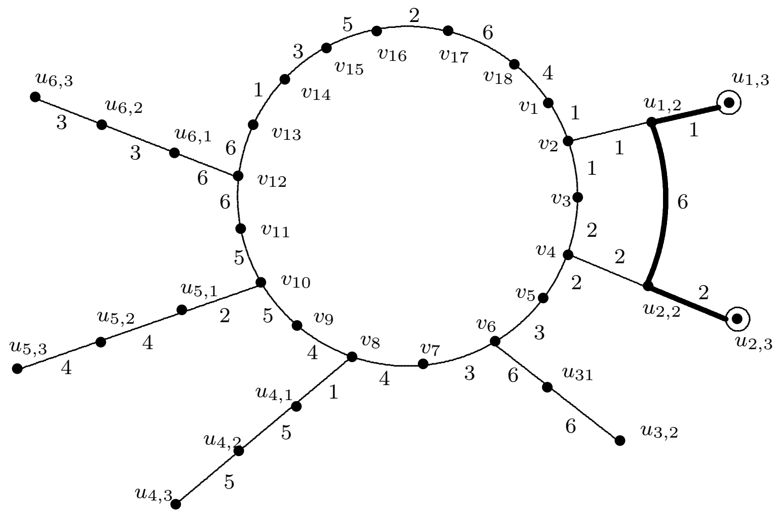


Figure 15: A graph $G \in \mathcal{W}_{33}$ with a detectable 6-coloring in Case 2

We have seen that $d_2(4) = d_2(5) = 2$. These observations together with Theorem 3.2 yield the following.

Corollary 3.3. *For each integer $n \ge 4$,*

$$d_2(n) = \begin{cases} 2 & \text{if } n = 4, 5 \\ \left\lceil \frac{-5+\sqrt{8n+9}}{2} \right\rceil & \text{otherwise.} \end{cases}$$

By Corollary 3.3, $d_2(n) \approx \sqrt{2n}$ for large values of n. We state the following result, which describes all pairs k, n of integers for which there exists a connected graph in $\mathcal{G}_{2,n}$ having detection number k. Since the proof is long and heavily case-oriented, we omit the proof.

Theorem 3.4. *Let $k \ge 2$ and $n \ge 4$ be integers. There exists a connected graph G of order n having cycle rank 2 and $\det(G) = k$ if and only if $d_2(n) \le k \le D_2(n)$.*

Since $d_2(14) = 3$ and $D_2(14) = 10$, there is, according to Theorem 3.4, a connected graph G_k of order 14 having cycle rank 2 and detection number k for each integer k with $3 \le k \le 10$. This is illustrated in Figure 16.

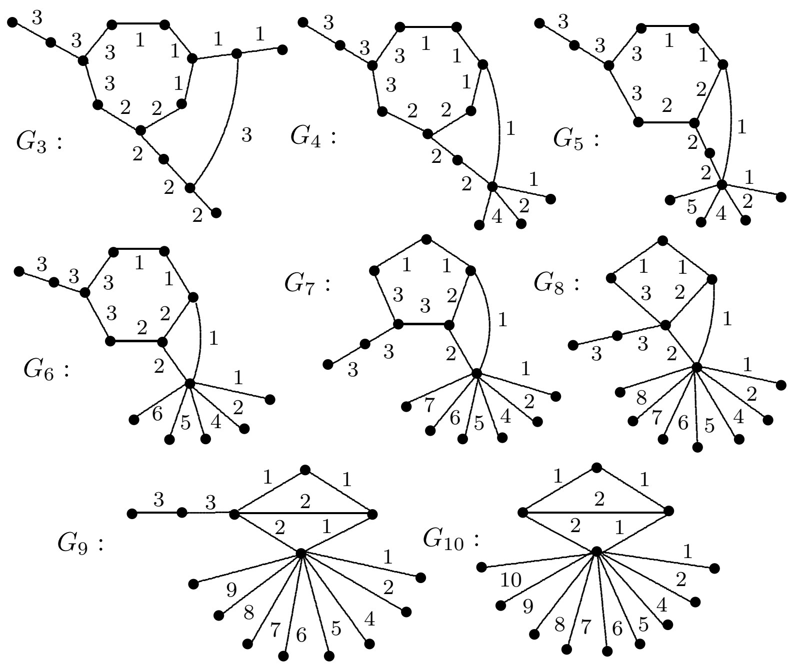


Figure 16: Graphs G_k $(3 \leq k \leq 10)$ in $\mathcal{W}_{14}$ with detection number k

4. Some Concluding Remarks

Recall for integers ψ and n, with $\psi \geq 0$ and $n \geq \left\lceil \frac{3+\sqrt{1+8\psi}}{2} \right\rceil$, that $D_\psi(n)$ denotes the maximum detection number among all connected graphs of order n having cycle rank ψ and $d_\psi(n)$ denotes the minimum detection number among all connected graphs of order n having cycle rank ψ. By Theorems D and E and Corollaries 2.9 and 3.3, we have the following:

$$\text{for } \psi = 0, \quad D_0(n) = n-1 \text{ for } n \geq 3 \text{ and } d_0(n) = \left\lceil \frac{-5+\sqrt{8n+41}}{2} \right\rceil \text{ for } n \geq 3;$$

$$\text{for } \psi = 1, \quad D_1(n) = n-3 \text{ for } n \geq 6 \text{ and } d_1(n) = \left\lceil \frac{-5+\sqrt{8n+25}}{2} \right\rceil \text{ for } n \geq 4;$$

$$\text{for } \psi = 2, \quad D_2(n) = n-4 \text{ for } n \geq 7 \text{ and } d_2(n) = \left\lceil \frac{-5+\sqrt{8n+9}}{2} \right\rceil \text{ for } n \geq 6.$$

For integers ψ, t, n with $t \geq 3$, $n \geq t+3$, and $\binom{t-2}{2} + 1 \leq \psi \leq \binom{t-1}{2}$, it is possible to construct a connected graph G of order n having cycle rank ψ such that $\det(G) \geq n-t$. For example, let T be a tree of order n with exactly $n-t$ end-vertices and let $V(T) = U_1 \cup U_2$, where U_1 is the set of end-vertices of T and $U_2 = V(T) - U_1$. Thus $|U_2| = t$, the subgraph $\langle U_2 \rangle$ induced by U_2 is connected, and $|E(\langle U_2 \rangle)| = t-1$. Since

$$\psi \leq \binom{t-1}{2} = \binom{t}{2} - (t-1),$$

we can constructs a connected graph $G \in \mathcal{G}_{\psi,n}$ from T by adding ψ edges to the vertices of U_2 in T. Since G contains exactly $n-t$ end-vertices, $\det(G) \geq n-t$. These observations yields the following.

Proposition 4.1. *For integers* $\psi \geq 1$, $t \geq 3$, *and* $n \geq t+3$,

$$D_\psi(n) \geq n - t \qquad \textit{if } \binom{t-2}{2} + 1 \leq \psi \leq \binom{t-1}{2}.$$

Let G be a nontrivial connected graph having maximum degree Δ and cycle rank ψ. Recall that if n_i is the number of vertices of degree i in G where $1 \leq i \leq \Delta$, then

$$n_1 = (2-2\psi) + n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta$$

Furthermore, if $\det(G) = k$, then G has at most k end-vertices and at most $\frac{k^2+k}{2}$ vertices of degree 2. Since

$$\begin{aligned} n_1 &= (2-2\psi) + n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta \\ &\geq (2-2\psi) + n_3 + n_4 + n_5 + \cdots + n_\Delta, \end{aligned}$$

it follows that

$$n_3 + n_4 + n_5 + \cdots + n_\Delta \leq n_1 - (2 - 2\psi) \leq k + 2\psi - 2$$

and so

$$n \leq k + \tfrac{k^2+k}{2} + (k + 2\psi - 2) = \tfrac{k^2+5k+4\psi-4}{2}$$

Hence the largest possible order of a connected graph having cycle rank ψ and detection number k is $\frac{k^2+5k+4\psi-4}{2}$. That is, if G is a connected graph of order n with

$$\tfrac{(k-1)^2+5(k-1)+4\psi-4}{2} + 1 \leq n \leq \tfrac{k^2+5k+4\psi-4}{2}$$

and having cycle rank ψ, then $\det(G) \geq k$, implying that $d_\psi(n) \geq k$. As a consequence of these observations, we have the following.

Proposition 4.2. *For integers* $\psi \geq 1$ *and* $n \geq 2 + 2\psi$,

$$d_\psi(n) \geq \left\lceil \frac{-5 + \sqrt{8n + (41 - 16\psi)}}{2} \right\rceil.$$

Note that for Proposition 4.2 to be nontrivial, we assume that

$$\left\lceil \tfrac{-5+\sqrt{8n+(41-16\psi)}}{2} \right\rceil \geq 2$$

and so $n \geq 2 + 2\psi$ as stated in the result.

We conclude this section with the following conjectures.

Conjecture 4.3. *For integers* $\psi \geq 1$, $t \geq 3$, *and* $n \geq t+3$,

$$D_\psi(n) = n - t \qquad \textit{if } \binom{t-2}{2} + 1 \leq \psi \leq \binom{t-1}{2}.$$

Conjecture 4.4. *For integers* $\psi \geq 1$ *and* $n \geq 2 + 2\psi$,

$$d_\psi(n) = \left\lceil \frac{-5 + \sqrt{8n + (41 - 16\psi)}}{2} \right\rceil.$$

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