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On Optimum Summable Graphs

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Abstract

For a graph G, let $\sigma(G)$ and $\delta(G)$ denote, respectively, its sum number and minimum degree. Trivially, $\sigma(G) \geq \delta(G)$. A nontrivial connected graph G is called a k*-optimum summable graph*, where $k \geq 1$, if $\sigma(G) = \delta(G) = k$. In this paper, we show that if G is a k-optimum summable graph of order n, $k \geq 3$, then (1) $n \geq 2k$; (2) the complete bipartite graph $K_{k,n-k}$ is not a spanning subgraph of G. We also describe new families of k-optimum summable graphs for $k \geq 1$.

Keywords: Sum graph, Sum number, Sum labeling, Minimum degree
Optimum summable graph, Unit graph.

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1. Introduction

All graphs considered here are finite simple graphs. For a graph G, $V(G)$ will denote its vertex set and $E(G)$ its edge set, while $n(G)$ and $e(G)$ respectively denote the order and size of G; that is, $n = n(G) = |V(G)|$ and $e(G) = |E(G)|$. A graph G is *nontrivial* if $n(G) \geq 2$. For other standard notation and terminology not explained here, refer to [1].

Let $\boldsymbol{N}$ denote the set of positive integers. Following Harary [2], the *sum graph* $G^+(S)$ of a finite subset $S \subset \boldsymbol{N}$ is the graph with vertex set S and edge set E such that for distinct $u, v \in S$, $uv \in E$ if and only if $u + v \in S$. By extension a graph G is called a *sum graph* if it is isomorphic to the sum graph $G^+(S)$ of $S \subset \boldsymbol{N}$.

The notion of sum graph can be defined equivalently as follows. For a graph G with minimum degree $\delta(G) \geq 1$ and a positive integer k, we write G_k for $G \cup \overline{K_k}$, the disjoint union of G and k isolated vertices. Then the graph G_k is a *sum* graph if there exists an injective labeling $L : V(G_k) \longrightarrow \boldsymbol{N}$ such that for any two distinct vertices u, v of G_k, $uv \in E(G_k)$ iff there exists $w \in V(G_k)$ with $L(w) = L(u) + L(v)$. In this case, L is called a *sum labeling* of G_k. Observe that, by definition, the vertex with the largest label in a sum graph cannot be adjacent to any other vertex. Thus, if G_k is a sum graph, then $k \geq 1$. For a connected graph G, its *sum number*, denoted by $\sigma(G)$, is defined as the *smallest* k for which G_k is a sum graph. Since the vertex with the largest label in G is adjacent to at least $\delta(G)$ vertices, we have $\sigma(G) \geq \delta(G)$. Motivated by this relation, we define a nontrivial connected graph G to be k*-optimum summable*, where $k \geq 1$, if $\sigma(G) = \delta(G) = k$. Following Harary [2], a nontrivial connected graph G is called a *unit graph* if G_1 is a sum graph. Thus, G is a unit graph iff it is 1-optimum summable.

The problem of characterizing k-optimum summable graphs (even when $k = 1$) is believed to be very difficult. In this paper, we shall first show in the next section that if G is a k-optimum summable graph of order n, $k \geq 3$, then (1) $n \geq 2k$; (2) the complete bipartite graph $K_{k,n-k}$ is not a spanning subgraph of G. In the remaining sections we describe new families of k-optimum summable graphs for $k \geq 1$.

2. Necessary Conditions

Let K_n denote the complete graph of order n. We have $\sigma(K_2) = 1$, $\sigma(K_3) = 2$ and so K_2 is 1-optimum summable and K_3 is 2-optimum summable. However, it is known [3] that $\sigma(K_n) = 2n - 3$ for $n > 4$, and therefore K_n is not $(n-1)$-optimum summable.

For the rest of this paper, let G be a k-optimum summable graph. Let L be a sum labeling of G_k. For convenience, throughout this paper, we shall refer to the vertices of G_k by their sum labels.

Let u be the largest vertex in $V(G)$. Since G is a k-optimum summable graph, we have $\deg(u) \geq k$. But since u is the vertex with the largest label, $\deg(u) \leq k$, and so $\deg(u) = k$. Denoting by $N(x)$ the set of vertices adjacent to a given vertex x, let

$A = N(u) = \{a_1, a_2, \ldots, a_k\}$, where $a_1 < a_2 < \cdots < a_k$. Then

$$C = \{u + a_1, u + a_2, \ldots, u + a_k\} = \{c_1, c_2, \ldots, c_k\}$$

is the set of the k isolated vertices in G_k, where $c_1 < c_2 < \cdots < c_k$. Let $B = V(G)\backslash(A \cup \{u\}) = \{b_1, b_2, \ldots, b_{n-k-1}\}$, where $b_1 < b_2 < \cdots < b_{n-k-1}$.

Lemma 2.1. $a_i + a_j \notin A$ *for* $1 \le i < j \le k$.

Proof. Suppose that there exist i, j with $1 \le i < j \le k$ such that $a_i + a_j \in A$. Then $k \ge 3$ and $a_i + a_j = a_p$ for some $p \in j + 1..k$. As $u + a_p \in V(G_k)$, $u + a_i$ is adjacent to a_j, contradicting the fact that $u + a_i$ is an isolated vertex. □

Lemma 2.2. $b_i + a_j \notin A$ *for every* $1 \le i \le n - k - 1$ *and* $1 \le j \le k$.

Proof. Suppose that $b_i + a_j \in A$ for some $j \in j + 1..k$. Then $k \ge 2$ and $u + b_i + a_j \in V(G_k)$. Hence $u + a_j$ is adjacent to b_i, a contradiction. □

Now let $X = N(a_1)\backslash\{u\} = \{x_1, x_2, \ldots, x_{k'-1}\}$, where $x_1 < x_2 < \cdots < x_{k'-1}$ and $k' \ge k$. Obviously, $X \subset A \cup B$.

Lemma 2.3. $x_i + a_1 \notin C$ *for every* $i \in 1..k' - 1$.

Proof. Obvious since $x_i + a_1 < u + a_1$ for $1 \le i \le k' - 1$. □

Recall that for $k \ge 3$ a k-optimum summable graph G cannot be a complete graph, and so $n(G) \ge \delta(G) + 2$. However, as the next theorem shows, we can find a much better general lower bound on the order of a k-optimum summable graph.

Theorem 2.1. *If* G *is a* k*-optimum summable graph for* $k \ge 3$*, then* $n(G) \ge 2k$.

Proof. Let G be a k-optimum summable graph with $V(G) = \{u\} \cup A \cup B$ and $V(G_k) = V(G) \cup C$ as described above.

Consider the edges between a_1 and its neighbours x_i, $i = 1, \ldots, k' - 1$, other than u. By Lemma 2.1 and Lemma 2.2, $a_1 + x_i \notin A$ for every $i \in 1..k' - 1$; by Lemma 2.3, $a_1 + x_i \notin C$ for every $i \in 1..k' - 1$. Hence, for every $i \in 1..k' - 1$, $a_1 + x_i \in B \cup \{u\}$. Since a_1 is also adjacent to u, this tells us that $\deg(a_1) = k \le |B| + 2$, hence that $|B| \ge k - 2$. Since $|B| = n - k - 1$, it follows that $n \ge 2k - 1$.

Next we show that $|B| \ne k - 2$, thus proving that $n \ge 2k$. If on the contrary we suppose that $|B| = k - 2$, then

(1) Every $a_i \in A$ is adjacent to at least one other $a_j \in A$.

(2) Every $a_i \in A$ is adjacent to some $x \ne u$ such that $a_i + x \notin B$.

(3) If $u = a_i + x$ for some $x \in A \cup B$, then by Lemma 2.3 for every $i' \in i+1..k$, $(a_{i'}, x) \notin E$.

The edges involving a_1 can only sum to $b_1, b_2, \ldots, b_{k-2}, u$ or $c_1 = u + a_1$ which implies that $\deg(a_1)$ is at most k, hence exactly k. Thus there exists some $x \in A \cup B$ such that $(a_1, x) \in E(G)$ and $a_1 + x = u$. Two cases then arise, depending on whether $x \in A$ or $x \in B$:

Case 1 $x \in A$

Suppose $x = a_j$ for some $j \in 2..k$. Denoting by x_i, $1 \le i \le k$, the vertices adjacent to a_1 in ascending order, and recalling that the vertices of A and B are also listed in ascending order, we must have

$$a_1 + x_1 = b_1, a_1 + x_2 = b_2, \ldots, a_1 + x_{k-2} = b_{k-2}, a_1 + a_j = u, a_1 + u = c_1,$$

where $x_{k-1} = a_j$ and $x_k = u$. Thus for some $m \ge 2$ we may arrange the vertices in ascending sequence as follows:

$$a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_{k-2}, a_j.$$

Now consider a_j. From (3) we know that for every $j' > 1$, $(a_{j'}, a_j) \notin E$. Thus a_j can be adjacent only to $a_1, b_1, b_2, \ldots, b_{k-2}$ and u, where $a_j + a_1 = u$; therefore, for $y \in B \cup \{u\}$, $a_j + y \in C$. Since $a_j + u = c_j$, it follows that $j = k-1$ or k.

(a) Suppose $j = k-1$.

Here for every $i \in 1..k-2$,

$$c_i = b_i + a_{k-1} = a_i + u = (a_i + a_1) + a_{k-1},$$

from which $b_i = a_1 + a_i$. Thus a_1 is adjacent to $a_2, a_3, \ldots, a_{k-2}$ as well as to a_{k-1} and u, but by (3) not to a_k. Hence a_1 must be adjacent to one vertex, say b_r, in B, and further, by Lemmas 2.1–2.3, $a_1 + b_r = b_q$ for some $q \in r+1..k-2$.

At the same time $b_q = a_1 + a_s$ for some a_s so that $a_s = b_r$, giving duplicate labels in G. Therefore $j \ne k-1$.

(b) Suppose $j = k$.

We conclude as in (a) that a_1 is adjacent to $a_2, a_3, \ldots, a_{k-2}$, and in addition to a_k and u. Suppose that $(a_1, a_{k-1}) \in E(G)$. But then $a_1 + a_k \in B$, as in (a) an impossibility since $b_i = a_1 + a_i$ for every $i \in 1..k-2$. Thus $j \ne k$.

We have shown that Case 1 is impossible.

Case 2 $x \in B$

Suppose $x = b_j$ for some $j \in 1..k-2$. Then $u = a_1 + b_j$, so that for every $i \in 1..k$, $c_i = a_1 + (a_i + b_j)$. Since $a_i + b_j > u$ for every $i > 1$, it follows that vertices $a_i + b_j$ cannot exist. Thus b_j is not adjacent to any of $a_2, a_3, \dots, a_k$, and so has degree at most $k-2$, contradicting the requirement that $\delta = k$. Thus $u \neq a_1 + b_j$ and Case 2 is impossible.

On the assumption that $|B| \leq k-2$, we have shown that $a_1 + x \neq u$ for any x. Hence $|B| \geq k-1$, as required. □

The next result gives us more insight into the structure of a k-optimum summable graph.

Theorem 2.2. *If G is a k-optimum summable graph, $k \geq 3$, then $K_{k,n-k}$ is not a spanning subgraph of G.*

Proof. Suppose to the contrary that there exists a k-optimum summable graph G such that G contains $K_{k,n-k}$ as a spanning subgraph. As before, let $V(G) = \{u\} \cup A \cup B$ and $V(G_k) = \{u\} \cup A \cup B \cup C$, where u is the largest label in G and $|A| = k$. As we have seen, u must have degree exactly k. If we suppose that u is in the bipartite set S_k of order k, then since u must be adjacent to every vertex in the bipartite set S_{n-k}, it follows that $n-k \leq k$. But since by Theorem 2.1, $n-k \geq k$, therefore $k = n-k$. Thus without loss of generality we may assume that u is a vertex of S_{n-k}, and so we may assume that $S_k = A = \{a_1, a_2, \dots, a_k\}$, where $a_i > a_j$ whenever $i > j$, and $S_{n-k} = B \cup \{u\} = \{b_1, b_2, \dots, b_{n-k-1}, u\}$, where $b_i > b_j$ whenever $i > j$.

From Lemma 2.2 we have $a_i + b_j \in B \cup C \cup \{u\}$ for every $i \in 1..k$, $j \in 1..n-k-1$. From Lemmas 2.2 and 2.3 it follows that $a_1 + b_j \in B \cup \{u\}$ for every $j \in 1..n-k-1$. Since $b_1 \neq a_1 + b_j$, we must have $a_1 + b_j = b_{j+1}$ for every $j \in 1..n-k-2$ and $a_1 + b_{n-k-1} = u$. But then

$$u = a_1 + b_{n-k-1} < a_2 + b_{n-k-1} < a_2 + u$$

which implies $a_2 + b_{n-k-1} = a_1 + u$.

However, since $u = a_1 + b_{n-k-1}$, it follows that $a_2 = 2a_1$, an impossibility as it would imply an edge between vertex a_1 and the isolate $u + a_1$. □

Observe that for $k = 1$, $K_2 = K_{1,1}$, while for $k = 2$, K_3 contains $K_{2,1}$. Thus Theorem 2.2 is sharp. On the other hand, we shall see in Section 5 that the lower bound for $n(G)$ in Theorem 2.1 is *not* sharp.

Remark 2.1. *Let $d_1, d_2, \dots, d_n$ be the degree sequence of a connected graph G of order $n \geq 2$, where $d_1 \leq d_2 \leq \cdots \leq d_n$. It was shown in [4] that $\sigma(G) > \max_{1 \leq i \leq n}(d_i - i)$. As a direct consequence of this result, we have another necessary condition, namely $d_i - i \leq k-1$ for each $i = 1, 2, \dots, n$, for G to be a k-optimum summable graph.*

3. Unit Graphs

It was pointed out in Section 1 that unit graphs and 1-optimum summable graphs are identical. Smyth [5] showed that if G is a unit graph of order n, then $e(G) \leq \lfloor n^2/4 \rfloor$; he established further that for all integers m and n with $1 \leq n-1 \leq m \leq \lfloor n^2/4 \rfloor$, there exists a unit graph of order n and size m. Ellingham [6] proved that any nontrivial tree is a unit graph, a conjecture of Harary [2]. Until now, however, the problem of characterizing unit graphs remains open. In this section, we describe a new family of unit graphs.

Given integers $p \geq 3$ and $q \geq 2$, let $Q(p,q)$ denote the graph obtained from the union of the cycle C_p of order p and the path P_q of order q by identifying one end-vertex of P_q with a vertex of C_p (see Figure 3.1). $Q(p,q)$ is called a *tadpole*.

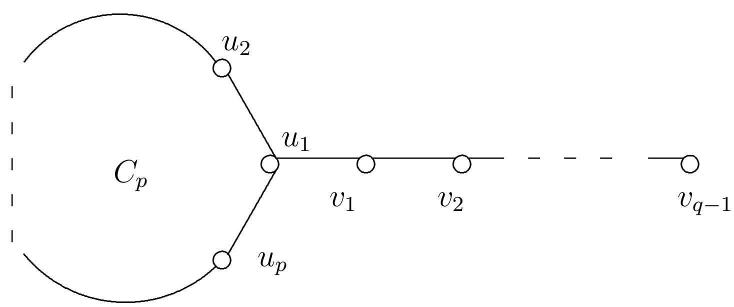

Figure 3.1. The tadpole $Q(p,q)$

Our aim in this section is to show that every tadpole is a unit graph. The following observation on a generalized Fibonacci sequence will be useful.

Lemma 3.1. *If an integer sequence* $\{a_i | i = 1, 2, \cdots\}$ *satisfies the following condition* ($*$)*:*

$$\begin{cases} a_2 > a_1 > 0 \\ a_i = a_{i-1} + a_{i-2} \quad \text{for} \quad i \geq 3, \end{cases}$$

then

$$a_k + a_j < a_{j+1} \quad \text{for} \quad j - k \geq 2 \quad \text{and} \quad k \geq 1.$$

Proof. Since $j - k \geq 2$ and $k \geq 1$, $a_k \leq a_{j-2}$. Now $a_{j+1} - a_j = a_{j-1}$. Thus, $a_k + a_j < a_{j+1}$. □

It follows from this result that if the label sequence $\{a_i | i = 1, 2, \cdots, p\}$ satisfies ($*$), then $G^+(\{a_i | i = 1, 2, \cdots, p\}) \cong P_{p-1} \cup \overline{K_1}$.

Theorem 3.1. *The tadpole* $Q(p,q)$ *is a unit graph for all* $p \geq 3$ *and* $q \geq 2$.

Proof. Since $\delta(Q(p,q)) = 1$, $\sigma(Q(p,q)) \geq 1$. Let $G = Q(p,q)$, where $V(G) = A \cup B$, $A = \{u_1, u_2, \ldots, u_p\}$, $B = \{v_1, v_2, \ldots, v_{q-1}\}$, and the subgraph induced by A is isomorphic to C_p. Let $V(G_1) = V(G) \cup \{w_1\}$. We consider two cases.

Case 1. $p = 3$ and $q \geq 2$.

Consider a labeling g of G_1 as follows:

$$\begin{cases} g(u_1) = 1,\ g(u_2) = 2,\ g(u_3) = 3,\ g(v_1) = 4; \\ g(v_2) = 5 \quad \text{for} \quad q \geq 3; \\ g(v_i) = g(v_{i-1}) + g(v_{i-2}) \quad \text{for} \quad 3 \leq i \leq q-1; \\ g(w_1) = \begin{cases} 5 & \text{when} \quad q = 2; \\ g(v_{q-1}) + g(v_{q-2}) & \text{when} \quad q > 2. \end{cases} \end{cases}$$

Let $H = G^+(\{g(x) | x \in V(G_1)\})$. We wish to prove that $H \cong G_1$.

Let $Y = \{g(v_i) | i = 1, 2, \ldots, q-1\}$. Since $Y \cup \{g(u_1)\}$ satisfies the condition ($*$) in Lemma 3.1, $G^+(Y \cup \{g(u_1)\}) \cong P_{q-1} \cup \overline{K_1}$. This, together with the value of $g(w_1)$, implies that $H[Y \cup \{g(w_1)\}] \cong P_q$. Clearly, $H[\{g(u_1), g(u_2), g(u_3)\}] \cong C_3$. It is now easy to see that $H \cong G_1$, as asserted. Hence $\sigma(Q(3,q)) = 1$ for $q \geq 2$.

Case 2. $p \geq 4$ and $q \geq 2$.

Consider a labeling g of G_1 as follows:

$$\begin{cases} g(u_1) = 1,\ g(u_2) = 3; \\ g(u_i) = g(u_{i-1}) + g(u_{i-2}) \quad \text{for} \quad 3 \leq i \leq p-1; \\ g(u_p) = g(u_{p-1}) + g(u_1),\ g(v_1) = g(u_p) + g(u_{p-2}); \\ g(v_2) = g(u_p) + g(u_{p-1}) \quad \text{for} \quad q \geq 3; \\ g(v_i) = g(v_{i-1}) + g(v_{i-2}) \quad \text{for} \quad 3 \leq i \leq q-1; \\ g(w_1) = \begin{cases} g(u_p) + g(u_{p-1}) & \text{when} \quad q = 2; \\ g(v_{q-1}) + g(v_{q-2}) & \text{when} \quad q > 2. \end{cases} \end{cases}$$

Let $J = G^+(\{g(x) | x \in V(G_1)\})$. We wish to prove that $J \cong G_1$.

The strictly increasing sequence

$$g(u_1), g(u_2), \cdots, g(u_{p-2}), g(u_p), g(u_{p-1}), g(v_1), g(v_2), \cdots, g(v_{q-1}), g(w_1)$$

has subsequence $X = \{g(u_1), g(u_2), \cdots, g(u_{p-2}), g(u_{p-1})\}$. Since X satisfies the condition ($*$) in Lemma 3.1, $G^+(X) \cong P_{p-2} \cup \overline{K_1}$. This, together with the values of $g(u_p)$, $g(v_1)$ and $g(v_2)$ (or $g(w_1)$), ensures that $J[X \cup \{g(u_p)\}] \cong C_p$.

Consider the sequence $Y = \{g(u_{p-3}), g(v_j) | j = 1, 2, \ldots, q-1\}$. Note that Y satisfies the condition ($*$) in Lemma 3.1, so that $G^+[\{g(u_{p-3})\} \cup Y] \cong P_{q-1} \cup \overline{K_1}$. This, together with the value of $g(w_1)$, ensures that $J[\{g(u_{p-3})\} \cup Y] \cong P_q$.

It is clear from the definition of g that $g(u_{p-3})$ is a vertex of degree 3 in J. Next we assert that no other adjacencies between $g(u_i)$ with $i \neq p-3$ and $g(v_j)$ exist. Suppose that there exist i, j with $i \neq p-3$ such that $g(u_i)+g(v_j) \in V(G_1)$. Then either $g(u_i)+g(v_j) = g(v_k)$ with $k > j$ or $g(u_i)+g(v_j) = g(w_1)$. For $q > 2$, however, $g(w_1)-g(v_j) \geq g(v_{q-2}) > g(u_p)$. Thus, $g(u_i)+g(v_j) = g(v_k)$ for some $k > j$. If $k > 2$, then $g(v_k)-g(v_j) \geq g(v_{k-2}) \geq g(v_1) > g(u_p)$, a contradiction. Thus $k \leq 2$, and we have $k = 2$ and $j = 1$. Hence $g(u_i) = g(u_{p-3})$ and so $i = p-3$, a contradiction.

It follows from the above discussion that $J[X \cup Y \cup \{g(u_p)\}] \cong G$. Clearly, $g(w_1)$ is isolated in J. Hence $J \cong G_1$, as required.

This completes the proof of Theorem 3.1. □

4. 2-Optimum Summable Graphs

It is known [2] that $\sigma(C_4) = 3$ and $\sigma(C_n) = 2$ for all $n \geq 3$ with $n \neq 4$. Thus $\{C_n | n \geq 3, n \neq 4\}$ is a family of 2-optimum summable graphs. In this section we introduce two new families of 2-optimum summable graphs.

Consider two tadpoles $Q = Q(p,q)$ and $Q' = Q'(p',q')$ with isolated vertices w_1 and w'_1, respectively. We first sum-label $Q \cup \{w_1\}$ and $Q' \cup \{w'_1\}$ as described in Section 3, using a labelling g. Observe that since under g each edge is represented by a unique vertex, we can multiply the labels by any positive integer and still retain a sum labeling. Now form a single graph $B = B(p,q,p',q')$ from Q and Q' by adding the edge $(v_{q-1}, v'_{q'-1})$. We multiply all the original labels of $Q' \cup \{w'_1\}$ by $g(w_1)$, yielding a sum labeling h, and then reassign $h(w_1) \leftarrow g(w_1)g(v'_{q'-1}) + g(v_{q-1})$ to represent the new edge. Since $h(u'_1) = g(w_1)$, $B \cup \{w_1, w'_1\}$ now has a sum labeling. We have proved

Theorem 4.1. *$B(p,q,p',q')$, $p,p' \geq 3$, $q,q' \geq 2$, is 2-optimum summable.*

We now construct another 2-optimum summable graph. Given integers p,q,r with $p \geq q \geq r \geq 2$ and $q \geq 3$, let $\theta(p,q,r)$ denote the graph obtained by connecting two vertices via three internally disjoint paths P_r, P_q and P_p as shown in Figure 4.1. We call the graph $\theta(p,q,r)$ a *generalized θ-graph.*

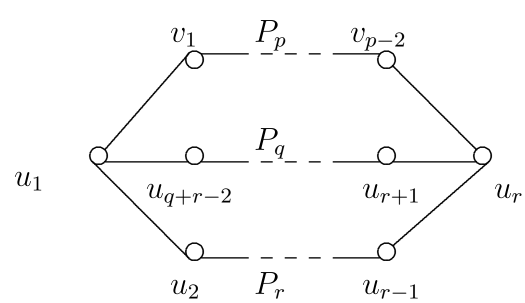

Figure 4.1. The generalized θ-graph $\theta(p,q,r)$

Theorem 4.2. *The generalized* θ*-graph* $\theta(p,q,r)$ *is a* 2*-optimum summable graph for all* p,q,r *with* $p \geq q \geq r \geq 2$ *and* $q \geq 3$ *except when* $(p,q,r) = (3,3,2)$ *or when* $(p,q,r) = (3,3,3)$.

Proof. Let $G = \theta(p,q,r)$ for $p \neq 3$ or $q \neq 3$. Let $V(G) = A \cup B$, where $A = \{u_1, u_2, \ldots, u_{q+r-2}\}$, $B = \{v_1, v_2, \ldots, v_{p-2}\}$ and the subgraphs induced by A and B are respectively isomorphic to C_{q+r-2} and P_{p-2}. Since $\delta(G) = 2$, $\sigma(G) \geq 2$. Let $V(G_2) = V(G) \cup \{w_1, w_2\}$.

Case 1. $r = 2$, $q = 3$ and $p \geq 6$.

Consider a labeling h of G_2 as follows:

$$\begin{cases} h(u_1) = 1, h(u_2) = 2, h(u_3) = 3; \\ h(v_1) = 4, h(v_2) = 5; \\ h(v_i) = h(v_{i-1}) + h(v_{i-2}) \quad \text{for} \quad 3 \leq i \leq p-2; \\ h(w_1) = h(v_{p-2}) + h(u_2); \\ h(w_2) = h(v_{p-2}) + h(v_{p-3}). \end{cases}$$

Let $H = G^+(\{h(x) | x \in V(G_2)\})$.

Clearly, u_1 and u_2 are two vertices of degree 3 in H. Since $p \geq 6$, $h(v_{p-3}) - h(u_2) > h(v_{p-4})$. Thus, $h(w_1)$ is isolated in H. By means of an argument similar to that given in Case 1 of the proof of Theorem 3.1, it is not difficult to verify that $H \cong G_2$. The result thus follows.

Case 2. $q + r \geq 6$ and $p \geq 6$.

Consider a labeling h of G_2 as follows:

$$\begin{cases} h(u_1) = 1, h(u_2) = 3; \\ h(u_i) = h(u_{i-1}) + h(u_{i-2}) \quad \text{for} \quad 3 \leq i \leq q+r-3; \\ h(u_{q+r-2}) = h(u_{q+r-3}) + h(u_1); \\ h(v_1) = h(u_{q+r-2}) + h(u_{q+r-4}), h(v_2) = h(u_{q+r-2}) + h(u_{q+r-3}); \\ h(v_i) = h(v_{i-1}) + h(v_{i-2}) \quad \text{for} \quad 3 \leq i \leq p-2; \\ h(w_1) = \begin{cases} h(v_{p-2}) + h(u_{q-2}) & \text{when} \quad r = 2, \\ h(v_{p-2}) + h(u_{q+1}) & \text{when} \quad r = 3, 4 \\ h(v_{p-2}) + h(u_{q-4}) & \text{when} \quad r \geq 5 \end{cases} \\ h(w_2) = h(v_{p-2}) + h(v_{p-3}). \end{cases}$$

Let $J = G^+(\{h(x) | x \in V(G_2)\})$.

Clearly, the degree of u_{q+r-5} is 3 and $u_1 u_2 \cdots u_{q+r-5} u_{q+r-4} u_{q+r-2} u_{q+r-3} u_1$ is a cycle of order $q + r - 2$ in J. Since $p \geq 6$,

$$h(v_{p-3}) - \max\{h(u_{q-2}), h(u_{q+1}), h(u_{q-4})\} > h(v_{p-4}).$$

Thus, $h(w_1)$ is isolated in J. By means of an argument similar to that given in Case 2 of the proof of Theorem 3.1, it is not difficult to verify that $J \cong G_2$. The result thus follows.

Case 3. $p \le 5$.

The following labeling-induced sum graphs show that this case is also covered.

$$G^+(\{1,3,4,7,11,18,29,30,48,59,107;108,166\}) \cong \theta(5,5,5) \cup \overline{K_2}$$
$$G^+(\{1,3,4,7,11,18,19,30,37,67;68,104\}) \cong \theta(5,5,4) \cup \overline{K_2}$$
$$G^+(\{1,3,4,7,11,12,19,23,42;43,65\}) \cong \theta(5,5,3) \cup \overline{K_2}$$
$$G^+(\{1,3,4,7,8,12,15,27;31,42\}) \cong \theta(5,5,2) \cup \overline{K_2}$$
$$G^+(\{1,3,4,7,11,18,19,30,37;38,67\}) \cong \theta(5,4,4) \cup \overline{K_2}$$
$$G^+(\{1,3,4,7,11,18,19,30;31,37\}) \cong \theta(5,4,3) \cup \overline{K_2}$$
$$G^+(\{1,3,4,5,8,9,17;20,26\}) \cong \theta(5,4,2) \cup \overline{K_2}$$
$$G^+(\{1,3,4,7,11,12,19;23,31\}) \cong \theta(5,3,3) \cup \overline{K_2}$$
$$G^+(\{1,3,4,7,8,12;13,15\}) \cong \theta(5,3,2) \cup \overline{K_2}$$
$$G^+(\{1,3,4,7,11,12,19,23;34,42\}) \cong \theta(4,4,4) \cup \overline{K_2}$$
$$G^+(\{1,3,4,7,8,12,15;22,27\}) \cong \theta(4,4,3) \cup \overline{K_2}$$
$$G^+(\{7,8,11,15,19,23;30,34\}) \cong \theta(4,4,2) \cup \overline{K_2}$$
$$G^+(\{1,3,4,7,8,12;15,20\}) \cong \theta(4,3,3) \cup \overline{K_2}$$
$$G^+(\{1,3,4,7,8;9,11\}) \cong \theta(4,3,2) \cup \overline{K_2}$$

This completes the proof of Theorem 4.2. □

Remark 4.1. *The two generalized θ-graphs not included in Theorem 4.2 are $\theta(3,3,2)$ and $\theta(3,3,3)$. They are, as a matter of fact, not 2-optimum summable graphs. Indeed, by Theorem 2.2, we have $\sigma(\theta(3,3,2)) \ge 3$ and $\sigma(\theta(3,3,3)) \ge 3$. These, together with the two labeling-induced sum graphs*

$$G^+(\{2,4,7,9;6,11,16\}) \cong \theta(3,3,2) \cup \overline{K_3},$$
$$G^+(\{1,2,3,8,10;4,11,18\}) \cong \theta(3,3,3) \cup \overline{K_3}.$$

show that $\sigma(\theta(3,3,2)) = \sigma(\theta(3,3,3)) = 3$.

5. k-Optimum Summable Graphs, $k \ge 3$

In this final section we shall establish two existence results, one for 3-optimum summable graphs and one for k-optimum summable graphs, where $k \ge 4$.

Theorem 5.1. *For each $l \ge 1$, there exists a 3-optimum summable graph of order $4l+3$.*

Proof. Given $l \geq 1$, our aim is to construct a subset S^l of N such that $G^+(S^l) \cong G_3$ and to show that G is a 3-optimum summable graph of order $4l+3$.

Let $A_i = \{a_{i1}, a_{i2}, a_{i3}\}$ for $1 \leq i \leq l+2$ and $B = \{b_1, b_2, \ldots, b_l\}$, where

$$\begin{cases} a_{11} = 1, a_{12} = 4 \quad \text{and} \quad a_{13} = 7; \\ a_{ij} = \sum_{p=1}^{3} a_{(i-1)p} - a_{(i-1)j} \quad \text{for} \quad 2 \leq i \leq l+2 \quad \text{and} \quad 1 \leq j \leq 3; \\ b_i = \sum_{p=1}^{3} a_{ip} \quad \text{for} \quad 1 \leq i \leq l. \end{cases}$$

Let $S^l = (\cup_{i=1}^{l+2} A_i) \cup B$ and $H = G^+(S^l)$. Clearly, $v(H) = 4l+6$.

For $i \geq 3$ and $1 \leq j \leq 3$, observe that

$$\begin{aligned} a_{ij} &= \sum_{p=1}^{3} a_{(i-1)p} - a_{(i-1)j} \\ &= 2\sum_{p=1}^{3} a_{(i-2)p} - (\sum_{p=1}^{3} a_{(i-2)p} - a_{(i-2)j}) \\ &= \sum_{p=1}^{3} a_{(i-2)p} + a_{(i-2)j} > a_{(i-1)j}. \qquad (\#) \end{aligned}$$

Clearly, $\min\{a_{(l+2)1}, a_{(l+2)2}, a_{(l+2)3}\} > b_i$ for $1 \leq i \leq l$. Thus, the three vertices in A_{l+2} are the three largest vertices in H. For $1 \leq j_1 \leq 3, 1 \leq j_2 \leq 3$ and $j_1 \neq j_2$,

$$a_{(l+2)j_1} - a_{(l+2)j_2} = a_{(l+1)j_2} - a_{(l+1)j_1} = \cdots = (-1)^{l+1}(a_{1j_1} - a_{1j_2}).$$

Now $A_2 = \{5, 8, 11\}$ and $a_{1j_1} - a_{1j_2}$ can only take one of the two positive integers 3 and 6. Thus $a_{1j_1} - a_{1j_2} \notin S^l$, and so the three vertices in A_{l+2} are isolated in H.

It follows from the values of the three integers in A_{i+1} that $H[A_i] \cong C_3$ for $1 \leq i \leq l+1$. Notice that

$$a_{ij} + a_{(i-1)j} = (\sum_{p=1}^{3} a_{(i-1)p} - a_{(i-1)j}) + a_{(i-1)j} = \sum_{p=1}^{3} a_{(i-1)p} = b_{i-1}$$

for $1 \leq i \leq l+1$. This implies that a_{ij} is adjacent to $a_{(i-1)j}$. Thus the degree of any vertex in $\cup_{i=1}^{l+1} A_i$ is at least 3.

For $1 \leq i \leq l$ and $1 \leq j \leq 3$, by $(\#)$, we have

$$a_{(i+2)j} = \sum_{p=1}^{3} a_{ip} + a_{ij} = b_i + a_{ij}.$$

Thus b_i is adjacent to a_{i1}, a_{i2}, a_{i3} for $1 \leq i \leq l$, and so the degree of any vertex in B is at least 3.

Let $G = H[S^l \backslash A_{l+2}]$. It follows from the above discussion that G is connected and $\delta(G) = 3$. Thus G is a 3-optimum summable graph of order $4l + 3$. The proof is thus complete. □

As an illustration of the construction used in the above proof, we present the graph $G^+(S^2)$ in Figure 5.1.

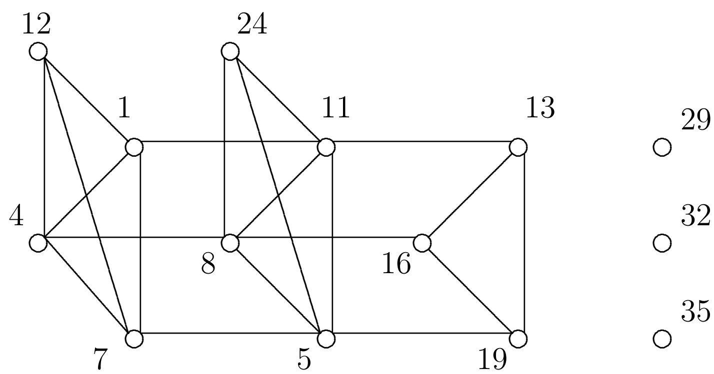

Figure 5.1

Finally, we have:

Theorem 5.2. *For each $k \geq 4$, there exists a k-optimum summable graph.*

Proof.

Given $k \geq 4$, our aim is to construct a subset $S^{(k)}$ of N such that $G^+(S^{(k)}) \cong G_k$ and to show that G is a k-optimum summable graph.

Let $I = \{1, 2, \ldots, k\}$ and $a_i = 10^{i-1}$ for $i \in I$. Define

$$\begin{cases} A_j = \{\sum_{p \in D} a_p | D \subseteq I \quad \text{and} \quad |D| = j\} \quad \text{for} \quad 1 \leq j \leq k; \\ B = \{a_i + \sum_{p=1}^{k} a_p | i \in I\}. \end{cases}$$

Let $S^{(k)} = (\cup_{j=1}^{k} A_j) \cup B$ and $H = G^+(S^{(k)})$.

Clearly, the k vertices of B are the k largest vertices in H. Since $u - v \notin S^{(k)}$ for any pair of distinct vertices $u, v \in B$, the k vertices in B are isolated in H.

It is obvious that $|A_k| = 1$ and the vertex in A_k is adjacent to all the k vertices of A_1. For any vertex $w \in A_j$, where $1 \leq j < k$, there exists a subset D of I with $|D| = j$ such that $w = \sum_{p \in D} a_p$. Clearly, w is adjacent to a_p for $p \in I \backslash D$. For a fixed $\alpha \in D$, by the fact that $w + (\sum_{p \in I \backslash D} a_p + a_\alpha) = \sum_{p \in I} a_p + a_\alpha$, w is adjacent to $\sum_{p \in I \backslash D} a_p + a_\alpha$ which is a vertex of A_{k-j+1}. Thus, $d(w) \geq |I \backslash D| + |D| = k$. Let $G = H[S^{(k)} \backslash A_k]$. It follows from the above discussion that G is connected and $\delta(G) = k$. Hence G is a k-optimum summable graph. □

References

[1] J.A. Bondy and U.S.R. Murty, *Graph theory with applications*, MacMillan 1976.

[2] F. Harary, Sum graphs and difference graphs, *Congressus Numerantium,* **72** (1990) 101-108.

[3] D. Bergstrand, F. Harary, K. Hodges, G. Jennings, L. Kuklinski and J. Wiener, The sum number of a complete graph, *Bull. Malaysian Math. Soc.,* **12** (1989) 25-28.

[4] Hao Tianbao, On sum graphs, *Journal of Combinatorial Mathematics & Combinatorial Computing,* **6** (1989) 207-212.

[5] W. F. Smyth, Sum graphs of small sum number, *Colloquia Mathematica Societatis János Bolyai,* **60** (1991) 669-678.

[6] Mark N. Ellingham, Sum graphs from trees, *Ars Combinatoria,* **35** (1993) 335-349.