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Coloring Fiber Product of Graphs

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Abstract

The fiber product of graphs over homomorphims, a notion introduced by P. Hell (1972) is used here as a new approach to S. Hedetniemi's conjecture (1966) since it's a subgraph of the cross product. We consider here only the fiber product over colorings which are special cases of homomorphisms of graphs. We show that Khelladi's conjecture (1991): "The fiber product over colorings of two n-chromatic graphs is also an n-chromatic graph" implying trivially Hedetniemi's is true for $n \leq 3$. Moreover, we propose an equivalent statement of Khelladi's conjecture using the class of graph colourings of a graph defined by El-Zahar and Sauer.

Keywords: Graph product, coloring, Hedetniemi's conjecture.

2000 Mathematics Subject Classification:

1. Introduction

The graphs we consider are finite and simple, that is, undirected and with no loops nor multiple edges. If G is a graph, the set of vertices (resp. edges) of G is denoted $V(G)$ (resp. $E(G)$) or simply V (resp. E) if there is no ambiguity. For standard notions and results on graphs the reader is referred to [1].

Let us recall that if G_1 and G_2 are graphs, a *homomorphism* $f: G_1 \longrightarrow G_2$ is a mapping f from $V(G_1)$ into $V(G_2)$ such that $f(u_1)$ and $f(u_2)$ are adjacent vertices of G_2 whenever u_1 and u_2 are adjacent vertices of G_1.

It is well known that a homomorphism $c: G \longrightarrow K_n$ is just an n-coloring (of the vertices) of G. The smallest integer n such that G admits an n-coloring is the chromatic number of G and is denoted $\chi(G)$. A graph G is n-chromatic if $\chi(G) = n$. It is also clear that if there exists a homomorphism $G_1 \longrightarrow G_2$ then $\chi(G_1) \leq \chi(G_2)$.

The *cross product* of the graphs G_1 and G_2 is the graph denoted $G_1 \times G_2$ whose vertex set is $V(G_1) \times V(G_2)$ and where (u_1, u_2) is adjacent to (v_1, v_2) whenever $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$.

Let H, G_1 and G_2 be graphs and $f_1 : G_1 \longrightarrow H$ and $f_2 : G_2 \longrightarrow H$ be homomorphisms. The *fiber product* of G_1 and G_2 over f_1 and f_2 is the graph denoted $G_1 \times_H G_2$ whose vertex set is $V(G_1 \times_H G_2) = \{(u_1, u_2) \in V(G_1) \times V(G_2) \mid f_1(u_1) = f_2(u_2)\}$ and where (u_1, u_2) is adjacent to (v_1, v_2) whenever $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. If there is no ambiguity, we call the fiber product of G_1 and G_2 over f_1 and f_2 simply the fiber product. It is clear that $G_1 \times_H G_2$ is an induced subgraph of $G_1 \times G_2$. For example, for a 3-coloring of two odd cycles see Figure 1, their fiber product is illustrated in Figure 2.

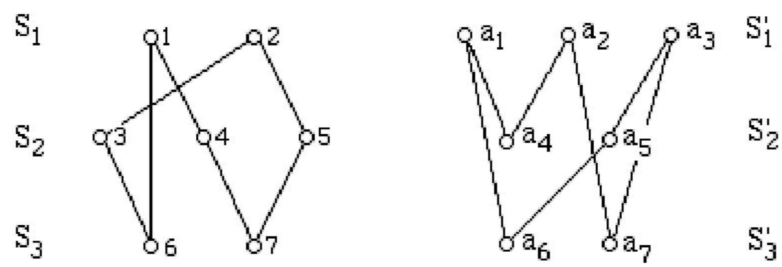

Figure 1: A 3-colorings of two odd cycles.

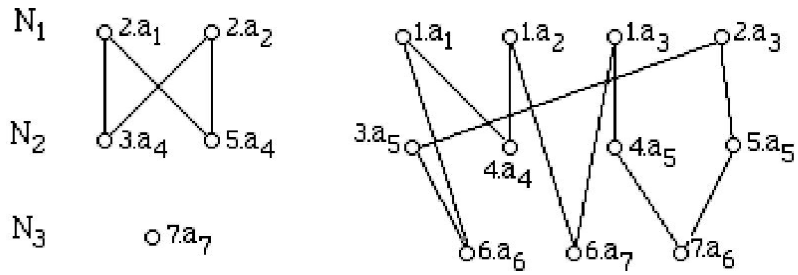

Figure 2: The fiber product of the two cycles of Fig. 1.

2. Coloring the Fiber Product

The following observation will be useful.

Observation 1. *Let G (resp. G') be an n-colorable graph whose vertex set V (resp. V') is partitioned into n independent sets S_i (resp. S'_i) where the vertices of S_i (resp. S'_i) have the color i ($i = 1, ..., n$). Then,*

- *a) the vertices of $G \times_{K_n} G'$ are partitioned into n independent sets $N_i = S_i \times S'_i$ $(i = 1, ...n)$,*
- *b) the set of edges between N_i and N_j in $G \times_{K_n} G'$ is $\{(u, u\prime)(v, v\prime) \in E(G \times_{K_n} G') \mid u \in S_i, v \in S_j, \; u' \in S'_i, \; v' \in S'_j\}$,*
- *c) if p is the number of edges between S_i and S_j , q the number of edges between S'_i and S'_j , and r the number of edges between N_i and N_j , then $r = p.q$,*
- *d) if G is a k-chromatic ($k \leq n$) graph then $G \times_{K_n} K_n$ is isomorphic to G for any n-coloring of G .*

A *semi-Eulerian graph* is a graph (not necessarily connected) in which each vertex has even degree. An *odd semi-Eulerian* graph is a semi-Eulerian graph with an odd number of edges.

Theorem 2. *If C and C' are two odd cycles, then $C \times_{K_3} C'$ is an odd semi-Eulerian graph with degrees equal to $0, 2$ or 4.*

Proof. Let S_i (resp. S'_i) for $i = 1, 2, 3$ be the three independent sets of C (resp. C') defined by a $3-$coloring of C (resp. C'). Let u be a vertex of S_i and u' a vertex of S'_i . The degree of (u, u') in $C \times_{K_3} C'$ is:

- if the two neighbors of u are in S_k, then the degree of (u, u') is:

  - four if the two neighbors of u' are in S'_k;
  - two if exactly one neighbor of u' is in S'_k;
  - zero if none neighbor of u' is in S'_k .

-Othewise, one may assume that u and u' have neighbors in S_k and S_t (resp. in S'_k and S'_t with $k \neq t$) , then (u, u') has degree two.

To complete the proof of Theorem 2, we need the following two lemmas.

Lemma 3. *If $G = (V, E)$ is a 3-edge colorable graph whose degrees are only 1 and 3 , then for every 3-edge coloring of G and for every subset X of vertices of degrees 3 of G , if E_i denotes the set of edges between X and $V \backslash X$ which are colored with color i , then $|X|$ and $|E_i|$ have the same parity.*

Proof. The vertex set V may be partitioned into $S \cup W$ where S is the set of vertices of degree 1 and W the set of vertices of degree 3 . If $|X| = 0$ the property is clear. Now, if $|X| > 0$, let v be a vertex of X and $X' = X \backslash \{v\}$. The cardinality of the set E'_i of edges colored i between X' and $V \backslash X'$ has the same parity as $|X'|$ by induction on $|X'|$. Let $e = vx$ be the unique edge colored i (e exists and is unique because v has degree 3 and G is 3-edge colored). If x belongs to X' , then e belongs to Ei' , hence $E_i = E'_i \backslash \{e\}$; otherwise $E_i = E'_i \cup \{e\}$. Therefore, in either case $|E_i| = |E'_i| \pm 1$ has the same parity as $|X| = |X'| + 1$. □

The following result seems well known in the folklore, but a proof is given.

Lemma 4. *For every* 3*-coloring of an odd cycle* C *into independent sets,* S_1, S_2 *and* S_3*, the number of edges between* S_i *and* S_j *is odd for* $i, j = 1, 2, 3,\ \ i \neq j$*.*

Proof. Let C be described as the sequence of vertices $u_1 u_2 ... u_{2r+1}$. Put a direction on the edges of G such that every vertex has just one entering and one leaving arc from it. If $c : C \longrightarrow K_3$ is a 3-coloring of C, an edge coloring of C is obtained via the arcs by putting the color $c(u)$ to the arc (u, v) of C (this just means that we walk in one direction of C and color suitably its edges). We define a new undirected graph denoted by G by adding an extra vertex u_i' for every vertex u_i of C and an extra edge $u_i u_{i'}$ $(i = 1, ..., 2r+1)$. The 3-edge coloring of C is then extended to G by using the missing color at the vertex u_i to color the edge $u_i u_{i'}$. Therefore, the number of edges colored 1 in G is equal to the number of edges between vertices colored 2 and 3 in the given 3-coloring of C. Applying Lemma 3 to G with $X = V(C)$, we obtain that the number of edges colored 1 in G and between X and $V(G \backslash X)$ (that is, $|E_1|$) is odd as $|X|$. The same argument for the colors 2 and 3 completes the proof. □

We now complete the proof of Theorem 2.

By Lemma 4 and Observation 1-c, the graph $C \times_{K_3} C'$ has an odd number of edges between two N_i. Furthermore, this graph has three independent sets of vertices, hence its total number of edges is odd. □

Corollary 5. *If* G *and* G' *are* 3*-chromatic graphs, then for any* 3*-colorings of* G *and* G'*, the chromatic number of* $G \times_{K_3} G'$ *is three.*

Proof. Let C (resp. C') be an odd cycle of G (resp. G'). Then C (resp. C') is colored by restriction of the coloring of G (resp. G'). We apply Theorem 2 and deduce that at least one connected component of $C \times_{K_3} C'$ is an odd connected Eulerian graph and hence contains an odd cycle. The proof is completed if we notice that $C \times_{K_3} C'$ is a subgraph of $G \times_{K_3} G'$. □

3. Conjectures on the Chromatic Number

One of the classical open question on the cross product is Hedetniemi's conjecture on the chromatic number.

Conjecture 6. *(Hedetniemi [4]) If* G_1 *and* G_2 *are two graphs satisfying* $\chi(G_1) > n$ *and* $\chi(G_2) > n$*, then* $\chi(G_1 \times G_2') > n$*.*

For more information on this problem, see [3].

In [2], El-Zahar and Sauer have given an equivalent conjecture to the one of Hedetniemi and have proved it for the case $n = 4$. They used properties of graph colorings defined as follow: let $G = (V, E)$ a graph. For each integer n, the *graph colorings* of G denoted by $C_n(G)$, is the graph whose vertex set is the set of all functions $f\ : V(G)\ \longrightarrow \{1, 2, .., n\}$ and two such functions f and g are connected by a edge whenever for all edges $ab \in\ E(G)$, $f\,(a) \neq g(b)$.

Conjecture 7. *(El-Zahar and Sauer [2]) If* G *is a graph such that* $\chi\ \ (G) > n$*, then* $\chi(C_n(G)) = n$*.*

Theorem 8. *(El-Zahar and Sauer [2])* $C_3(G)$ *is* 3*-chromatic for each* 4*-chromatic graph* G *.*

Another approach to Hedetniemi's conjecture could be to develop more general products (see for instance [7]). On the other hand, Khelladi suggests studying the fiber product and proposes in [6] a stronger conjecture than Hedetniemi's.

Conjecture 9. *(Khelladi [6]) For any* n*-chromatic graphs* G *and* G' *and any* n*-colorings* $G \longrightarrow K_n$ *and* $G' \longrightarrow K_n$*, the graph* $G \times_{K_n} G'$ *is also* n*-chromatic.*

Indeed, this conjecture implies Hedetniemi's and the difficulty of conjecture 9 as an approach to Hedetniemi's lies in the fact that we have to look at all n-colorings of the graphs G and G'. We show easily that Khelladi's conjecture is true for bipartite graph, n-partite complete graphs. Corollary 5 gives a proof of Khelladi's conjecture whenever $n = 3$.

A weaker version of Conjecture 9 that also implies Hedetniemi's conjecture is the following:

Conjecture 10. *For any graphs* G *and* H *satisfying* $\chi(G) > n$ *and* $\chi(H) > n$ *and* $p \geq Max\{\chi(G), \chi(H)\}$*, there exist* p*-colorings* $g : G \longrightarrow K_p$ *and* $h : H \longrightarrow K_p$ *such that* $\chi(G \times_{K_p} H) > n$*.*

We propose here an equivalent formulation of Conjecture 10 based on graph colorings as given by El-Zahar and Sauer.

Let $p \geq \chi(G)$. For any homomorphism $h : G \longrightarrow K_p$, let $C_n(G, h)$ be the graph with vertex set $V(C_n(G, h)) = \cup A_i$, $i = 1, ..., p$, where

$$A_i = \{ \ \varphi \ / \ \varphi : V_i = \{h^{-1}(i)\} \ \longrightarrow K_n \ \},$$

and edge set

$$E(C_n(G, h)) = \{ \ \varphi\varphi', \varphi \in A_i, \ \varphi' \in A_j \ / \ \forall \ (u, v) \in V_i \times V_j \ \mid uv \in E(G) \\ \text{we have} \varphi(u) \neq \varphi'(v)\}.$$

Observe first that $V_i = \{h^{-1}(i)\}$ is a graph without loops since the sets $V_i = \{h^{-1}(i)\}$ are independent. Therefore, the sets A_i are independent in $C_n(G, h)$. Thus $p \geq \chi(C_n(G, h))$.

Moreover, if $p = \chi(G)$, then $\chi(C_n(G, h)) \geq Min\ \{n, p\}$. Indeed, if $p < n$, the p constant maps φ_i from V_i to K_n induce a complete subgraph of order p. This follows by the fact that for every $(u, v) \in V_i \times V_j$ such that $uv \in E(G)$, we have $\varphi_i(u) \neq \varphi_j(v)$ since h is an optimal coloration of G . Now if $n \leq p$, we will have a complete subgraph of order n formed by the n constant maps, and hence $\chi(C_n(G, h)) \geq Min\ \{n, p\}$. On the other hand, the mapping c defined by $c(\varphi) = i$, for all $\varphi \in A_i$ and all $i = 1, ..., p$, is a p-coloring of $C_n(G, h)$.

We give below a conjecture equivalent to Conjecture 10.

Conjecture 11. *If* $\chi(G) > n$*, then for every map* $g : G \longrightarrow K_p$ *with* $\chi(G) = p$ *we have* $\chi(C_n(G, g)) = n$*.*

Proposition 12. *The Conjectures 10 and 11 are equivalent.*

Proof. Conjecture 11 $\Longrightarrow$ Conjecture 10.

Let G and H be a counterexample of Conjecture 10. One may assume that $\chi(G) = \chi(H) = p$ and let $g : G \longrightarrow K_p$, $h : H \longrightarrow K_p$ and $\chi(G \times_{K_p} H) = n < p$. We will show that $\chi(C_n(G, g) \geq p > n$.

Let c be an n-coloring of $G \times_{K_p} H$ and consider the graph $C_n(G, g)$ with $\{g^{-1}(i)\} = V_i \subseteq V(G)$. For every vertex of H, let $\varphi_x \in A_{h(x)}$ be the map from $V_{h(x)}$ into K_n such that $\varphi_x(u) = c(u, x)$, and let ϕ be the map from H into $C_n(G, g)$ which associates to x the map φ_x.

Let xy an edge of H. Then $h(x) \neq h(y)$. Let $(u, v) \in V_{h(x)} \times V_{h(y)}$ and uv an edge of G. Then $g(u) = h(x)$, $g(v) = h(y)$ and since $xy \in E(H)$ and $uv \in E(G)$ imply that $(u, x)(v, y)$ is an edge of $G \times_{K_p} H$, we have $c(u, x) \neq c(v, y)$. Therefore ϕ is an homomorphism which implies $\chi(C_n(G, g)) \geq \chi(H) = p > n$.

Conjecture 10 $\Longrightarrow$ Conjecture 11.

Assume that there is a counterexample of Conjecture 11, that is, G a graph with $\chi(G) > n$ and there exist a map $g : G \longrightarrow K_p$ with $p = \chi(G)$ such that $\chi(C_n(G, g)) \neq n$. Since $\chi(C_n(G, h)) \geq Min\ \{n, p\}$ and $p > n$, then $\chi(C_n(G, g)) > n$.

Let h be a p-coloring of $C_n(G, g)$ such that $h(\varphi) = i$ if and only if $\varphi \in A_i$. Let us show that $\chi(G \times_{K_p} C_n(G, g)) \leq n$. Let $\varphi \in A_i$. By the definition of the fiber product, for $(u, \varphi) \in G \times_{K_p} C_n(G, g)$ we have $g(u) = h(\varphi) = i$, and by the definition of the map g, $u \in g^{-1}(i)$. Let c be the map from $G \times_{K_p} C_n(G, g)$ into K_n defined by $c(u, \varphi) = \varphi(u)$, and let $(u, \varphi)(v, \varphi')$ an edge of $G \times_{K_p} C_n(G, g)$ where $\varphi \in A_i$, $\varphi' \in A_j$. This means that uv and $\varphi\varphi'$ are edges of G and $C_n(G, g)$ respectively, with $g(u) = \phi(\varphi)$ and $g(v) = \phi(\varphi')$. This is equivalent to saying for $i \neq j$, $\forall (u, v) \in g^{-1}(i) \times g^{-1}(j)$ such that $uv \in E(G)$ we have $\varphi(u) \neq \varphi'(v)$ and so $C(u, \varphi) \neq C(v, \varphi')$. Thus C is an n-coloring of $G \times_{K_p} C_n(G, g)$. $\square$

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