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Eternally Secure Sets, Independence Sets and Cliques

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Abstract

Goddard, Hedetniemi, and Hedetniemi conjectured that if the independence number of a graph is equal to the eternal security number, then the independence number is equal to the chromatic number of the complement of the graphs (a.k.a, the clique covering number of the graph) [*JCMCC*, vol. 52, pp. 160-180]. We prove the conjecture is true when the independence number is two and provide counterexamples when the independence number is greater than two.

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1. Introduction

Let $G = (V, E)$ be a simple graph with independence number $\beta(G)$. The *open neighborhood* of vertex v is denoted by $N(v)$, and its *closed neighborhood* $N(v) \cup \{v\}$ is denoted by $N[v]$. A dominating set of G is a set $D \subseteq V$ such that, for all $v \in V$, $N[v] \cap D \neq \emptyset$.

Several recent papers study problems concerned with protecting the vertices in a graph from a series of one or more attacks, see for example [1, 2, 5]. In such a problem, guards are located at vertices. The guards can protect the vertices at which they are located and can move to a neighboring vertex to defend an attack there. Thus having a guard located at each vertex of a dominating set suffices to defend a graph against a single attack. Variations of this problem have been proposed including Roman Domination [3], Weak Roman Domination [4] and k-secure sets/eternal secure sets [1, 2, 5].

Let R denote a sequence of vertices, with first element $R(1)$ and i^{th} element $R(i)$. The elements of R are interpreted as the locations of a sequence of consecutive attacks at vertices, each of which must be defended by a guard. At most one guard is allowed to move to defend each attack.

Let D_0 be the set of initial locations of the guards and let D_i be the set of locations of the guards after $R(i)$ is defended (so $R(i) \in D_i$). We refer to D_i as a *configuration* of the guards.

If $R(i) \notin D_i$, then $D_i = (D_{i-1} \setminus \{v\}) \cup \{R(i)\}$, where $v \in D_{i-1}$ and $R(i) \in N(v)$. We say that the guard at v has *moved* to $R(i)$.

A set D is an *eternal secure set* if, for all possible sequences R, there exists a sequence $D_0, D_1, \ldots$ such that $D_i = D_{i-1} \setminus \{v\} \cup R(i)$ (possibly $v = R(i)$), $R(i) \in N[v]$, and each D_i is a dominating set. The size of a smallest eternal secure set in G is denoted $\gamma_\infty(G)$, or simply γ_∞ [2].

Let θ to denote the clique-covering number (i.e., the minimum number of cliques needed to cover the vertices of G). Of course, $\theta(G) = \chi(\overline{G})$. It is not hard to prove that $\beta(G) \leq \gamma_\infty(G) \leq \theta(G)$ for all graphs G [2, 5]. There exist graphs for which $\beta(G) < \gamma_\infty(G) < \theta(G)$ [5]. Goddard et al. proved that if $\beta = 2$ then $\gamma_\infty \leq 3$ [5]. Klostermeyer and MacGillivray proved that $\gamma_\infty \leq \binom{\beta+1}{2}$, for all graphs with independence number $\beta \geq 1$. It is not known if this bound is best possible [6]. Burger et al. proved that if $\theta(G) \leq 3$, then $\theta(G) = \gamma_\infty(G)$ [2].

Goddard et al. conjecture that if $\beta(G) = \gamma_\infty(G)$ then $\beta(G) = \theta(G)$ [5]. This conjecture is trivially true for perfect graphs. We prove the conjecture is true when the independence number is two and provide counterexamples when the independence number is greater than two.

2. Results

We implicitly use the following fact in our arguments.

Fact 1. *Let H be an induced subgraph of G. Then $\gamma_\infty(G) \geq \gamma_\infty(H)$.*

Proof. Suppose to the contrary that $\gamma_\infty(G) < \gamma_\infty(H)$. Consider a sufficiently long sequence of attacks on graph G containing only vertices in H. Assume without loss of generality that after some number of requests, all $\gamma_\infty(G)$ guards are located at vertices in H (else the argument proceeds in a similar fashion, since any other guards can be ignored). By definition, the remaining attacks can be served by these guards. But this contradicts the fact that $\gamma_\infty(G) < \gamma_\infty(H)$. □

Fact 2. *If $\beta(G) \leq 2$ and $\beta(G) = \gamma_\infty(G)$, then $\beta(G) = \theta(G)$.*

Proof. The theorem is obviously true when β=1, since $\theta = 1$ in such cases. Assume $\beta = 2$. Suppose $\theta > 2$. Then G is not perfect, so by the Strong Perfect Graph Theorem it contains an odd hole (i.e., an induced chordless cycle) or odd anti-hole (i.e., the complement of a chordless cycle of odd length at least five). If the odd hole or odd anti-hole is C_5, then it has $\gamma_\infty > 2$. If the odd hole is C_{2k+1} for $k > 2$, then $\beta > 2$. And if the odd anti-hole is $\overline{C_{2k+1}}$ then by Burger et al. we know that $\gamma_\infty > 2$. [2] □

Corollary 3. *If $\beta(G) \leq 2$, then $\beta(G) = \gamma_\infty(G)$ if and only if G is perfect.*

We are now ready to state the main result of the paper.

Theorem 4. *Let G be graph with $\beta(G) = c, \theta(G) = k \geq c$, and $\gamma_\infty(G) = g$. Add independent vertices $v_1, v_2, \ldots v_p, p \leq g$, and connect each v_i to each vertex of G. Call the resulting graph G'. Then $\gamma_\infty(G') = g$ (and clearly $\beta(G') = MAX(c,p)$ and $\theta(G') = k$).*

Proof. Let R be a sequence of vertices (attacks) of G'. If $v_i \notin R$ for all $i, 1 \leq i \leq p$, then the theorem is trivial. Suppose otherwise. In G', initially locate g guards at the vertices of an eternal secure set of G. We shall run a "simulation" in which the same sequence of attacks occurs on graph G, with each vertex in R that is a v_i-type vertex replaced by a "null" request requiring no action in the simulation. We maintain the invariant that each guard g_i in G' is either on v_i or on the vertex of G corresponding to g_i's location in the simulation on G. If an attack at vertex v in G is served by guard g_i, then the same attack can be served in G' by g_i in one of two ways: if g_i is located at the same vertex in G' as in G, then it services the attack as in G. Else g_i is at v_i, in which case g_i can move from v_i to v, since v_i is adjacent to each $v \in V(G)$. If an attack occurs at v_i (and no guard is at v_i), then we move g_i to v_i. □

Observe that Theorem 4 can be easily generalized to the case when $p > \gamma_\infty(G)$, in which case $\gamma_\infty(G') = p$.

Corollary 5. *Let G be graph with $\beta(G) = c, \theta(G) = k \geq c$, and $\gamma_\infty(G) = g$. Add independent vertices $v_1, v_2, \ldots v_p$, and connect each v_i to each vertex of G. Call the resulting graph G'. Then $\gamma_\infty(G') = MAX(p, g)$ (and clearly $\beta(G') = MAX(c, p)$ and $\theta(G') = k$).*

To see that this provides a counterexample to conjecture of Goddard, Hedetniemi, and Hedetniemi, let G be a graph with $\beta(G) = 2$ and $\theta(G) = k > 3$ (such graphs exist because there are triangle-free graphs with chromatic number k for any $k \geq 1$). Then $\gamma_\infty = 3$, by Fact 2. Add three independent vertices v_1, v_2, v_3 and connect v_1, v_2, v_3 to each vertex of G. Call this G'. Hence $\gamma_\infty(G') = 3$ (and clearly $\beta(G') = 3$ and $\theta(G') = k$. We formalize this example as follows.

Corollary 6. *For any $c > 2$ there exists a graph G with independence number c, $\gamma_\infty(G) = c$, and $\theta = k$, for any $k > c$.*

Proof. It is well-known that there exists a triangle-free graph, G, with chromatic number k, for any $k \geq 1$. Then $\beta(\overline{G}) = 2$, $\theta(\overline{G}) = k$, and, from Fact 2, $\gamma_\infty(\overline{G}) = 3$. Letting $p = c$ in Corollary 6, we obtain the desired result. □

Note that this implies $\frac{\theta}{\gamma_\infty}$ cannot, in general, be bounded by a constant.

3. Conclusions

Though we have resolved the conjecture stated in [5], we are far from a complete understanding of the relationship between β, θ, and γ_∞. We state some problems for future study.

Is the Goddard,Hedetniemi, and Hedetniemi conjecture true for any special classes of graphs, such as planar or circular-arc graphs?

As mentioned above, Goddard, Hedetniemi, and Hedetniemi ask whether $\frac{\gamma_\infty}{\beta}$ can be bounded by a constant (in particular, they asked about the case when $\beta = 3$). Klostermeyer and MacGillivray proved that $\gamma_\infty \leq \binom{\beta+1}{2}$, but of course this does not provide a constant bound on $\frac{\gamma_\infty}{\beta}$, in general.

As a broader direction, we would like to characterize when the equality $\beta = \gamma_\infty$ implies $\beta = \theta$. It is easy to find graphs that are not perfect for which this is the case: consider C_5 and add a vertex which is adjacent to two adjacent vertices from the cycle. The resulting graph has $\beta = 3$ and $\theta = 3$, but is not perfect.

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