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Hamiltonicity of the cartesian product of two directed cycles minus a subgroup

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Abstract

Let $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$ be the product of two directed cycles minus a subgroup. Also, let $A = \frac{a}{c}$ and $B = \frac{b}{d}$. We show that this digraph is Hamiltonian if and only if there exist positive integers m and n such that $Am + Bn = AB - 1$ and $\gcd(dm, cn) = 1$.

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1. Introduction

Improving on the work of [3] Curran [2] used the theory of torus knots to prove that the Cartesian product $\mathbb{Z}_a \times \mathbb{Z}_b$ of two directed cycles is Hamiltonian if and only if there exists a pair of relatively prime positive integers m and n such that $am + bn = ab$. Penn and Witte [5] used the same method to prove a similar result for digraphs of the form $\mathbb{Z}_a \times \mathbb{Z}_b - \{(0,0)\}$, the digraph $\mathbb{Z}_a \times \mathbb{Z}_b$ with one vertex removed. This digraph $\mathbb{Z}_a \times \mathbb{Z}_b - \{(0,0)\}$ is Hamiltonian if and only if there exists a pair of relatively prime positive integers m and n such that $am + bn = ab - 1$.

Note that removing one vertex from $\mathbb{Z}_a \times \mathbb{Z}_b$ as in [5] is equivalent to removing the subgroup $\mathbb{Z}_1 \times \mathbb{Z}_1$. In this paper, we consider a natural generalization by removing a subgroup of the form $\mathbb{Z}_c \times \mathbb{Z}_d$ and find necessary and sufficient conditions for the digraph $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$ to be Hamiltonian. Here, $\mathbb{Z}_c$ is a subgroup of $\mathbb{Z}_a$ and $\mathbb{Z}_d$ is a subgroup of $\mathbb{Z}_b$. See Figure 1 for a picture of the digraph $(\mathbb{Z}_8 \times \mathbb{Z}_6) - (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

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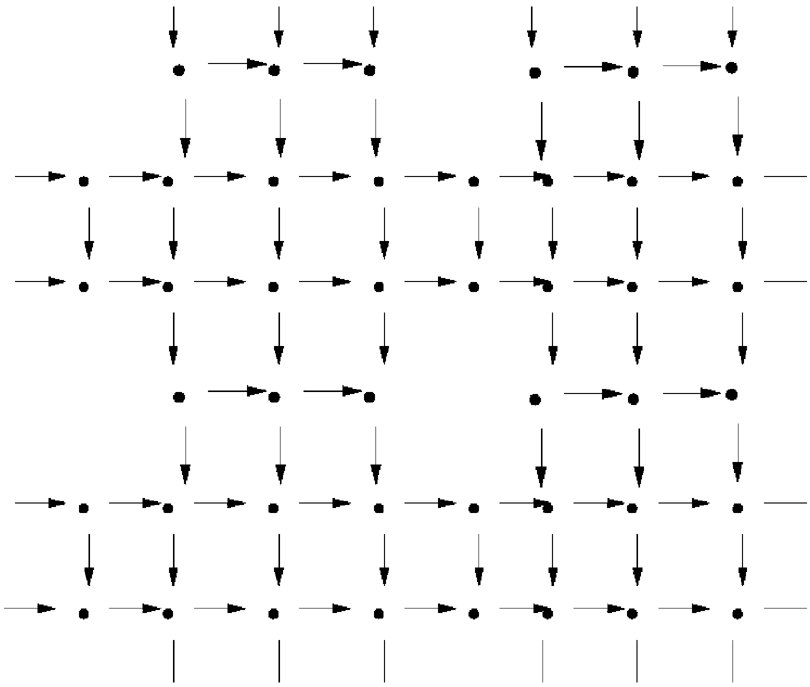

Figure 1. The digraph $(\mathbb{Z}_8 \times \mathbb{Z}_6) - (Z_2 \times Z_2)$.

We recall some definitions and results that will be useful. We refer the reader to [1] for the basic language of digraphs. We are considering digraphs of the form $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$, where $\mathbb{Z}_c$ and $\mathbb{Z}_d$ are subgroups of $\mathbb{Z}_a$ and $\mathbb{Z}_b$, respectively. Thus, we start with $\mathbb{Z}_a \times \mathbb{Z}_b$, remove vertices that belong to $\mathbb{Z}_c \times \mathbb{Z}_d$, and consider the induced subgraph on the remaining vertices.

Definition 1.1. *Let H be a Hamiltonian circuit on $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$. Then a vertex (x, y)* ***travels by*** *$(1, 0)$ if the out-edge of (x, y) is an in-edge of $(x + 1, y)$ in H. Similarly, a vertex (x, y)* ***travels by*** *$(0, 1)$ if the out-edge of (x, y) is an in-edge of $(x, y + 1)$ in H.*

Note that in a Hamiltonian circuit, each vertex travels by either $(1, 0)$ or by $(0, 1)$.

Definition 1.2. *Let $g = \gcd(a, b)$. For any integer p, let $\langle p\rangle$ be the subset of $\mathbb{Z}_a \times \mathbb{Z}_b$ consisting of pairs (x, y) such that $x + y = p$ modulo g.*

The subset $\langle 0\rangle$ is the subgroup of $\mathbb{Z}_a \times \mathbb{Z}_b$ generated by $(1, -1)$, and $\langle p\rangle$ is the coset $(p, 0) + \langle 0\rangle$. The subgroup $\langle 0\rangle$ has index g, so there are exactly g distinct cosets.

The following lemma shows why such cosets are useful.

Lemma 1.3. *Let H be a spanning circuit of $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$. Suppose that (x, y), $(x+1, y)$, and $(x + 1, y - 1)$ all belong to $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$. Then (x, y) travels by $(1, 0)$ if and only if $(x + 1, y - 1)$ travels by $(1, 0)$. Also, (x, y) travels by $(0, 1)$ if and only if $(x + 1, y - 1)$ travels by $(0, 1)$.*

Curran and Witte [2] give a proof this lemma. We recall the following result from [5].

Theorem 1.4. *The digraph* $\mathbb{Z}_a \times \mathbb{Z}_b - \{(0,0)\}$ *is Hamiltonian if and only if there exist relatively prime positive integers* m *and* n *such that* $am + bn = ab - 1$.

In the previous theorem, the numbers m and n have useful geometric interpretations. If we embed $\mathbb{Z}_a \times \mathbb{Z}_b$ in the torus in the obvious way, then a Hamiltonian circuit is an embedded loop, and its knot class is equal to (m,n) . See [6] for more details on knot classes.

2. The Main Theorem

We now come to our main result.

Theorem 2.1. *Let* $A = \frac{a}{c}$ *and* $B = \frac{b}{d}$. *Then* $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$ *is Hamiltonian if and only if there exist positive integers* m *and* n *such that*

1. $Am + Bn = AB - 1$, *and*
2. $\gcd(dm, cn) = 1$.

In order to prove this theorem, we need the following facts about Hamiltonian digraphs.

Lemma 2.2. *If* $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$ *is Hamiltonian, then* $\gcd(A,B) = 1$.

Proof. Suppose for contradiction that the coset $\langle -1 \rangle$ contains no vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$. The vertex $(-1,0)$ must travel by $(0,1)$, so a repeated application of Lemma 2.3 implies that every vertex in $\langle -1 \rangle$ must also travel by $(0,1)$. This is impossible because the vertex $(0,-1)$ certainly travels by $(1,0)$. By contradiction, $\langle -1 \rangle$ contains a vertex $(\alpha a, \beta b)$ for some integers α and β .

From the definition of a coset we find that $\alpha A + \beta B = -1 \mod g$. There exists an integer γ such that $\alpha A + \beta B = -1 + g\gamma$. Now $\gcd(A,B)$ divides g , so $\gcd(A,B)$ divides $\alpha A + \beta B - g\gamma = -1$. □

Corollary 2.3. *If* $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$ *has a Hamiltonian circuit, then it is unique.*

Proof. We will show that every coset contains a vertex $(\alpha A, \beta B)$ for some α and β in $\mathbb{Z}$. The repeated application of Lemma 2.3 then gives the result.

From the proof of Lemma 3.2, we know that $\alpha A + \beta B = -1 \mod g$, for some integers α and β . Multiplying this equation by $-p$ we get $(-\alpha p)A + (-\beta p)B = \ p \mod g$. From the definition of a coset, we know $((-\alpha p)A, (-\beta p)B)$ is in $\langle p \rangle$. □

The following corollary tells us that any Hamiltonian circuit of $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$ must be periodic.

Corollary 2.4. *Suppose* $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$ *has a Hamiltonian circuit. Let* α *and* β *be integers. Then* (x,y) *and* $(x + \alpha A, y + \beta B)$ *travel in the same direction.*

Proof. Let H be a Hamiltonian circuit of $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$. Then $\phi(x,y) = (x + \alpha A, y + \beta B)$ is an automorphism of $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$. So $\phi(H)$ is a Hamiltonian circuit of $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$ as well. But the Hamiltonian circuit is unique (by Corollary 3.3), so $\phi(H) = H$. Thus, any vertex v travels in the same direction as the vertex $\phi(v)$. □

Now we are ready to prove Theorem 3.1.

Proof. First suppose that $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$ is Hamiltonian. Let H be the Hamiltonian circuit, and let (m_0, n_0) be its knot class. We know that m_0 and n_0 are relatively prime positive integers [6].

We also know that, in H, there are exactly am_0 vertices that travel by $(1,0)$ and exactly bn_0 vertices that travel by $(0,1)$, and that H contains each vertex in $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$. By counting vertices, $am_0 + bn_0 = ab - cd$. We divide this equation by cd and get $A\frac{m_0}{d} + B\frac{n_0}{c} = AB - 1$.

Consider the sets of vertices $U = \{(0,y) \mid y \in \mathbb{Z}_b\}$ and $V = \{(x,0) \mid x \in \mathbb{Z}_a\}$. By Corollary 3.4, the number of vertices in U traveling by $(1,0)$ is a multiple of d. Since the knot class of H is (m_0, n_0), there are exactly m_0 vertices in U that travel by $(1,0)$. Therefore, d divides m_0. The same argument shows that c divides n_0.

Now let $m = \frac{m_0}{d}$ and $n = \frac{n_0}{c}$. We know that m and n are integers. Since $m_0 = md$ and $n_0 = nc$, we know md and nc are relatively prime. This proves conditions (1) and (2).

Now suppose that conditions (1) and (2) hold. By Theorem 1.4, our assumptions imply that there is a Hamiltonian circuit H' on $\mathbb{Z}_A \times \mathbb{Z}_B - \mathbb{Z}_1 \times \mathbb{Z}_1$. Note that the knot class of H' is (m,n). We construct a spanning subgraph H of $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$ by requiring each vertex (x,y) in $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$ to travel in the same direction as the vertex $(\overline{x}, \overline{y})$ in $\mathbb{Z}_A \times \mathbb{Z}_B - \mathbb{Z}_1 \times \mathbb{Z}_1$. Here, $\overline{x}$ and $\overline{y}$ represent the residue classes of x and y, modulo A and modulo B, respectively. By construction, we know the knot class of H is (dm, cn). The subgraph H is connected because we assumed that $\gcd(dm, cn) = 1$ [6]. Therefore, H is a Hamiltonian circuit of $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$. ☐

Example 2.5. *We give two examples illustrating the theorem. First, consider* $(\mathbb{Z}_8 \times \mathbb{Z}_6) - (\mathbb{Z}_2 \times \mathbb{Z}_2)$ *as shown in Figure 1. This digraph is not Hamiltonian because condition (2) of Theorem 3.1 can never be satisfied.*

On the other hand, $(\mathbb{Z}_8 \times \mathbb{Z}_6) - (\mathbb{Z}_1 \times \mathbb{Z}_2)$ *is Hamiltonian; take* $m = 1$ *and* $n = 5$.

3. Questions

We end with a few open questions related to our theorem. The first question extends our problem to larger dimensions.

Question 3.1. *When does the digraph* $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \cdots \times \mathbb{Z}_{a_n} - \mathbb{Z}_{c_1} \times \mathbb{Z}_{c_2} \times \cdots \times \mathbb{Z}_{c_n}$ *have a Hamiltonian circuit?*

Not every subgroup of $\mathbb{Z}_a \times \mathbb{Z}_b$ is of the form $\mathbb{Z}_c \times \mathbb{Z}_d$. This leads to our next question.

Question 3.2. *Let* A *be any subgroup of* $\mathbb{Z}_a \times \mathbb{Z}_b$. *When does the digraph* $\mathbb{Z}_a \times \mathbb{Z}_b - A$ *have a Hamiltonian circuit?*

Instead of just removing a subgroup, it is also possible to remove more than one coset of a subgroup.

Question 3.3. *Choose a positive number* r. *When is it possible to remove* r *distinct cosets of* $\mathbb{Z}_c \times \mathbb{Z}_d$ *from* $\mathbb{Z}_a \times \mathbb{Z}_b$ *and obtain a Hamiltonian graph?*

The digraph $\mathbb{Z}_a \times \mathbb{Z}_b$ is said to be hyperhamiltonian if there is a spanning circuit which passes through one vertex exactly twice and all others exactly once. The digraph $\mathbb{Z}_a \times \mathbb{Z}_b$ is hyperhamiltonian if and only if there exist positive integers m and n such that $am + bn = ab + 1$ and $\gcd(m, n) = 1$ or 2 [4].

This idea can be generalized in the following way.

Question 3.4. *When is there a spanning circuit of $\mathbb{Z}_a \times \mathbb{Z}_b$ that passes through each vertex in the subgroup $Z_c \times \mathbb{Z}_d$ exactly twice and all others exactly once?*

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