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Vertex-Disjoint Cycles Containing Specified Vertices in a Graph

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Abstract

Let $k \geq 2$ and $n \geq 1$ be integers, and let G be a graph of order n with minimum degree at least $k+1$. Let $v_1, v_2, \dots v_k$ be k distinct vertices of G, and suppose that there exist vertex-disjoint cycles $C_1, C_2, \dots, C_k$ in G such that $\sum_{i=1}^{k} |V(C_i)| \geq \frac{399}{100}k$ and $v_i \in V(C_i)$ for each $1 \leq i \leq k$. Suppose further that the minimum value of the sum of the degrees of two nonadjacent distinct vertices is greater than or equal to n. Under these assumptions, we show that there exist vertex-disjoint cycles $D_1, D_2, \dots, D_k$ such that $v_i \in V(D_i)$ for each $1 \leq i \leq k$ and $V(G) = \bigcup_{i=1}^{k} V(D_i)$.

Keywords: Vertex-disjoint cycle, Specified vertex.

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1. Introduction

In this paper, all graphs considered are finite, undirected, simple graphs with no loops and no multiple edges. Let $G = (V(G), E(G))$ be a graph. For $v \in V(G)$, we let $d_G(v)$ denote the degree of v in G, and we define $\delta(G)$ by $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$. We further define $\sigma_2(G)$ by $\sigma_2(G) = \min\{d_G(x) + d_G(y) \mid x, y \in V(G), xy \notin E(G)\}$. (when G is a complete graph, we let $\sigma_2(G) = \infty$).

The following theorem appears in [1].

Theorem 1.1. *Let k, n be integers with $k \geq 2$ and $n \geq 3k$, and let G be a graph of order n such that $\sigma_2(G) \geq n$, $\delta(G) \geq k+1$ and $\sigma_2(G) + \delta(G) \geq n + 3k - 2$. Let $v_1, v_2, \dots, v_k$ be distinct vertices of G, and suppose that there exist vertex-disjoint cycles $C_1, C_2, \dots, C_k$ such that $v_i \in V(C_i)$ for each $1 \leq i \leq k$. Then there exist vertex-disjoint cycles $D_1, D_2, \dots, D_k$ such that $v_i \in V(D_i)$ for each $1 \leq i \leq k$ and $V(G) = \bigcup_{i=1}^{k} V(D_i)$.*

In Theorem 1.1, the condition $\sigma_2(G) + \delta(G) \geq n + 3k - 2$ is of technical nature, and it was once conjectured that Theorem 1.1 holds even if we drop the condition $\sigma_2(G) + \delta(G) \geq n + 3k - 2$. In connection with the conjecture, Sakai[2] proved the following theorem, which says that if we replace the condition $\sigma_2(G) \geq n$ by the stronger condition $\sigma_2(G) \geq n + \frac{k-4}{3}$, then we can drop the condition $\sigma_2(G) + \delta(G) \geq n + 3k - 2$.

Theorem 1.2. *Let k, n be integers with $k \geq 2$ and $n \geq 3k$, and let G be a graph of order n such that $\sigma_2(G) \geq n + \frac{k-4}{3}$ and $\delta(G) \geq k + 1$. Let $v_1, v_2, \ldots, v_k$ be distinct vertices of G, and suppose that there exist vertex-disjoint cycles $C_1, C_2, \ldots, C_k$ such that $v_i \in V(C_i)$ for each $1 \leq i \leq k$. Then there exist vertex-disjoint cycles $D_1, D_2, \ldots, D_k$ such that $v_i \in V(D_i)$ for each $1 \leq i \leq k$ and $V(G) = \bigcup_{i=1}^k V(D_i)$.*

In [2], it is also shown that the condition $\sigma_2(G) \geq n + \frac{k-4}{3}$ is best possible. This means that for $k \geq 5$, the conjecture mentioned in the paragraph preceding Theorem 1.2 is false (note that for $2 \leq k \leq 4$, the condition $\sigma_2(G) \geq n + \frac{k-4}{3}$ is equiequivalent to the condition $\sigma_2(G) \geq n$, and hence the conjecture holds). The purpose of this paper is to show that the conjecture is true in the case where $\sum_{i=1}^k |V(C_i)|$ is somewhat large. Specifically we prove the following theorem.

Theorem 1.3. *Let k, n be integers with $k \geq 2$ and $n \geq 3k$, and let G be a graph of order n such that $\sigma_2(G) \geq n$ and $\delta(G) \geq k + 1$. Let $v_1, v_2, \ldots, v_k$ be distinct vertices of G, and suppose that there exist vertex-disjoint cycles $C_1, C_2, \ldots, C_k$ such that $\sum_{i=1}^k |V(C_i)| \geq 4k$ and $v_i \in V(C_i)$ for each $1 \leq i \leq k$. Then there exist vertex-disjoint cycles $D_1, D_2, \ldots, D_k$ such that $v_i \in V(D_i)$ for each $1 \leq i \leq k$ and $V(G) = \bigcup_{i=1}^k V(D_i)$.*

As is shown in [2], the lower bound $k + 1$ on $\delta(G)$ is best possible. On the other hand, the lower bound $4k$ on $\sum_{i=1}^k |V(C_i)|$ is not best possible. In fact, we also prove the following theorem.

Theorem 1.4. *Let k, n be integers with $k \geq 2$ and $n \geq 3k$, and let G be a graph of order n such that $\sigma_2(G) \geq n$ and $\delta(G) \geq k + 1$. Let $v_1, v_2, \ldots, v_k$ be distinct vertices of G, and suppose that there exist vertex-disjoint cycles $C_1, C_2, \ldots, C_k$ such that $\sum_{i=1}^k |V(C_i)| \geq \frac{399}{100}k$ and $v_i \in V(C_i)$ for each $1 \leq i \leq k$. Then there exist vertex-disjoint cycles $D_1, D_2, \ldots, D_k$ such that $v_i \in V(D_i)$ for each $1 \leq i \leq k$ and $V(G) = \bigcup_{i=1}^k V(D_i)$.*

The lower bound $\frac{399}{100}k$ will still not be best possible, but it seems difficult even to determine the best possible coefficient of k in the lower bound on $\sum_{i=1}^k |V(C_i)|$.

Our notation is standard except possibly for the following. For a vertex v of a graph G, the neighborhood of v in G is denoted by $N_G(v)$; thus $d_G(v) = |N_G(v)|$. For a subgraph H of G and a vertex v of G with $v \in V(G) - V(H)$, let $N_G(v) \cap V(H)$ be denoted by $N_H(v)$, and let $d_H(v) = |N_H(v)|$. For a subgraph H of G and a subset S of $V(G) - V(H)$, we let $N_H(S) = \bigcup_{x \in S} N_H(x)$. For a subset S of $V(G)$, we let $\langle S \rangle_G$

denote the subgraph induced by S in G, and let $G-S=\langle V(G)-S\rangle_G$. For a subgraph H of G, we write $G-H$ for $G-V(H)$. For disjoint subsets X, Y of $V(G)$, we let $E_G(X,Y)$ denote the set of edges of G between X and Y.

Cycles and paths are considered to have a fixed direction. For a cycle $C=x_1x_2\cdots x_nx_1$ and for $x_i,x_j\in V(C)$ with $i<j<i+n$, we define segments $C[x_i,x_j]$ and $C^-[x_i,x_j]$ of C by $C[x_i,x_j]=x_ix_{i+1}\cdots x_{j-1}x_j$ and $C^-[x_i,x_j]=x_ix_{i-1}\cdots x_{j+1}x_j$ (indices are to be read modulo n). Further we define $C[x_i,x_j)=C[x_i,x_j]-\{x_j\}$, $C(x_i,x_j]=C[x_i,x_j]-\{x_i\}$ and $C(x_i,x_j)=C[x_i,x_j]-\{x_i,x_j\}$. For $v\in V(C)$, we let v^+ (resp. v^-) denote the successor (resp. the predecessor) of v on C, and let $v^{++}=(v^+)^+$ and $v^{--}=(v^-)^-$; thus if $v=x_i$, then $v^+=x_{i+1}$, $v^-=x_{i-1}$, $v^{++}=x_{i+2}$ and $v^{--}=x_{i-2}$. For a path $P=x_1x_2\cdots x_{n-1}x_n$, and for $x_i,x_j\in V(P)$ with $1\le i<j\le n$, we let $P[x_i,x_j]=x_ix_{i+1}\cdots x_{j-1}x_j$, $P^-[x_j,x_i]=x_jx_{j-1}\cdots x_{i+1}x_i$, $P[x_i,x_j)=P[x_i,x_j]-\{x_j\}$, $P(x_i,x_j]=P[x_i,x_j]-\{x_i\}$ and $P(x_i,x_j)=P[x_i,x_j]-\{x_i,x_j\}$.

2. Admissible Paths

This section contains results concerning two disjoint paths in a graph. Results proved in this section are not used in the proof of Theorem 1.3, and thus readers not interested in the proof of Theorem 1.4 are advised to skip this section. Before stating results about two paths, we prove three easy lemmas concerning a path.

Lemma 2.1. *Let R be a path with $|V(R)|\ge 2$, and let $v\in V(R)$. Let M, N be subsets of $V(R)$ with $|M|,|N|\ge 2$. Then there exists a segment T of R with one endvertex in M and the other in N such that either $v\in V(T)$ and $|V(T)|\ge 2$, or $v\notin V(T)$ and v is not an isolated vertex in $R-T$ (we allow the possibility that $|V(T)|=1$).*

Proof. Write $R=r_1r_2\cdots r_\alpha$. Take $r_a\in M$ and $r_b\in N$ so that $|b-a|$ is as large as possible. By the symmetry of the role of M and N, we may assume $a\le b$. Then $|V(R[r_a,r_b])|\ge 2$. Thus we may assume that v is an isolated vertex in $R-R[r_a,r_b]$. By reversing the direction of R if necessary, we may assume $v=r_1$ and $r_a=r_2$. By maximality of $|b-a|$, $r_1\notin M$. Take $r_c\in M-\{r_2\}$, and let $T=R[r_c,r_b]$ or $R[r_b,r_c]$ according as $c\le b$ or $c>b$. Then $v\notin V(T)$ and v is not an isolated vertex in $R-T$. □

Lemma 2.2. *Let R be a path with $|V(R)|\ge 2$, and let $v\in V(R)$. Let M, N be subsets of $V(R)$ with $|M|,|N|\ge 2$. Then there exists a segment T of R with one endvertex in M and the other in N such that $V(R)-V(T)\ne\emptyset$, and either $v\in V(T)$ or $v\notin V(T)$ and v is not an isolated vertex in $R-T$ (we allow the possibility that $|V(T)|=1$).*

Proof. Fix $r_c\in M$ (if $v\in M$, we let $r_c=v$). Take $r_b\in N$ so that $|b-c|$ is as small as possible, and let $T'=R[r_c,r_b]$ or $R[r_b,r_c]$ according as $c\le b$ or $c>b$. Then $V(R)-V(T')\ne\emptyset$. Hence if v is not an endvertex of R or $v=r_c$, then T' has the desired properties. Thus we may assume $v=r_1\notin M$. Similarly we may assume $r_1\notin N$.

Now take $r_a \in M - \{r_2\}$ and $r_b \in N - \{r_2\}$, and let $T = R[r_a, r_b]$ or $R[r_b, r_a]$. Then T has the desired properties. □

Lemma 2.3. *Let R be a path with $|V(R)| \geq 2$, and let $v, x, y \in V(R)$ with $x \neq y$. Then there exists $z \in \{x, y\}$ such that v is not an isolated vertex in $R - \{z\}$ (we allow z to be equal to v).*

Proof. As in the proof of Lemma 2.1, we may assume $v = r_1$. Thus if we take $z \in \{x, y\} - \{r_2\}$, then z has the desired property. □

In the rest of this section, we let P, Q be paths in a graph H such that $|V(P)| \geq 3$, $|V(Q)| \geq 3$, $V(P) \cap V(Q) = \emptyset$ and $V(P) \cup V(Q) = V(H)$, and let $l = |V(P)|$ and $m = |V(Q)|$, and write $P = p_1 p_2 \cdots p_l$ and $Q = q_1 q_2 \cdots q_m$. Further we fix $u \in V(P)$ and $w \in V(Q)$. We sometimes refer to u and w as malignant vertices (see the last paragraph of the proof of Lemma 2.4; see also the first paragraph of Section 4). We consider the following five conditions concerning a (p_1, p_l)-path P' and a (q_1, q_m)-path Q'.

(i) $V(P') \cap V(Q') = \emptyset$ and $(V(P) \cup V(Q)) - (V(P') \cup V(Q')) \neq \emptyset$.

(ii)
- (a) If $u \notin V(P') \cup V(Q')$ and $w \in V(P') \cup V(Q')$, then u is not an isolated vertex in $P - (V(P') \cup V(Q'))$.
- (b) If $u \in V(P') \cup V(Q')$ and $w \notin V(P') \cup V(Q')$, then w is not an isolated vertex in $Q - (V(P') \cup V(Q'))$.
- (c) If $u, w \notin V(P') \cup V(Q')$, then at least one of u and w is not an isolated vertex in $(P \cup Q) - (V(P') \cup V(Q'))$.

(iii) $|V(P')| \geq 3$, $|V(Q')| \geq 3$.

(iv) If $V(P') \cap \{u, w\} \neq \emptyset$ and $|V(P')| = 3$, or $V(Q') \cap \{u, w\} \neq \emptyset$ and $|V(Q')| = 3$, then $|(V(P) \cup V(Q)) - (V(P') \cup V(Q'))| \geq 2$.

(v) If u is an endvertex of P, then the following hold.
- (a) If $w \in V(P')$, then $|V(P')| \geq 4$.
- (b) If $w \in V(Q')$, then $|V(Q')| \geq 4$.
- (c) If $w \in V(P')$ and $|V(P')| = 4$, or $w \in V(Q')$ and $|V(Q')| = 4$, then $|(V(P) \cup V(Q)) - (V(P') \cup V(Q'))| \geq 2$.

We say that P', Q' are strongly $(P, Q; u, w)$-admissible if P' and Q' satisfy (i), (ii), (iii), (iv), (v); we say that P', Q' are $(P, Q; u, w)$-admissible if P' and Q' satisfy (i), (ii), (iii), (iv). Note that the condition (v) is not symmetric with respect to P and Q. Note also that in the case where u is not an endvertex of P, the notion of strongly $(P, Q; u, w)$-admissible paths is no different from that of $(P, Q; u, w)$-admissible paths.

We now prove four lemmas concerning the existence of admissible paths.

Lemma 2.4. *Suppose that* $l \geq 4$, $m \geq 4$, *and* u *is an endvertex of* P . *Suppose further that* $|E_H(V(P), V(Q))| \geq 3(|V(P)|+|V(Q)|-3)$. *Then* H *contains* $(P, Q; u, w)$ *-admissible paths.*

Proof. We first consider the case where $l = 4$ or $m = 4$.

Claim 2.1. *Suppose that* $m = 4$. *Then the conclusion of the lemma holds.*

Proof. Set $X = \{x \in V(P) \,|\, d_Q(x) = 4\}$ and $Y = \{x \in V(P) \,|\, d_Q(x) = 3\}$. Since $|E_H(V(P), V(Q))| \geq 3|V(P)| + 3$ by assumption, we have $|X| \geq 3$ and $|X \cup Y| \geq 4$. Take $Z \subseteq X \cup Y$ with $|Z| = 4$ and $|Z \cap X| \geq 3$, and write $Z = \{p_a, p_b, p_c, p_d\}$ with $a < b < c < d$. By reversing the direction of P if necessary, we may assume $p_d \in X$. By reversing the direction of Q if necessary, we may also assume $q_2 \in N_Q(p_a)$. Take $z \in \{p_b, p_c\} \cap X$. Assume first that $w \in \{q_2, q_3\}$. Let $P' = P[p_1, p_a]q_2q_3P[p_d, p_l]$ and $Q' = q_1zq_4$. Then $u, w \in V(P')$ and $|V(P')| \geq 4$, and hence P' , Q' are $(P, Q; u, w)$ - admissible. Assume now that $w \in \{q_1, q_4\}$. Let $P' = P[p_1, p_a]q_2P[p_d, p_l]$ and $Q' = q_1zq_4$. Then $|(V(P) \cup V(Q)) - (V(P') \cup V(Q'))| \geq |\{p_b, p_c, q_3\} - \{z\}| = 2$, and hence P' , Q' are $(P, Q; u, w)$ -admissible. □

Claim 2.2. *Suppose that* $l = 4$. *Then the conclusion of the lemma holds.*

Proof. Set $X = \{x \in V(Q) \,|\, d_P(x) = 4\}$ and $Y = \{x \in V(Q) \,|\, d_P(x) = 3\}$. Then $|X| \geq 3$ and $|X \cup Y| \geq 4$. Take $Z \subseteq X \cup Y$ with $|Z| = 4$ and $|Z \cap X| \geq 3$, and write $Z = \{q_a, q_b, q_c, q_d\}$ with $a < b < c < d$. Assume first that $w \in V(Q(q_a, q_d))$. By Lemma 2.3, there exists $z \in \{q_b, q_c\}$ such that w is not an isolated vertex in $Q(q_a, q_d) - \{z\}$. If $z \notin X$, then by reversing the direction of P if necessary, we may assume $p_1, p_3 \in N_P(z)$; if $z \in X$, then we may similarly assume $p_2 \in N_P(q_a) \cap N_P(q_d)$. In either case, we let $P' = p_1zp_3p_4$ and $Q' = Q[q_1, q_a]p_2Q[q_d, q_m]$. Then $u \in V(P')$, $w \notin V(Q')$ and $|V(P')| = 4$, and hence P' , Q' are $(P, Q; u, w)$ -admissible. Assume now that $w \notin V(Q(q_a, q_d))$. We may assume $p_2 \in N_P(q_a) \cap N_P(q_d)$. Take $z \in \{q_b, q_c\} \cap X$, and let $P' = p_1zp_4$ and $Q' = Q[q_1, q_a]p_2Q[q_d, q_m]$. Then $|(V(P) \cup V(Q)) - (V(P') \cup V(Q'))| \geq |\{p_3, q_b, q_c\} - \{z\}| = 2$, and hence P' , Q' are $(P, Q; u, w)$ -admissible. □

We now complete the proof of the lemma by induction on $l + m$. In view of Claims 2.1 and 2.2, we may assume $l, m \geq 5$. Assume for the moment that $d_P(q_1) \leq 3$ or $d_P(q_m) \leq 3$. By symmetry, we may assume $d_P(q_1) \leq 3$. Then P and $Q - \{q_1\}$ satisfy the assumptions of the lemma (if $w = q_1$, we let $w' = q_2$ be the new malignant vertex on $Q - \{q_1\}$; if $w \neq q_1$, we simply let $w' = w$). Hence $H - \{q_1\}$ contains $(P, Q - \{q_1\}; u, w')$ - admissible paths P' , Q'' . If we let $Q' = qQ''$, P' , Q' are $(P, Q; u, w)$ -admissible. Thus we may assume $d_P(q_1), d_P(q_m) \geq 4$. Similarly we may assume $d_Q(p_1), d_Q(p_l) \geq 4$. Now apply Lemma 2.2 with $R = Q - \{q_1, q_m\}$, $v = w$, $M = N_{Q-\{q_1,q_m\}}(p_1)$ and

$N = N_{Q-\{q_1,q_m\}}(p_l)$, and let T be as in Lemma 2.2 (in the case where $w \in \{q_1, q_m\}$, we apply Lemma 2.2 with v being any fixed vertex on $Q - \{q_1, q_m\}$). By reversing the direction of T if necessary, we may assume that p_1 is adjacent to the initial vertex of T and p_l is adjacent to the terminal vertex of T. Since $d_{P-\{p_1,p_l\}}(q_1), d_{P-\{p_1,p_l\}}(q_m) \geq 2$, there exists a segment S of $P - \{p_1, p_l\}$ with one endvertex adjacent to q_1 and the other adjacent to q_m such that $V(P) - \{p_1, p_l\} - V(S) \neq \emptyset$. We may assume that q_1 and q_m are adjacent to the initial vertex and terminal vertex of S, respectively. Let $P' = p_1 T p_l$ and $Q' = q_1 S q_m$. Then $|(V(P) \cup V(Q)) - (V(P') \cup V(Q'))| = |V(P - \{p_1, p_l\}) - V(S)| + |V(Q - \{q_1, q_m\}) - V(T)| \geq 2$, and hence P', Q' are $(P, Q; u, w)$-admissible.

□

Lemma 2.5. *Suppose that $l \geq 5$ and $m \geq 4$. Suppose further that $|E_H(V(P - \{u\}), V(Q))| \geq 3(|V(P - \{u\})| + |V(Q)| - 3)$. Then H contains $(P, Q; u, w)$-admissible paths.*

Proof. We first consider the case where $l = 5$ or $m = 4$.

Claim 2.3. *Suppose that $m = 4$. Then the conclusion of the lemma holds.*

Proof. Set $X = \{x \in V(P - \{u\}) \mid d_Q(x) = 4\}$ and $Y = \{x \in V(P - \{u\}) \mid d_Q(x) = 3\}$. Take $Z \subseteq X \cup Y$ with $|Z| = 4$ and $|Z \cap X| \geq 3$, and write $Z = \{p_a, p_b, p_c, p_d\}$ with $a < b < c < d$. By Lemma 2.3, there exists $z \in \{p_b, p_c\}$ such that u is not isolated vertex in $P(p_a, p_d) - \{z\}$ (even if $u \in V(P(p_a, p_d))$). If $z \notin X$, we may assume $q_1, q_3 \in N_Q(z)$; if $z \in X$, we may assume $q_2 \in N_Q(p_a) \cap N_Q(p_d)$. In either case, we let $P' = P[p_1, p_a] q_2 P[p_d, p_l]$ and $Q' = q_1 z q_3 q_4$. Then $|V(Q')| = 4$. If $u \in V(P(p_a, p_d))$, then since $u \notin X \cup Y$ by the definition of X and Y, we have $d - a \geq 4$, and hence $|V(P) - (V(P') \cup V(Q'))| = (d - a - 1) - 1 \geq 2$; if $u \notin V(P(p_a, p_d))$, then we have $a \geq 2$ or $d \leq l - 1$, and hence $|V(P')| \geq 4$. Consequently P', Q' are $(P, Q; u, w)$-admissible. □

Claim 2.4. *Suppose that $l = 5$ and u is not an endvertex of P. Then the conclusion of the lemma holds.*

Proof. Set $X = \{x \in V(Q) \mid d_{P-\{u\}}(x) = 4\}$. Take $Z \subseteq X$ with $|Z| = 3$ and write $Z = \{q_a, q_b, q_c\}$ with $a < b < c$. By Lemma 2.3, there exists $z \in \{p_2, p_3, p_4\} - \{u\}$ such that u is an isolated vertex in $P - \{p_1, p_5\} - \{z\}$. Let $P' = p_1 q_b p_5$ and $Q' = Q[q_1, q_a] z Q[q_c, q_m]$. Then $u \in V(P - (V(P') \cup V(Q')))$ and u is not an isolated vertex in $P - (V(P') \cup V(Q'))$. This implies that $|V(P) - (V(P') \cup V(Q'))| = 2$. This also implies that the condition (ii) in the definition of the term "*admissible*" is satisfied even if w is an isolated vertex in $Q - (V(P') \cup V(Q'))$. Consequently P', Q' are $(P, Q; u, w)$-admissible. □

We complete the proof of the lemma by induction. If u is an endvertex of P, we obtain desired paths by applying Lemma 2.4 to $P - \{u\}$ and Q. Thus we may assume u is not an endvertex of P. In view of Claims 2.3 and 2.4, we may also assume $l \geq 6$ and $m \geq 5$. If $d_Q(p_1) \leq 3$ or $d_Q(p_l) \leq 3$, we obtain desired paths by applying the induction hypothesis to

$P-\{p_1\}$ and Q or $P-\{p_l\}$ and Q (note that we can apply the induction hypothesis even if u is an endvertex of $P-\{p_1\}$ or $P-\{p_l\}$). Thus we may assume $d_Q(p_1), d_Q(p_l) \geq 4$. Similarly we may assume $d_{P-\{u\}}(q_1), d_{P-\{u\}}(q_m) \geq 4$. By Lemma 2.2, there exists a segment S of $P-\{p_1,p_l\}$ with one endvertex adjacent to q_1 and the other adjacent to q_m such that $V(P-\{p_1,p_l\})-V(S) \neq \emptyset$ and u is not an isolated vertex in $P-\{p_1,p_l\}-V(S)$, and there exists a segment T of $Q-\{q_1,q_m\}$ with one endvertex adjacent to p_1 and the other adjacent to p_l and such that $V(Q-\{q_1,q_m\})-V(T) \neq \emptyset$ and w is not an isolated vertex in $Q-\{q_1,q_m\}-V(T)$. We may assume that q_1 and q_m are adjacent to the initial vertex and the terminal vertex of S, respectively, and that p_1 and p_l are adjacent to the initial vertex and the terminal vertex of T, respectively. Let $P' = p_1Tp_l$ and $Q' = q_1Sq_m$. Then $|(V(P)\cup V(Q)) - (V(P')\cup V(Q'))| = |V(P-\{p_1,p_l\})-V(S)| + |V(Q-\{q_1,q_l\})-V(T)| \geq 2$, and hence P', Q' are $(P,Q;u,w)$-admissible. $\square$

Lemma 2.6. *Suppose that $l \geq 4$, $m \geq 4$, $l+m \geq 10$, and u is an endvertex of P. Suppose further that $|E_H(V(P),V(Q))| \geq 3(|V(P)|+|V(Q)|-3)$. Then H contains strongly $(P,Q;u,w)$-admissible paths.*

Proof. We first consider the case where $l=4$ or $m=4$. For a technical reason, we find it convenient to include the case where $l+m=9$. Specifically we prove the following two claims.

Claim 2.5. *Suppose that $m=4$ and $l \geq 5$. In the case where $l=5$, suppose further that $d_Q(p_1)+d_Q(p_l) \geq 7$. Then the conclusion of the lemma holds.*

Proof. Set $X = \{x \in V(P) \,|\, d_Q(x)=4\}$ and $Y = \{x \in V(P) \,|\, d_Q(x)=3\}$. Then $|X| \geq 3$, and we also have $|X \cup Y| \geq 5$ in the case where $l \geq 6$. We divide proof into the following two cases.

Case 1. $w \in \{q_2,q_3\}$.

By symmetry, we may assume $w=q_2$. Take $p_a,p_b,p_c \in X$ with $a<b<c$. Assume first that $c-a=2$. Let $P' = P[p_1,p_a]q_2P[p_c,p_l]$ and $Q' = q_1p_bq_4$. Then $u,w \in V(P')$ and $|V(P')| \geq 5$, and hence P', Q' are strongly $(P,Q;u,w)$-admissible. Assume now that $c-a \geq 3$. Let $P' = P[p_1,p_a]q_2q_3P[p_c,p_l]$ and $Q' = q_1p_bq_4$. Then $u,w \in V(P')$ and $|V(P')| \geq 4$. Further if $|V(P')|=4$, then $p_a=p_1$ and $p_c=p_l$, and hence $|V(P)-(V(P')\cup V(Q'))| = l-3 \geq 2$. Consequently P', Q' are strongly $(P,Q;u,w)$-admissible.

Case 2. $w \in \{q_1,q_4\}$.

We show that there exist $p_a,p_c \in X\cup Y$ with $c-a \geq 4$ such that $\{p_a,p_c\}\cap X \neq \emptyset$ and $V(P(p_a,p_c))\cap X \neq \emptyset$. If $l=5$, then let $p_a=p_1$ and $p_c=p_5$. If $l \geq 6$ and $|X| \geq 4$, then take $Z \subseteq X\cup Y$ with $|Z|=5$ and $|Z\cap X| \geq 4$, write $Z=\{z_1,\dots,z_5\}$ so that $z_1,\dots,z_5$ occur on P in this order, and let $p_a=z_1$ and $p_c=z_5$. Assume now that $l \geq 6$ and

$|X| = 3$. Then $V(P) = X \cup Y$. Take $Z \subseteq V(P)$ with $|Z| = 6$ and $Z \supseteq X$, and write $Z = \{z_1, \ldots, z_6\}$ so that $z_1, \ldots, z_6$ occur on P in this order. If $\{z_1, z_6\} \cap X \neq \emptyset$, then let $p_a = z_1$ and $p_c = z_6$; if $z_1, z_6 \in Y$, then noting that $\{z_2, z_5\} \cap X \neq \emptyset$, let $p_a = z_2$ and $p_c = z_6$ if $z_2 \in X$, and let $p_a = z_1$ and $p_c = z_5$ if $z_2 \notin X$. Thus in any case, there exist such p_a, p_c. Then $N_Q(p_a) \cap N_Q(p_c) \cap \{q_2, q_3\} \neq \emptyset$. We may assume $q_2 \in N_Q(p_a) \cap N_Q(p_c)$. Now take $p_b \in V(P(p_a, p_c)) \cap X$, and let $P' = P[p_1, p_a]q_2P[p_c, p_l]$ and $Q' = q_1p_bq_3q_4$. Then $w \in V(Q')$, $|V(Q')| = 4$, and $|V(P) - (V(P') \cup V(Q'))| = (c - a - 1) - 1 \geq 2$. Hence P', Q' are strongly $(P, Q; u, w)$-admissible. □

Claim 2.6. *Suppose that $l = 4$ and $m \geq 5$. In the case where $m = 5$, suppose further that $d_P(q_1) \geq 3$ and $d_P(q_m) \geq 3$. Then the conclusion of the lemma holds.*

Proof. Set $X = \{x \in V(Q) \mid d_P(x) = 4\}$ and $Y = \{x \in V(Q) \mid d_P(x) = 3\}$. Then $|X| \geq 3$, and we have $|X| + |Y| \geq 5$ in the case where $l \geq 6$. If $l \geq 6$, then take $Z \subseteq X \cup Y$ with $|Z| = 5$ and $|Z \cap X| \geq 3$, and write $Z = \{q_a, q_b, q_c, q_d, q_e\}$ with $a < b < c < d < e$; if $l = 5$, simply write $Q = q_aq_bq_cq_dq_e$.

Case 1. $w \in V(Q(q_a, q_e))$.

Subcase 1.1. $d_P(q_a) = d_P(q_e) = 3$.

In this case, $\{q_b, q_c, q_d\} \subseteq X$. Applying Lemma 2.3 with $R = Q(q_a, q_e)$, $v = w$ and $\{x, y\} \subseteq \{q_b, q_c, q_d\} - \{w\}$, we see that there exists $z \in \{q_b, q_c, q_d\} - \{w\}$ such that w is not an isolated vertex in $Q(q_a, q_e) - \{z\}$. Let $P' = p_1zp_4$. Note that $N_P(q_a) \cap \{p_2, p_3\} \neq \emptyset$ and $N_P(q_e) \cap \{p_2, p_3\} \neq \emptyset$. By symmetry, we may assume $p_2 \in N_P(q_a)$. Now let $Q' = Q[q_1, q_a]p_2Q[q_e, q_m]$ or $Q' = Q[q_1, q_a]p_2p_3Q[q_e, q_m]$ according as $p_2 \in N_P(q_e)$ or $p_2 \notin N_P(q_e)$. Then $w \notin V(P') \cup V(Q')$ and $|V(Q) - (V(P') \cup V(Q'))| \geq 2$, and hence P', Q' are strongly $(P, Q; u, w)$-admissible.

Subcase 1.2. $d_P(q_a) + d_P(q_e) \geq 7$ and $|\{q_b, q_c, q_d\} \cap X| \geq 2$.

We may assume $p_2 \in N_P(q_a) \cap N_p(q_e)$. By Lemma 2.2, there exists $z \in \{q_b, q_c, q_d\} \cap X$ such that w is not an isolated vertex in $Q(q_a, q_e) - \{z\}$. Let $P' = p_1zp_3p_4$ and $Q' = Q[q_1, q_a]p_2Q[q_e, q_m]$. Then $w \notin V(Q')$ and $|V(P')| = 4$, and $|V(Q) - (V(P') \cup V(Q'))| \geq 2$, and hence P', Q' are strongly $(P, Q; u, w)$-admissible.

Subcase 1.3. $d_P(q_a) + d_P(q_e) \geq 7$ and $|\{q_b, q_c, q_d\} \cap X| = 1$.

In this case, $d_P(q_a) = d_P(q_e) = 4$ and $\{q_b, q_c, q_d\} \subseteq X \cup Y$. By Lemma 2.3, there exists $z \in \{q_b, q_c, q_d\} - \{w\}$ such that w is not an isolated vertex in $Q(q_a, q_e) - \{z\}$. By symmetry, we may assume $p_1, p_3 \in N_P(z)$. Let $P' = p_1zp_3p_4$ and $Q' = Q[q_1, q_a]p_2Q[q_e, q_m]$. Then $w \notin V(P') \cup V(Q')$ and $|V(Q) - (V(P') \cup V(Q'))| \geq 2$, and hence P', Q' are strongly $(P, Q; u, w)$-admissible.

Case 2. $w \notin V(Q(q_a, q_e))$.

Subcase 2.1. $d_P(q_a) = d_P(q_e) = 4$.

Take $z \in \{q_b, q_c, q_d\} \cap X$, and let $P' = p_1 z p_4$ and $Q' = Q[q_1, q_a]p_2 p_3 Q[q_e, q_m]$. Then $w \in V(Q')$ and $|V(Q')| \geq 4$, and $|V(Q) - (V(P') \cup V(Q'))| \geq 2$, and hence P' , Q' are strongly $(P, Q; u, w)$ -admissible.

Subcase 2.2. $d_P(q_a) + d_P(q_e) \leq 7$.

In this case, $|\{q_b, q_c, q_d\} \cap X| \geq 2$. We also have $d_P(q_a) + d_P(q_d) \geq 7$ or $d_P(q_b) + d_P(q_e) \geq 7$. We may assume $d_P(q_a) + d_P(q_d) \geq 7$. We may also assume $p_2 \in N_P(q_a) \cap N_P(q_d)$. Now take $z \in \{q_b, q_c\} \cap X$, and let $P' = p_1 z p_4$ and $Q' = Q[q_1, q_a]p_2 Q[q_d, q_m]$. Then $w \in V(Q')$ and $|V(Q')| \geq 4$, and $|(V(P) \cup V(Q)) - (V(P') \cup V(Q'))| \geq |\{p_3, q_b, q_c\} - \{z\}| \geq 2$, and hence P' , Q' are strongly $(P, Q; u, w)$ -admissible. □

We turn to the case where $l, m \geq 5$.

Claim 2.7. *Suppose that* $l \geq 5$ *,* $m \geq 5$ *,* $d_P(q_1) \geq 4$ *,* $d_P(q_m) \geq 4$ *,* $d_Q(p_1) \geq 4$ *and* $d_Q(p_l) \geq 4$ *. Then the conclusion of the lemma holds.*

Proof. **Case 1.** $w \notin \{q_1, q_m\}$.

By Lemma 2.1, there exists a segment T of $Q - \{q_1, q_m\}$ with one endvertex adjacent to p_1 and the other adjacent to p_l such that either $w \in V(T)$ and $|V(T)| \geq 2$, or $w \notin V(T)$ and w is not an isolated vertex in $Q - \{q_1, q_m\} - V(T)$. We may assume that p_1 and p_l are adjacent to the initial vertex and the terminal vertex of T , respectively. Let $P' = p_1 T p_l$. Then either $w \in V(P')$ and $|V(P')| \geq 4$, or $w \notin V(P')$ and w is not an isolated vertex in $Q - \{q_1, q_m\} - V(P')$. Now since $|V(P - \{p_1, p_l\})| \geq 3$ and $d_{P-\{p_1,p_l\}}(q_1), d_{P-\{p_1,p_l\}}(q_m) \geq 2$, there exists a segment S of $P - \{p_1, p_l\}$ with one endvertex adjacent to q_1 and the other adjacent to q_m such that $|V(P - \{p_1, p_l\}) - V(S)| \geq 2$. We may assume that q_1 and q_m are adjacent to the initial vertex and the terminal vertex of S , respectively. Let $Q' = q_1 S q_m$. Then $|V(P) - (V(P') \cup V(Q'))| \geq 2$. Consequently P' , Q' are strongly $(P, Q; u, w)$ -admissible.

Case 2. $w \in \{q_1, q_m\}$.

Since $d_{P-\{p_1,p_l\}}(q_1)$, $d_{P-\{p_1,p_l\}}(q_m) \geq 2$, there exists a segment S of $P - \{p_2, p_l\}$ with one vertex adjacent to q_1 and the adjacent to q_m such that $|V(S)| \geq 2$. Also as in Case 1, there exists a segment T of $Q - \{q_1, q_m\}$ with one endvertex adjacent to p_1 and the other adjacent to p_l such that $|V(Q - \{q_1, q_m\}) - V(T)| \geq 2$. We may assume that q_1 and q_m are adjacent to the initial vertex and the terminal vertex of S , respectively, and that p_1 and p_l are adjacent to the initial vertex and the terminal vertex of T , respectively. Let $P' = p_1 T p_l$ and $Q' = q_1 S q_m$. Then $w \in V(Q')$ and $|V(Q')| \geq 4$, and $|V(Q) - (V(P') \cup V(Q'))| \geq 2$, and hence P' , Q' are strongly $(P, Q; u, w)$ -admissible. □

We now focus our attention on the case where $l = m = 5$. We first settle the following special case.

Claim 2.8. *Suppose that* $l = m = 5$, $d_{Q-\{q_1,q_5\}}(p_1) + d_{Q-\{q_1,q_5\}}(p_5) \geq 4$, $d_{P-\{p_1,p_5\}}(q_1) \geq 1$ *and* $d_{P-\{p_1,p_5\}}(q_5) \geq 2$. *Then the conclusion of the lemma holds.*

Proof. **Case 1.** $w \in \{q_2, q_3, q_4\}$.

If $d_{Q-\{q_1,q_5\}}(p_1), d_{Q-\{q_1,q_5\}}(p_5) \geq 2$, then by Lemma 2.1, there exists a segment T of $Q - \{q_1, q_5\}$ with one endvertex adjacent to p_1 and the other adjacent to p_5 such that either $w \in V(T)$ and $|V(T)| \geq 2$, or $w \notin V(T)$ and w is not an endvertex of $Q - \{q_1, q_5\} - V(T)$; if $d_{Q-\{q_1,q_5\}}(p_1) = 1$ and $d_{Q-\{q_1,q_5\}}(p_5) = 3$, or $d_{Q-\{q_1,q_5\}}(p_5) = 1$ and $d_{Q-\{q_1,q_5\}}(p_1) = 3$, then we easily see that there exists a segment T of $Q - \{q_1, q_5\}$ with one endvertex adjacent to p_1 and the other adjacent to p_5 such that $w \in V(T)$ and $|V(T)| \geq 2$. We may assume that p_1 and p_5 are adjacent to the initial vertex of T and the terminal vertex of T, respectively. There also exists a segment S of $P - \{p_1, p_5\}$ with one endvertex adjacent to q_1 and the other adjacent to q_5 such that $|V(S)| \leq 2$. We may assume that q_1 and q_5 are adjacent to the initial vertex of and the terminal vertex of S, respectively. Let $P' = p_1Tp_5$ and $Q' = q_1Sq_5$. Then either $w \in V(P')$ and $|V(P')| \geq 4$ or $w \notin V(P') \cup V(Q')$ and w is not an isolated vertex in $Q - (V(P') \cup V(Q'))$. Further if $|V(P')| \leq 4$, then $|V(T)| \leq 2$, and hence $|(V(P) \cup V(Q)) - (V(P') \cup V(Q'))| = (3 - |V(S)|) + (3 - |V(T)|) \geq 2$. Consequently P', Q' are strongly $(P, Q; u, w)$-admissible.

Case 2. $w \in \{q_1, q_5\}$.

There exists a segment S of $P - \{p_1, p_5\}$ with one endvertex adjacent to q_1 and the other adjacent to q_5 such that $|V(S)| \geq 2$. We may assume that q_1 and q_5 are adjacent to the initial vertex and the terminal vertex of S, respectively. Let $Q' = q_1Sq_5$. Take $z \in N_{Q-\{q_1,q_5\}}(p_1) \cap N_{Q-\{q_1,q_5\}}(p_5)$ and let $P' = p_1zp_5$. Then $w \in V(Q')$ and $|V(Q')| \geq 4$, and $|V(Q) - (V(P') \cup V(Q'))| \geq 2$, and hence P', Q' are strongly $(P, Q; u, w)$-admissible. □

Claim 2.9. *Suppose that* $l = m = 5$. *Then the conclusion of the lemma holds.*

Proof. In view of Claim 2.7, we may assume that one of $d_P(q_1)$, $d_P(q_5)$, $d_Q(p_1)$ and $d_Q(p_5)$ is less than or equal to 3. We first consider the case where $d_Q(p_1) \leq 3$ or $d_Q(p_5) \leq 3$. By symmetry, we may assume $d_Q(p_1) \leq 3$. Then $|E_H(V(P - \{p_1\}), V(Q))| \geq 3(|V(P - \{p_1\})| + |V(Q)| - 3) = 18$. If $d_{P-\{p_1\}}(q_1), d_{P-\{p_1\}}(q_5) \geq 3$, then we can apply Claim 2.6 to $P - \{p_1\}$ and Q to get desired paths. Thus by symmetry, we may assume $d_{P-\{p_1\}}(q_1) \leq 2$. Then $|E_H(V(P - \{p_1\}), V(Q - \{q_1\}))| \geq 16$, and hence equality holds throughout. In particular, $d_Q(p_1) = 3$, $d_{P-\{p_1\}}(q_1) = 2$ and $d_{Q-\{q_1\}}(p_5) = d_{P-\{p_1\}}(q_5) = 4$. Consequently the desired conclusion follows from Claim 2.8.

Thus we may assume $d_Q(p_1), d_Q(p_5) \geq 4$, and hence we have $d_P(q_1) \leq 3$ or $d_P(q_5) \leq 3$. We may assume $d_P(q_1) \leq 3$. Then $|E_H(V(P), V(Q - \{q_1\}))| \geq 18$. Since $d_Q(p_1), d_Q(p_5) \geq 4$, we also have $d_{Q-\{q_1\}}(p_1), d_{Q-\{q_1\}}(p_5) \geq 3$. Now if

$d_{Q-\{q_1\}}(p_1) = 4$ or $d_{Q-\{q_1\}}(p_5) = 4$, then we can apply Claim 2.5 to P and $Q - \{q_1\}$ to get desired paths. Thus we may assume $d_{Q-\{q_1\}}(p_1) = d_{Q-\{q_1\}}(q_5) = 3$. Then $|E_H(V(P - \{p_1, p_5\}), V(Q - \{q_1\}))| \geq 12$, and hence equality holds throughout. In particular, $d_P(q_1) = 3$ and $d_{P-\{p_1,p_5\}}(q_5) = 3$. Consequently the desired conclusion follows from Claim 2.8. □

We can now easily complete the proof of Lemma 2.6 by induction on $l + m$. In view of Claims 2.5, 2.6 and 2.9, we may assume $l \geq 5$, $m \geq 5$ and $l + m \geq 11$. If $d_Q(p_1) \leq 3$ or $d_Q(p_5) \leq 3$, then we can apply the induction hypothesis to $P - \{p_1\}$ and Q or $P - \{p_5\}$ and Q. Thus we may assume $d_Q(p_1), d_Q(p_5) \geq 4$. Similarly we may assume $d_P(q_1), d_P(q_5) \geq 4$. The desired conclusion now follows from Claim 2.7. □

Lemma 2.7. *Suppose that* $l \geq 6$, $m = 3$, $p_1p_l \in E(H)$ *and* $q_1q_3 \in E(H)$.

(i) *If* u *is an endvertex in* P *and* $|E_H(V(P), V(Q))| \geq 2|V(P)|+3$, *then* H *contains paths which are* $(P, q_1q_2q_3; u, w)$ *-admissible or* $(P, q_1q_3q_2; u, w)$ *-admissible.*

(ii) *If* $l \geq 7$ *and* $|E_H(V(P - \{u\}), V(Q))| \geq 2|V(P - \{u\})|+ 3$, *then* H *contains paths which are* $(p_1p_2 \cdots p_{l-1}p_l, q_1q_2q_3; u, w)$ *-admissible, or* $(p_1p_lp_{l-1} \cdots p_2, q_1q_2q_3; u, w)$ *-admissible, or* $(p_1p_2 \cdots p_{l-1}p_l, q_1q_3q_2; u, w)$ *-admissible, or* $(p_1p_lp_{l-1} \cdots p_2, q_1q_3q_2; u, w)$ *-admissible.*

Proof. We prove only (ii) (the proof of (i) is similar to and easier than that of (ii)). By replacing P by $p_1p_lp_{l-1} \cdots p_2$ if necessary, we may assume $u \notin \{p_2, p_{l-1}\}$. Set $X = \{x \in V(P - \{u\}) \mid d_Q(x) = 3\}$ and $Y = \{x \in V(P - \{u\}) \mid d_Q(x) = 2\}$. Then $|X| \geq 3$ and $|X \cup Y| \geq 5$. Our first aim is to show that there exist $p_a, p_b, p_c \in X \cup Y$ with $a < b < c$ and $c - a \geq 4$ such that $\{p_a, p_c\} \cap X \neq \emptyset$, $p_b \in X$ and u is not an isolated vertex in $P(p_a, p_c) - \{p_b\}$. If there exist $y_1, y_2 \in X \cup Y$ such that y_1, y_2 occur on P in this order, $|V(P(y_1, y_2))| \geq 3$, $\{y_1, y_2\} \cap X \neq \emptyset$ and $|V(P(y_1, y_2)) \cap X| \geq 2$, then by Claim 2.3, there exists $z \in V(P(y_1, y_2)) \cap X$ such that u is not an isolated vertex in $P(y_1, y_2) - \{z\}$, and hence we see that there exist vertices with the required properties by letting $p_a = y_1$, $p_b = z$ and $p_c = y_2$. Thus we may assume there exist no such vertices y_1, y_2. Suppose that $|X| \geq 4$. Take $Z \subseteq X \cup Y$ with $|Z| = 5$ and $|Z \cap X| \geq 4$, and write $Z = \{z_1, z_2, \dots, z_5\}$ so that $z_1, z_2, \dots, z_5$ occur on P in this order. Now if we let $y_1 = z_1$ and $y_2 = z_2$, we get $|V(P(y_1, y_2))| \geq 3$, $\{y_1, y_2\} \cap X \neq \emptyset$ and $|V(P(y_1, y_2)) \cap X| \geq 2$. But this contradicts the assumption that there exist no such vertices y_1, y_2. Thus $|X| = 3$. This implies $V(P - \{u\}) = X \cup Y$. Write $X = \{x_1, x_2, x_3\}$ so that x_1, x_2, x_3 occur on P in this order. Suppose that $V(P[p_1, x_1)) - \{u\} \neq \emptyset$. Let $p_d = p_1$ or p_2 according as $u \neq p_1$ or $u = p_1$. Then $p_d \in V(P[p_1, x_1)) \cap Y$. If $|V(P[p_d, x_3)) \cap Y| \geq 2$, we get a contradiction by letting $y_1 = p_d$ and $y_2 = x_3$. Hence $|V(P[p_d, x_3)) \cap Y| \leq 1$, which implies $|V(P(x_3, p_l]) \cap Y| \geq 2$. Consequently we obtain a contradiction by letting $y_1 = x_1$ and letting $y_2 = p_l$ or p_{l-1} according as $u \neq p_l$ or $u = p_l$. Thus $V(P[p_1, x_1)) - \{u\} = \emptyset$. Hence we have $x_1 = p_1$, or $u = p_1$ and $x_1 = p_2$. Similarly $x_3 = p_l$, or $u = p_l$ and

$x_3 = p_{l-1}$. Now if we let $p_a = x_1$, $p_b = x_2$ and $p_c = x_3$, p_a, p_b, p_c have the required properties because $u \notin \{p_2, p_{l-1}\}$. This completes the proof of the assertion that there exist $p_a, p_b, p_c \in X \cup Y$ with $a < b < c$ and $c - a \geq 4$ such that $\{p_a, p_c\} \cap X \neq \emptyset$, $p_b \in X$ and u is not an isolated vertex in $P(p_a, p_c) - \{p_b\}$.

By replacing Q by $q_1 q_3 q_2$ if necessary, we may assume $q_2 \in N_Q(p_a) \cap N_Q(p_c)$. Let $P' = P[p_1, p_a] q_2 P[p_c, p_l]$ and $Q' = q_1 p_b q_3$. Then $|V(P) - (V(P') \cup V(Q'))| \geq 2$, and hence P', Q' are $(P, Q; u, w)$-admissible. □

3. Initial Reduction and the Proof of Theorem 1.3

Throughout the rest of this paper, let k, n, G, v_i and C_i be as in Theorem 1.4. In view of Theorem 1.2, we may assume $k \geq 5$. We may also assume that we have chosen k cycles C_i so that $\sum_{i=1}^{k} |V(C_i)|$ is maximum. Let $L = \langle \bigcup_{i=1}^{k} V(C_i) \rangle_G$, $H = G - L$, and write $|V(L)| = l$. If $V(H) = \emptyset$, then there is nothing to be proved. Thus suppose that

$$V(H) \neq \emptyset. \tag{1}$$

Let H_0 be a connected component of H. The first five claims, Claims 3.1 through 3.5, are proved in [1] and [2].

Claim 3.1. *Let D_i, $1 \leq i \leq k$, be vertex-disjoint subgraphs of L such that D_1 is a cycle or a path (we allow D_1 to consist of a single vertex), and D_i is a cycle for each i with $2 \leq i \leq k$. Suppose that D_i contains exactly one vertex in $\{v_1, v_2, \ldots, v_k\}$ for each i with $1 \leq i \leq k$. Suppose further that $d_L(u) \geq l - \frac{1}{2} \sum_{i=1}^{k} |V(D_i)|$ for all $u \in V(L) - \bigcup_{i=1}^{k} V(D_i)$. Then there exist vertex-disjoint subgraphs D_i^*, $1 \leq i \leq k$, of L such that D_1^* is a cycle or a path according as D_1 is a cycle or a path, D_i^* is a cycle for each i with $2 \leq i \leq k$, $V(D_i^*) \supseteq V(D_i)$ for each i with $1 \leq i \leq k$, and $\bigcup_{i=1}^{k} V(D_i^*) = V(L)$, and such that D_1^* has the same initial vertex and the same terminal vertex as D_1 in the case where D_1 is a path (thus if D_1 consists of a single vertex, then $D_1^* = D_1$).*

Claim 3.2.

(i) $|N_{C_i}(H_0)| \leq 1$ *for each i with $1 \leq i \leq k$.*

(ii) $H = H_0$.

Set $S = \{u \in V(L) \,|\, d_H(u) \geq 2\}$ and write $|S| = s$. By Claim 3.2(i), $|V(C_i) \cap S| \leq 1$ for each i with $1 \leq i \leq k$. Set $J = \{i \,|\, 1 \leq i \leq k, V(C_i) \cap S \neq \emptyset\}$ and $\overline{J} = \{1, 2, \ldots, k\} - J$. Then $|J| = s$. Write $V(C_i) \cap S = \{u_i\}$ for each $i \in J$.

Claim 3.3.

(i) $u_i \neq v_i$ *for each* $i \in J$.

(ii) $|V(H)| > k - s$.

Claim 3.4. $d_L(u) \geq l - s + 1$ *for all* $u \in V(L) - N_L(H)$.

For each $u \in V(L) - N_L(H)$, $d_L(u) \geq l - s + 1 = |(V(L) - \{u\}) - S| + 2$ by Claim 3.4, and hence $|N_G(u) \cap S| \geq 2$. In view of Claims 3.2(i) and 3.3(i), this in particular implies that $s \geq 2$, and

$$N_G(v_i) \cap (S - \{u_i\}) \neq \emptyset \text{ for each } i \in J. \tag{2}$$

Claim 3.5. *There exist no vertex-disjoint subgraphs* P , C'_i , $2 \leq i \leq k$, *in* L *such that*

$$\begin{cases} P \text{ is a path joining two distinct vertices in } \{u_1, u_2, \dots, u_s\},\ C'_i \text{ is a} \\ \text{cycle for each } i \text{ with } 2 \leq i \leq k, \text{ each of } P \text{ and the } C'_i, \text{ contains exactly} \\ \text{one vertex in } \{v_1, v_2, \dots, v_k\}, \text{ and } |(V(P) \cup (\bigcup_{i=2}^{k} V(C'_i))) \cap S| \geq s - 1. \end{cases} \tag{3}$$

Claim 3.6. $d_L(u) \geq l - s$ *for all* $u \in N_L(H) - S$.

Proof. Since $S \neq \emptyset$, we have $|V(H)| \geq 2$. Let $u \in N_L(H) - S$. Then $d_H(u) = 1$ by the definition of S. Write $N_H(u) = \{w\}$. Since $\sum_{h \in V(H)-\{w\}} d_G(h) \leq (|V(H)| - 1)(|V(H)| - 1) + s(|V(H)| - 1) + k - s - 1$ by Claim 3.2, there exists $h \in V(H) - \{w\}$ such that $d_G(h) \leq (|V(H)| - 1) + s + \frac{k-s-1}{|V(H)|-1}$. Then $d_G(h) \leq (|V(H)| - 1) + s$ by Claim 3.3(ii), and hence $d_L(u) = d_G(u) - 1 \geq n - ((|V(H)| - 1) + s) - 1 = l - s$. □

Claim 3.7. $d_L(u) \geq l - k + 1$ *for all* $u \in V(L) - S$.

Proof. If $s \leq k - 1$, then we obtain the desired conclusion simply by combining Claims 3.4 and 3.6; if $s = k$, then $S = N_L(H)$ by Claim 3.2, and hence the desired conclusion immediately follows from Claim 3.4. □

Claim 3.8. *There exist no vertex-disjoint cycles* C'_i , $1 \leq i \leq k$, *in* L *such that*

$$\begin{cases} |V(C'_i) \cap \{v_1, v_2, \dots, v_k\}| = 1 \text{ and } |V(C'_i) \cap S| \leq 1 \text{ for each } 1 \leq i \leq k, \\ s - 2 \leq |(\bigcup_{i=1}^{k} V(C'_i)) \cap S| \leq s - 1, \text{ and there exists } u \in S - (\bigcup_{i=1}^{k} V(C'_i)) \\ \text{such that } u \text{ is not an isolated vertex in } L - (\bigcup_{i=1}^{k} V(C'_i)) - (S - \{u\}). \end{cases} \tag{4}$$

Proof. Suppose that there exist vertex-disjoint cycles C'_i, $1 \leq i \leq k$, in L which satisfy (4). Then there exists $u \in S - \bigcup_{i=1}^{k} V(C'_i)$ such that u is not an isolated vertex in $L - (\bigcup_{i=1}^{k} V(C'_i)) - (S - \{u\})$. Take $w \in L - (\bigcup_{i=1}^{k} V(C'_i)) - S$ with $uw \in E(G)$. By Claim 3.7, $d_L(w) \geq l - k + 1 > \sum_{i=1}^{k}(|V(C'_i)| - 1) + 1$. Therefore there exists i with $1 \leq i \leq k$ such that $d_{C_i}(w) = |V(C_i)|$. Now letting $P' = uwC_i[v_i, u_i]$ and $C'_j = C_j$ for each $1 \leq j \leq k$ with $j \neq i$, we get a contradiction to Claim 3.5. □

Claim 3.9. *Let D_i, $1 \leq i \leq k$, be vertex-disjoint cycles in L such that $|V(D_i) \cap \{v_1, v_2, \ldots, v_k\}| = 1$ for each $1 \leq i \leq k$, and $\bigcup_{i=1}^{k} V(D_i) \supseteq S$. Then there exist vertex-disjoint cycles D_i^*, $1 \leq i \leq k$, in L such that $V(D_i^*) \supseteq V(D_i)$ for each $1 \leq i \leq k$, and $\bigcup_{i=1}^{k} V(D_i^*) = V(L)$.*

Proof. This follows from Claims 3.1 and 3.7. □

We may assume $1 \in J$. By (2), we may assume $2 \in J$ and $u_2 \in N_G(v_1)$. For $i = 1, 2$, take $w_i \in V(C_i) - \{u_i, v_i\}$, and redefine the direction of C_i so that $w_i \in V(C_i(v_i, u_i))$. Under this notation, we prove several claims concerning the degree of w_1, v_2 and w_2.

Claim 3.10.

(i) *Let $3 \leq i \leq k$. Then $d_{C_i}(w_1) + d_{C_i}(v_2) \leq |V(C_i)| + 3$. Further if $d_{C_i}(w_1) + d_{C_i}(v_2) = |V(C_i)| + 3$, then there exist $x_1, x_2 \in V(C_i)$ with $v_i \in V(C_i[x_1, x_2))$ such that $N_{C_i}(w_1) = V(C_i[x_2, x_1])$, $N_{C_i}(v_2) \supseteq V(C_i[x_1, x_2])$ and $|N_{C_i(x_2,x_1)}(v_2)| = 1$.*

(ii) *Let $3 \leq i \leq k$. Then $d_{C_i}(v_2) + d_{C_i}(w_2) \leq |V(C_i)| + 3$. Further if $d_{C_i}(v_2) + d_{C_i}(w_2) = |V(C_i)| + 3$, then there exist $x_3, x_4 \in V(C_i)$ with $v_i \in V(C_i[x_3, x_4))$ such that $N_{C_i}(w_2) = V(C_i[x_4, x_3])$, $N_{C_i}(v_2) \supseteq V(C_i[x_3, x_4])$ and $|N_{C_i(x_4,x_3)}(v_2)| = 1$.*

Proof. Suppose that $d_{C_i}(w_1) + d_{C_i}(v_2) \geq |V(C_i)| + 3$. This implies $d_{C_i}(w_1) \geq 3$. Choose $x_1, x_2 \in N_{C_i}(w_1)$ so that $v_i \in V(C_i[x_1, x_2))$ and $N_{C_i}(w_1) \cap V(C_i(x_1, x_2)) = \emptyset$. Then $d_{C_i}(w_1) \leq |V(C_i[x_2, x_1])|$. Suppose that there exist $y_1, y_2 \in N_{C_i}(v_2) \cap V(C_i(x_2, x_1))$ with $y_1 \neq y_2$. We may assume y_1, y_2 occur on C_i in this order. Let $P' = C_1[u_1, v_1]u_2$, $C'_2 = v_2C_i[y_1, y_2]v_2$ and $C'_i = w_1C_i[x_1, x_2]w_1$, and let $C'_j = C_j$ for each $3 \leq j \leq k$ with $j \neq i$. Then P' and the C'_i $(2 \leq i \leq k)$ are vertex-disjoint subgraphs satisfying (3), which contradicts Claim 3.5. Thus $|N_{C_i}(v_2) \cap V(C_i(x_2, x_1))| \leq 1$, which implies that $d_{C_i}(v_2) \leq |V(C_i[x_1, x_2])| + 1$. Consequently $d_{C_i}(w_1) + d_{C_i}(v_2) \leq |V(C_i)| + 3$. Further if $d_{C_i}(w_1) + d_{C_i}(v_2) = |V(C_i)| + 3$, then equality holds throughout, and hence $N_{C_i}(w_1) = V(C_i[x_2, x_1])$, $N_{C_i}(v_2) \supseteq V(C_i[x_1, x_2])$ and $|N_{C_i}(v_2) \cap V(C_i(x_2, x_1))| = 1$. Thus (i) is proved, and (ii) can be verified in a similar way. □

Claim 3.11. $d_{\langle V(C_1)\rangle_G}(w_1) + d_{C_1}(v_2) + d_{C_1}(w_2) \le 2|V(C_1)| + 1$.

Proof. Suppose that there exists $x \in N_{C_1}(v_2) \cap N_{C_1}(w_2)$ with $x \ne u_1, v_1$. We seek for a contradiction. Let $P' = C_1[u_1, v_1]u_2$ or $u_2C_1[v_1, u_1]$ according as $x \in V(C_1(v_1, u_1))$ or $x \in V(C_1(u_1, v_1))$. Also let $C_2' = xC_2[v_2, w_2]x$, and let $C_i' = C_i$ for each i with $3 \le i \le k$. Then we get a contradiction to Claim 3.5. Thus $N_{C_1}(v_2) \cap N_{C_1}(w_2) \subseteq \{u_1, v_1\}$, which implies $d_{C_1}(v_2) + d_{C_1}(w_2) \le |V(C_1)| + 2$. Since we clearly have $d_{\langle V(C_1)\rangle_G}(w_1) \le |V(C_1)| - 1$, the desired inequality immediately follows from this. □

Claim 3.12. $d_{C_1}(v_2) \le 4$.

Proof. Suppose that $d_{C_1}(v_2) \ge 5$. Then, by reversing the direction of C_i if necessary, we may assume that there exist $y_1, y_2 \in N_{C_1(v_1,u_1)}(v_2)$ such that y_1 , y_2 occur on C_1 in this order (note that we do not make use of w_1 in the proof of this claim). Now if letting $P' = C_1[u_1, v_1]u_2$, $C_2' = v_2C_1[y_1, y_2]v_2$, and $C_i' = C_i$ for each i with $3 \le i \le k$, we get a contradiction to Claim 3.5.

□

Claim 3.13. $N_{\langle V(C_2)\rangle_G}(v_2) \subseteq \{v_2^+, v_2^-, u_2\}$.

Proof. If $N_{\langle V(C_2)\rangle_G}(v_2) - \{v_2^+, v_2^-, u_2\} \ne \emptyset$, then letting $P' = C_1[u_1, v_1]u_2$, C_2' be a cycle through v_2 in $\langle V(C_2) - \{u_2\}\rangle_G$, and $C_i' = C_i$ for each i with $3 \le i \le k$, we get a contradiction to Claim 3.5. □

Claim 3.14. $d_{\langle V(C_2)\rangle_G}(v_2) \le 3$.

Proof. Proof. This follows from Claim 3.13. □

Claim 3.15. $d_{C_2}(w_1) + d_{\langle V(C_2)\rangle_G}(v_2) + d_{\langle V(C_2)\rangle_G}(w_2) \le 2|V(C_2)|$.

Proof. Clearly $d_{\langle V(C_2)\rangle_G}(w_2) \le |V(C_2)| - 1$.

Case 1. $v_2^- \ne u_2$.

If $|N_{C_2}(w_1) \cap \{v_2, v_2^+, v_2^-\}| \ge 2$, then letting $P' = C_1[u_1, v_1]u_2$, C_2' be a cycle through v_2 in $\langle \{v_2, v_2^+, v_2^-, w_1\}\rangle_G$, and $C_i' = C_i$ for each i with $3 \le i \le k$, we get a contradiction to Claim 3.5. Thus $|N_{C_2}(w_1) \cap \{v_2, v_2^+, v_2^-\}| \le 1$. Hence $d_{C_2}(w_1) \le |V(C_2)| - 2$, which together with Claim 3.14 implies the desired inequality.

Case 2. $v_2^- = u_2$.

Arguing as in Case 1, we obtain $|N_{C_2}(w_1) \cap \{v_2, v_2^+\}| \le 1$, and hence $d_{C_2}(w_1) \le |V(C_2)| - 1$. Since Claim 3.13 implies $d_{\langle V(C_2)\rangle_G}(v_2) \le 2$ by the assumption of Case 2, this implies the desired inequality.

□

Claim 3.16. *Let* $3 \le i \le k$.

(i) *Suppose that* $w_1 v_i \notin E(G)$. *Then*

$$d_{C_i}(w_1) + 2d_{C_i}(v_2) + d_{C_i}(w_2) \le \begin{cases} 2|V(C_i)| + 6 & (\textit{if } d_{C_i}(v_2) = |V(C_i)| \ge 6) \\ 2|V(C_i)| + 5 & (\textit{otherwise}). \end{cases}$$

(ii) *Suppose that* $w_1 v_i \in E(G)$ *and* $d_{C_i}(v_2) \ge 2$. *Then* $d_{C_i}(w_1) + d_{C_i}(v_2) + d_{C_i}(w_2) \le |V(C_i)| + 4$.

Proof. (i) Suppose that $d_{C_i}(w_1) + 2d_{C_i}(v_2) + d_{C_i}(w_2) \ge 2|V(C_i)| + 6$. We aim at showing $d_{C_i}(w_1) + 2d_{C_i}(v_2) + d_{C_i}(w_2) = 2|V(C_i)| + 6$ and $d_{C_i}(v_2) = |V(C_i)| \ge 6$. In view of Claim 3.10, we have $d_{C_i}(w_1) + 2d_{C_i}(v_2) + d_{C_i}(w_2) = 2|V(C_i)| + 6$, $d_{C_i}(w_1) + d_{C_i}(v_2) = |V(C_i)| + 3$ and $d_{C_i}(v_2) + d_{C_i}(w_2) = |V(C_i)| + 3$. Since $w_1 v_i \notin E(G)$, it follows from Claim 3.10(i) that there exist $x_1, x_2, x_3 \in V(C_i)$ with $v_i \in V(C_i(x_1, x_2))$ and $x_3 \in V(C_i(x_2, x_1))$ such that $N_{C_i}(w_1) = V(C_i[x_2, x_1])$ and $N_{C_i}(v_2) = V(C_i[x_1, x_2]) \cup \{x_3\}$. Suppose that $N_{C_i}(w_2) \cap V(C_i(x_2, x_1)) \ne \emptyset$. Take $b \in N_{C_i}(w_2) \cap V(C_i(x_2, x_1))$. Let $C_2' = C_2[v_2, w_2]C_i[b, x_3]v_2$ or $C_2[v_2, w_2]C_i^-[b, x_3]v_2$ according as $b \in V(C_i(x_2, x_3])$ or $b \in V(C_i(x_3, x_1))$. Further let $P' = C_1[u_1, v_1]u_2$, $C_i' = w_1 C_i[x_1, x_2]w_1$, and $C_j' = C_j$ for each $3 \le j \le k$ with $j \ne i$, Then we get a contradiction to Claim 3.5. Thus $N_{C_i}(w_2) \cap V(C_i(x_2, x_1)) = \emptyset$. Since Claim 3.10(ii) in particular implies that $N_{C_i}(w_2) \cup N_{C_i}(v_2) = V(C_i)$, this means $V(C_i(x_2, x_3)) = V(C_i(x_3, x_1)) = \emptyset$, i.e., $N_{C_i}(v_2) = V(C_i)$. Hence it follows from Claim 3.10(ii) that $N_{C_i}(w_2)$ consists of three consecutive vertices. Since $x_3 \notin N_{C_i}(w_2)$, we also have $|V(C_i)| \ge 4$. Now suppose that $v_i \in N_{C_i}(w_2)$. By reversing the direction of C_i if necessary, we may assume $v_i^+ \in N_{C_i}(w_2)$. Then letting $P' = C_1[u_1, v_1]u_2$, $C_2' = v_2 C_i[v_i^{++}, v_i^-]v_2$, $C_i' = w_2 v_i v_i^+ w_2$, and $C_j' = C_j$ for each $3 \le j \le k$ with $j \ne i$, we get a contradiction to Claim 3.5. Thus $v_i \notin N_{C_i}(w_2)$. Consequently we have $N_{C_i}(w_2) \subseteq V(C_i[v_i^+, x_2])$ or $N_{C_i}(w_2) \subseteq V(C_i[x_1, v_i^-])$. Therefore $|V(C_i)| \ge 6$, as desired.

(ii) **Case 1.** $(N_{C_i}(v_2) \cap N_{C_i}(w_2)) - \{v_i\} \ne \emptyset$.

Suppose that $|N_{C_i}(v_2) \cap N_{C_i}(w_2)| + |N_{C_i}(w_1)| \ge 5$. Then since $|N_{C_i}(v_2) \cap N_{C_i}(w_2)| \le 3$ by Claim 3.10, we have $|N_{C_i}(w_1)| \ge 2$. Thus $N_{C_i}(w_1) - \{v_i\} \ne \emptyset$, $(N_{C_i}(v_2) \cap N_{C_i}(w_2) - \{v_i\}) \ne \emptyset$ and $|N_{C_i}(w_1) - \{v_i\}| + |(N_{C_i}(v_2) \cap N_{C_i}(w_2) - \{v_i\})| \ge 3$. Hence we can take $x_1 \in N_{C_i}(w_1) - \{v_i\}$ and $x_2 \in N_{C_i}(v_2) \cap N_{C_i}(w_2) - \{v_i\}$ so that $x_1 \ne x_2$. Without loss of generality, we may assume that x_1, x_2 occur on C_i in this order. Now letting $P' = C_1[u_1, v_1]u_2$, $C_2' = w_2 x_2 C_2[v_2, w_2]$, $C_i' = w_1 C_i[v_i, x_1]w_1$, and $C_j' = C_j$ for each

$3 \le j \le k$ with $j \ne i$, we get a contradiction to Claim 3.5. Consequently $|N_{C_i}(v_2) \cap N_{C_i}(w_2)| + |N_{C_i}(w_1)| \le 4$. Since $|N_{C_i}(v_2)| + |N_{C_i}(w_2)| \le |V(C_i)| + |N_{C_i}(v_2) \cap N_{C_i}(w_2)|$, this implies the desired inequality.

Case 2. $(N_{C_i}(w_2) \cap N_{C_i}(v_2)) - \{v_i\} = \emptyset$.

If $d_{C_i}(w_2) \le 1$, then since $d_{C_i}(w_1) + d_{C_i}(v_2) \le |V(C_i)| + 3$ by Claim 3.10, we immediately obtain the desired inequality. Thus we may assume that $d_{C_i}(w_2) \ge 2$. Choose $y_1 \in N_{C_i}(v_2) - \{v_i\}$ and $y_2 \in N_{C_i}(w_2) - \{v_i\}$ so that the distance between y_1 and y_2 on $C_i - \{v_i\}$ is as small as possible. We may assume that y_1, y_2 occur on C_i in this order. Then $N_{C_i(y_1,y_2)}(v_2) = N_{C_i(y_1,y_2)}(w_2) = \emptyset$. Since $|N_{C_i[y_2,y_1]}(w_2)| + |N_{C_i[y_2,y_1]}(v_2)| \le |V(C_i[y_2,y_1])| + 1$ by the assumption of Case 2, this implies $d_{C_i}(v_2) + d_{C_i}(w_2) \le |V(C_i[y_2,y_1])| + 1$. Now if $N_{C_i}(w_1) \cap V(C_i(v_i,y_1)) \ne \emptyset$, then taking $a \in N_{C_i}(w_1) \cap V(C_i(v_i,y_1))$ and letting $P' = C_1[u_1,v_1]u_2$, $C'_2 = v_2C_i[y_1,y_2]C_2^-[w_2,v_2]$, $C'_i = w_1C_i[v_i,a]w_1$, and $C'_j = C_j$ for each $3 \le j \le k$ with $j \ne i$, we get a contradiction to Claim 3.5. Thus $N_{C_i}(w_1) \cap V(C_i(v_i,y_1)) = \emptyset$. Similarly $N_{C_i}(w_1) \cap V(C_i(y_2,v_i)) = \emptyset$. Consequently $N_{C_i}(w_1) \subseteq V(C_i[y_1,y_2]) \cup \{v_i\}$, and hence $d_{C_i}(w_1) \le |V(C_i[y_1,y_2])| + 1$. Therefore $d_{C_i}(w_1) + (d_{C_i}(v_2) + d_{C_i}(w_2)) \le (|V(C_i[y_1,y_2])| + 1) + (|V(C_i[y_2,y_1])| + 1) = |V(C_i)| + 4$, as desired. □

For later reference, we reformulate Claim 3.16 in the following forms.

Claim 3.17. *Let* $3 \le i \le k$. *Then*

$$d_{C_i}(w_1) + d_{C_i}(v_2) + d_{C_i}(w_2) \le \begin{cases} 2|V(C_i)| + 2 & (\textit{if } w_1v_i \notin E(G) \textit{ and } d_{C_i}(v_2) = 3) \\ 2|V(C_i)| + 1 & (\textit{otherwise}). \end{cases}$$

Proof. Assume for the moment that $w_1v_i \in E(G)$. If $d_{C_i}(v_2) \le 1$, then clearly $d_{C_i}(w_1) + d_{C_i}(v_2) + d_{C_i}(w_2) \le |V(C_i)| + 1 + |V(C_i)|$. If $d_{C_i}(v_2) \ge 2$, then by Claim 3.16(ii), $d_{C_i}(w_1) + d_{C_i}(v_2) + d_{C_i}(w_2) \le |V(C_i)| + 4 \le 2|V(C_i)| + 1$. Thus we may assume $w_1v_i \notin E(G)$. If $d_{C_i}(v_2) \le 2$, then clearly $d_{C_i}(w_1) + d_{C_i}(v_2) + d_{C_i}(w_2) \le (|V(C_i)| - 1) + 2 + |V(C_i)|$. If $d_{C_i}(v_2) = 3$, then by Claim 3.16(i), $d_{C_i}(w_1) + d_{C_i}(v_2) + d_{C_i}(w_2) \le 2|V(C_i)| + 5 - d_{C_i}(v_2) \le 2|V(C_i)| + 2$. If $4 \le d_{C_i}(v_2) \le 5$, then by Claim 3.16(i), $d_{C_i}(w_1) + d_{C_i}(v_2) + d_{C_i}(w_2) \le 2|V(C_i)| + 5 - d_{C_i}(v_2) \le 2|V(C_i)| + 1$. If $d_{C_i}(v_2) \ge 6$, then by Claim 3.16(i), $d_{C_i}(w_1) + d_{C_i}(v_2) + d_{C_i}(w_2) \le 2|V(C_i)| + 6 - d_{C_i}(v_2) \le 2|V(C_i)| + 1$. □

Claim 3.18. *Let* $i \in \overline{J}$. *Then* $d_{C_i}(w_1) + d_{C_i}(v_2) + d_{C_i}(w_2) \le 2|V(C_i)| + 2$.

Proof. This follows from Claim 3.17. □

Claim 3.19. *Let* $i \in J - \{1, 2\}$. *Then* $d_{C_i}(w_1) + d_{C_i}(v_2) + d_{C_i}(w_2) \le 2|V(C_i)| + 1$.

Proof. Suppose that $d_{C_i}(w_1) + d_{C_i}(v_2) + d_{C_i}(w_2) \ge 2|V(C_i)| + 2$. Then by Claim 3.17, we have $w_1v_i \notin E(G)$ and $d_{C_i}(v_2) = 3$, which forces $N_{C_i}(w_1) = V(C_i) - \{v_i\}$ and $N_{C_i}(w_2) = V(C_i)$. Take $x_1, x_2 \in N_{C_i}(v_2) - \{v_i\}$ so that x_1, x_2 occur on C_i in this order. If $u_i \in V(C_i(v_i, x_1])$, then letting $P' = C_2[u_2, v_2]C_i^-[x_1, u_i]$, $C_2' = C_1$, $C_i' = w_2C_i[x_2, v_i]w_2$, and $C_j' = C_j$ for each $3 \le j \le k$ with $j \ne i$, we get a contradiction to Claim 3.5. Thus we may assume $u_i \in V(C_i(x_1, v_i))$. Let $P' = C_2[u_2, v_2]C_i^-[x_2, u_i]$ or $C_2[u_2, v_2]C_i[x_2, u_i]$ according as $u_i \in V(C_i(x_1, x_2])$ or $u_i \in V(C_i(x_2, v_i))$. Further let $C_2' = C_1$, $C_i' = w_2C_i[v_i, x_1]w_2$, and $C_j' = C_j$ for each $3 \le j \le k$ with $j \ne i$. Then we get a contradiction to Claim 3.5. □

Claim 3.20. *Let $3 \le i \le k$, and suppose that $d_{C_i}(v_2) \ge 3$. Then*

$$d_{C_i}(w_1) + 2d_{C_i}(v_2) + d_{C_i}(w_2) \le \begin{cases} 2|V(C_i)| + 6 & (\textit{if } d_{C_i}(v_2) = |V(C_i)| \ge 6) \\ 2|V(C_i)| + 5 & (\textit{otherwise}). \end{cases}$$

Proof. In view of Claim 3.16(i), we may assume $w_1v_i \in E(G)$. Then by Claim 3.16(ii), $d_{C_i}(w_1) + 2d_{C_i}(v_2) + d_{C_i}(w_2) = (d_{C_i}(w_1) + d_{C_i}(v_2) + d_{C_i}(w_2)) + d_{C_i}(v_2) \le (|V(C_i)| + 4) + |V(C_i)|$. □

Claim 3.21. $l \le 2k + 2s - 4$.

Proof. By Claims 3.11, 3.15, 3.18 and 3.19,

$$\begin{aligned} d_G(w_1) + d_G(v_2) + d_G(w_2) &\le 2|V(C_1)| + 1 + 2|V(C_2)| + \sum_{i \in J - \{1,2\}} (2|V(C_i)| + 1) \\ &\quad + \sum_{i \in J - \{1,2\}} (2|V(C_i)| + 2) \\ &= \sum_{i=1}^{k} 2|V(C_i)| + 1 + (s-2) + 2(k-s) \\ &= 2l + 2k - s - 1. \end{aligned} \tag{5}$$

By Claim 3.4,

$$d_G(w_1) + d_G(v_2) + d_G(w_2) \ \ge \ 3(l - s + 1). \tag{6}$$

Combining (5) and (6), we obtain $3(l - s + 1) \le 2l + 2k - s - 1$, which is equivalent to the desired inequality. □

Proof of Theorem 1.3. Since $s \le k$, it follows from Claim 3.21 that $l \le 4k - 4$. But this contradicts the assumption that $l \ge 4k$.

4. Edges in L

We proceed to the proof of Theorem 1.4. Set

$$J(3) = \{i \mid i \in J, |V(C_i)| = 3\},$$
$$J(4) = \{i \mid i \in J, |V(C_i)| = 4\}.$$

Throughout the rest of this paper, we assume that we have chosen $C_1, C_2, \ldots, C_k$ so that $|J(3)|$ is minimum, and so that $|J(4)|$ is minimum, subject to the condition that $|J(3)|$ is minimum. In this section, we prove claims concerning upper bounds on $E(L)$. In obtaining upper bounds on $|E_G(V(C_i), V(C_j))|$, we make use of results proved in Section 2. Here for $i \in J$, u_i plays the role of the malignant vertex on $C_i[v_i, v_i^-]$; for $i \in \overline{J}$, we let $u_i = v_i$ and regard v_i as the malignant vertex on $C_i[v_i, v_i^-]$. We add that in this section, we do not make use of the vertices w_i (this means that we can reverse the direction of C_i when it is necessary).

Claim 4.1. *Suppose that* $|J(3)| + |J(4)| \geq \frac{1}{2}s$*, and let* $1 \leq i \leq k$*. Then* $d_{\langle V(C_i)\rangle_G}(a) \leq 4$ *for all* $a \in V(C_i)$*.*

Proof. If $|V(C_i)| \leq 5$, there is nothing to be proved. Thus we may assume $|V(C_i)| \geq 6$. Suppose that there exists $a_1 \in V(C_i)$ such that $d_{\langle V(C_i)\rangle_G}(a_1) \geq 5$. Then a_1 is adjacent to a vertex $a_2 \in V(C_i) - \{a_1^+, a_1^{++}, a_1^-, a_1^{--}\}$. By reversing the direction of C_i if necessary, we may assume $v_i \in V(C_i[a_1, a_2])$. Let $C_i' = C_i[a_1, a_2]a_1$, and let $C_j' = C_j$ for each j with $j \neq i$. Since $a_2 \neq a_1^+, a_1^{++}$, $|V(C_i')| \geq 4$. If $u_i \notin V(C_i')$, then $i \in J$ (because $u_j = v_j$ for every $j \in \overline{J}$ by definition) and, since $a_2 \neq a_1^-, a_1^{--}$, u_i is not an isolated vertex in $C_i - C_i'$, which contradicts Claim 3.8. Thus $u_i \in V(C_i')$.

For convenience, define K by letting $K = \{i\}$ if $i \in J$ and $|V(C_i')| = 4$, and letting $K = \emptyset$ otherwise. Further let $J'(4) = J(4) \cup K$ and $L' = \langle \bigcup_{j \in J(3) \cup J'(4)} V(C_j') \rangle_G$. Then by Claims 3.4 and 3.6,

$$d_{L'}(b) \geq |V(L')| - (s-1) \tag{7}$$

for each $b \in V(C_i - C_i')$. Assume first that $K = \emptyset$. Fix $b \in V(C_i - C_i')$. Then $d_{L'}(b) > |V(L')| - 2|J(3) \cup J(4)|$ by (7), and hence there exists $p \in J(3) \cup J(4)$ such that $d_{C_p'}(b) \geq |V(C_p')| - 1$. Then we can insert b into C_p' to get a cycle C_p'' with $V(C_p'') = V(C_p') \cup \{b\}$. Now by Claim 3.9, there exist vertex-disjoint cycles $D_1^*, D_2^*, \ldots, D_k^*$ such that $V(D_p^*) \supseteq V(C_p'')$, $V(D_j^*) \supseteq V(C_j')$ for each j with $j \neq p$, and $\bigcup_{j=1}^k V(D_j^*) = V(L)$. Set $J''(3) = \{j \mid V(D_j^*) \cap S \neq \emptyset, |V(D_j^*)| = 3\}$ and $J''(4) = \{j \mid V(D_j^*) \cap S \neq \emptyset, |V(D_j^*)| = 4\}$. Then $J''(3) \subseteq J(3)$ and $J''(3) \cup J''(4) \subseteq (J(3) \cup J(4)) - \{p\}$, which contradicts the minimality of $|J(3)|$ or $|J(4)|$.

Assume now that $K = \{i\}$. Then it follows from (7) that for each $b \in V(C_i - C_i')$, $d_{L'}(b) > |V(L')| - 2(|J(3) \cup J'(4)| - 1)$, and hence there exist two indices $p \in J(3) \cup J'(4)$

for which $d_{C_p}(b) \geq |V(C_p)| - 1$. Fix $b_1, b_2 \in V(C_i - C_i')$ with $b_1 \neq b_2$. Then there exist $p_1, p_2 \in J(3) \cup J'(4)$ with $p_1 \neq p_2$ such that $|N_G(b_\alpha) \cap V(C'_{p_\alpha})| \geq |V(C'_{p_\alpha})| - 1$ for each $\alpha = 1, 2$. For each α, let C''_{p_α} be a cycle with $V(C''_{p_\alpha}) = V(C'_{p_\alpha}) \cup \{b_\alpha\}$. Now by Claim 3.9, there exist vertex-disjoint cycles $D_1^*, D_2^*, \ldots, D_k^*$ such that $V(D^*_{p_\alpha}) \supseteq V(C''_{p_\alpha})$ for each $\alpha = 1, 2$, $V(D_j^*) \supseteq V(C_j')$ for each j with $j \neq p_1, p_2$, and $\bigcup_{j=1}^k V(D_j^*) = V(L)$. Let $J''(3)$, $J''(4)$ be as above. Then $J''(3) \subseteq J(3)$ and $J''(3) \cup J''(4) \subseteq (J(3) \cup J'(4)) - \{p_1, p_2\}$, which again contradicts the minimality of $|J(3)|$ or $|J(4)|$. □

Claim 4.2. *Suppose that* $|J(3)| + |J(4)| \geq \frac{1}{2}s$. *Let* $1 \leq i, j \leq k$ *with* $i \neq j$, *and suppose that* $i \in \overline{J}$, $|V(C_i)| \geq 4$, $|V(C_j)| \geq 4$ *and* $|V(C_i)| + |V(C_j)| \geq 10$. *Then* $|E_G(V(C_i), V(C_j))| \leq 3(|V(C_i)| + |V(C_j)|) - 10$.

Proof. Suppose that $|E_G(V(C_i), V(C_j))| \geq 3(|V(C_i)| + |V(C_j)|) - 9$. Then by Lemma 2.4, $\langle V(C_i) \cup V(C_j) \rangle_G$ contains strongly $(C_i[v_i, v_i^-], C_j[v_j, v_j^-]; v_i, u_j)$-admissible paths P', Q'. Let $C_i' = P'v_i$ and $C_j' = Q'v_j$, and let $C_h' = C_h$ for each h with $h \neq i, j$. If $u_j \notin V(C_i') \cup V(C_j')$, then u_j is not an isolated vertex in $C_j - (V(C_i') \cup V(C_j'))$, which contradicts Claim 3.8. Thus $u_j \in V(C_i') \cup V(C_j')$. It follows from (a) and (b) of (v) in the definition of strongly admissible paths that if $h \in \{i, j\}$ and $u_j \in V(C_h')$, then $|V(C_h')| \geq 4$.

If $j \in J$, let $K = \{h \mid h \in \{i, j\}, u_j \in V(C_h'), |V(C_h')| = 4\}$; if $j \in \overline{J}$, let $K = \emptyset$. Then $|K| \leq 1$, and $|(V(C_i) \cup V(C_j)) - (V(C_i') \cup V(C_j'))| \geq |K| + 1$ by the conditions (i) and (v)(c) in the definition of strongly admissible paths. Let $J'(4) = J(4) \cup K$ and $L' = \langle \bigcup_{h \in J(3) \cup J'(4)} V(C_h') \rangle_G$. Then $|J'(4) - J(4)| \leq |K|$. By Claims 3.4 and 3.6, for each $b \in (V(C_i) \cup V(C_j)) - (V(C_i') \cup V(C_j'))$, $d_{L'}(b) \geq |V(L')| - (s-1) > |V(L')| - 2|J(3) \cup J(4)| = |V(L')| - 2(|J(3) \cup J'(4)| - |J'(4) - J(4)|)$, and hence there exist $|J'(4) - J(4)| + 1$ indices $p \in J(3) \cup J'(4)$ for which $d_{C_p'}(b) \geq |V(C_p')| - 1$. Fix $|J'(4) - J(4)| + 1$ distinct vertices $b_\alpha \in (V(C_i) \cup V(C_j)) - (V(C_i') \cup V(C_j'))$ $(1 \leq \alpha \leq |J'(4) - J(4)| + 1)$. Then we can take $p_\alpha \in J(3) \cup J'(4)$ such that $|N_G(b_\alpha) \cap V(C'_{p_\alpha})| \geq |V(C'_{p_\alpha})| - 1$ for each α, so that the p_α $(1 \leq \alpha \leq |J'(4) - J(4)| + 1)$ are distinct. For each α, let C''_{p_α} be a cycle with $V(C''_{p_\alpha}) = V(C'_{p_\alpha}) \cup \{b_\alpha\}$. Now by Claim 3.9, there exist vertex-disjoint cycles $D_1^*, D_2^*, \ldots, D_k^*$ such that $V(D^*_{p_\alpha}) \supseteq V(C''_{p_\alpha})$ for each α, $V(D_h^*) \supseteq V(C_h')$ for each h with $h \neq p_\alpha$ $(1 \leq \alpha \leq |J'(4) - J(4)| + 1)$, and $\bigcup_{h=1}^k V(D_h^*) = V(L)$. Set $J''(3) = \{h \mid V(D_h^*) \cap S \neq \emptyset, |V(D_h^*)| = 3\}$ and $J''(4) = \{h \mid V(D_h^*) \cap S \neq \emptyset, |V(D_h^*)| = 4\}$. Then $J''(3) \subseteq J(3)$ and $J''(3) \cup J''(4) \subseteq (J(3) \cup J'(4)) - \{p_\alpha \mid 1 \leq \alpha \leq |J'(4) - J(4)| + 1\}$, which contradicts the minimality of $|J(3)|$ or $|J(4)|$. □

Claim 4.3. *Suppose that* $|J(3)| \geq \frac{1}{2}s$. *Let* $1 \leq i, j \leq k$ *with* $i \neq j$, *and suppose that* $|V(C_i)| \geq 5$ *and* $|V(C_j)| \geq 4$.

(i) *If* $i \in \overline{J}$, *then* $|E_G(V(C_i), V(C_j))| \leq 3(|V(C_i)| + |V(C_j)|) - 10$.

(ii) *If* $i \in J$, *then* $|E_G(V(C_i) - \{u_i\}, V(C_j))| \leq 3(|V(C_i) - \{u_i\}| + |V(C_j)|) - 10$.

Proof. Suppose that either $i \in \overline{J}$ and $|E_G(V(C_i), V(C_j))| \geq 3(|V(C_i)| + |V(C_j)|) - 9$, or $i \in J$ and $|E_G(V(C_i) - \{u_i\}, V(C_j))| \geq 3(|V(C_i) - \{u_i\}| + |V(C_j)|) - 9$. Then by Lemmas 2.4 and 2.5, $\langle V(C_i) \cup V(C_j)\rangle_G$ contains $(C_i[v_i, v_i^-], C_j[v_j, v_j^-]; u_i, u_j)$-admissible paths P', Q'. Let $C'_i = P'v_i$ and $C'_j = Q'v_j$, and let $C'_h = C_h$ for each h with $h \neq i, j$. If $\{u_i, u_j\} - (V(C'_i) \cup V(C'_j)) \neq \emptyset$, then by the condition (ii) in the definition of admissible paths, there exists $h \in \{i, j\}$ with $u_h \notin V(C'_i) \cup V(C'_j)$ such that u_h is not an isolated vertex in $(C_i \cup C_j) - (V(C'_i) \cup V(C'_j)) - (\{u_i, u_j\} - \{u_h\})$, which contradicts Claim 3.8. Thus $u_i, u_j \in V(C'_i) \cup V(C'_j)$.

Let $S' = \{u_h \,|\, h \in \{i, j\} \cap J\}$ and $K = \{h \,|\, h \in \{i, j\}, V(C'_h) \cap S' \neq \emptyset, |V(C'_h)| = 3\}$. If $|K| \leq 1$, then $|(V(C_i) \cup V(C_j)) - (V(C'_i) \cup V(C'_j))| \geq |K| + 1$ by the conditions (i) and (iv) in the definition of admissible paths; if $K = \{i, j\}$, then $|(V(C_i) \cup V(C_j)) - (V(C'_i) \cup V(C'_j))| = |V(C_i)| + |V(C_j)| - 6 \geq 3 = |K| + 1$. Let $J'(3) = J(3) \cup K$ and $L' = \langle \bigcup_{h \in J'(3)} V(C'_h)\rangle_G$. Then by Claims 3.4 and 3.6, $d_{L'}(b) \geq |V(L')| - (s-1) > |V(L')| - 2|J(3)| = |V(L')| - 2(|J'(3)| - |K|)$ for each $b \in (V(C_i) \cup V(C_j)) - (V(C'_i) \cup V(C'_j))$. Fix $|K|+1$ distinct vertices $b_\alpha \in (V(C_i) \cup V(C_j)) - (V(C'_i) \cup V(C'_j))$ $(1 \leq \alpha \leq |K|+1)$. Then we can take $p_\alpha \in J'(3)$ such that $|N_G(b) \cap V(C'_{p_\alpha})| \geq 2$ for each α, so that the p_α $(1 \leq \alpha \leq |K|+1)$ are distinct. For each α, let C''_{p_α} be a cycle with $V(C''_{p_\alpha}) = V(C'_{p_\alpha}) \cup \{b_\alpha\}$. Now by Claim 3.9, there exist vertex-disjoint cycles $D^*_1, D^*_2, \ldots, D^*_k$ such that $V(D^*_{p_\alpha}) \supseteq V(C''_{p_\alpha})$ for each α, $V(D^*_h) \supseteq V(C'_h)$ for each h with $h \neq p_\alpha$ $(1 \leq \alpha \leq |K|+1)$, and $\bigcup_{h=1}^{k} V(D^*_h) = V(L)$. Set $J''(3) = \{h \,|\, V(D^*_h) \cap S \neq \emptyset, |V(D^*_h)| = 3\}$. Then $J''(3) \subseteq J'(3) - \{p_\alpha \,|\, 1 \leq \alpha \leq |K|+1\}$, which contradicts the minimality of $|J(3)|$. □

Claim 4.4. *Suppose that* $|J(3)| + |J(4)| \geq \frac{1}{2}s$. *Let* $1 \leq i, j \leq k$ *with* $i \neq j$, *and suppose that* $i \in \overline{J}$, $|V(C_i)| \geq 6$ *and* $|V(C_j)| = 3$. *Then* $|E_G(V(C_i), V(C_j))| \leq 2|V(C_i)| + 2$.

Proof. Suppose that $|E_G(V(C_i), V(C_j))| \geq 2|V(C_i)| + 3$. In view of Lemma 2.7(i), we may assume that $\langle V(C_i) \cup V(C_j)\rangle_G$ contains $(C_i[v_i, v_i^-], C_j[v_j, v_j^-]; v_i, u_j)$-admissible paths P', Q' by reversing the direction of C_j if necessary. Let $C'_i = P'v_i$ and $C'_j = Q'v_j$, and let $C'_h = C_h$ for each h with $h \neq i, j$. We have $u_j \in V(C'_i) \cup V(C'_j)$ by Claim 3.8.

If $j \in J$, let $K = \{h \,|\, h \in \{i, j\}, u_j \in V(C'_h), |V(C'_h)| = 3\}$; if $j \in \overline{J}$, let $K = \emptyset$. Let $J'(3) = (J(3) - \{j\}) \cup K$. Then $|J'(3)| = |J(3)| - 1$ or $|J(3)|$. Assume first that $|J'(3)| = |J(3)| - 1$. By Claim 3.9, there exist vertex-disjoint cycles $D^*_1, D^*_2, \ldots, D^*_k$ such that $V(D^*_h) \supseteq V(C'_h)$ for each h, and $\bigcup_{h=1}^{k} V(D^*_h) = V(L)$. Set $J''(3) = \{h \,|\, V(D^*_h) \cap S \neq \emptyset, |V(D^*_h)| = 3\}$. Then $J''(3) \subseteq J'(3)$, which contradicts the minimality of $|J(3)|$. Assume now that $|J'(3)| = |J(3)|$. Let $L' = \langle \bigcup_{h \in J'(3) \cup J(4)} V(C'_h)\rangle_G$. Fix $b \in (V(C_i) \cup V(C_j)) - (V(C'_i) \cup V(C'_j))$. Then by Claims 3.4 and 3.6, $d_{L'}(b) \geq |V(L')| - (s-1) > |V(L')| - 2|J(3) \cup J(4)| = |V(L')| - 2|J'(3) \cup J(4)|$. Hence there exists $p \in J'(3) \cup J(4)$ such that $|N_G(b) \cap V(C'_p)| \geq |V(C'_p)| - 1$. Let C''_p be a cycle with $V(C''_p) = V(C'_p) \cup \{b\}$. Now by Claim 3.9, there exist vertex-disjoint cycles $D^*_1, D^*_2, \ldots, D^*_k$ such that $V(D^*_p) \supseteq V(C''_p)$, $V(D^*_h) \supseteq V(C'_h)$ for each h with $h \neq p$, and $\bigcup_{h=1}^{k} V(D^*_h) = V(L)$. Set

$J''(3) = \{h \mid V(D_h^*) \cap S \neq \emptyset, |V(D_h^*)| = 3\}$ and $J''(4) = \{h \mid V(D_h^*) \cap S \neq \emptyset, |V(D_h^*)| = 4\}$. Then $J''(3) \subseteq J'(3)$ and $J''(3) \cup J''(4) \subseteq (J'(3) \cup J(4)) - \{p\}$, which contradicts the minimality of $|J(3)|$ or $|J(4)|$. □

Claim 4.5. *Suppose that* $|J(3)| \geq \frac{1}{2}s$. *Let* $1 \leq i, j \leq k$ *with* $i \neq j$, *and suppose that* $|V(C_i)| \geq 7$ *and* $|V(C_j)| = 3$.

(i) *If* $i \notin J$, *then* $|E_G(V(C_i), V(C_j))| \leq 2|V(C_i)| + 2$.

(ii) *If* $i \in J$, *then* $|E_G(V(C_i) - \{u_i\}, V(C_j))| \leq 2|V(C_i) - \{u_i\}| + 2$.

Proof. Suppose that either $i \notin J$ and $|E_G(V(C_i), V(C_j))| \geq 2|V(C_i)| + 3$, or $i \in J$ and $|E_G(V(C_i) - \{u_i\}, V(C_j))| \geq 2|V(C_i) - \{u_i\}| + 3$. In view of Lemma 2.7, we may assume that $\langle V(C_i) \cup V(C_j)\rangle_G$ contains $(C_i[v_i, v_i^-], C_j[v_j, v_j^-]; u_i, u_j)$-admissible paths P', Q' by reversing the direction of C_i and C_j if necessary. Let $C_i' = P'v_i$ and $C_j' = Q'v_j$, and let $C_h' = C_h$ for each h with $h \neq i, j$. We have $u_i, u_j \in V(C_i') \cup V(C_j')$ by Claim 3.8.

Let $S' = \{u_h \mid h \in \{i, j\} \cap J\}$, $K = \{h \mid h \in \{i, j\}, V(C_h') \cap S \neq \emptyset, |V(C_h')| = 3\}$, and $J'(3) = (J(3) - \{j\}) \cup K$. Note that if $j \notin J(3)$, then $|K| \leq |S'| \leq 1$. Hence $|J(3)| - 1 \leq |J'(3)| \leq |J(3)| + 1$. If $|J'(3)| = |J(3)| - 1$, then arguing as in the first half of the second paragraph of the proof of Claim 4.4, we get a contradiction to the minimality of $|J(3)|$. Thus $|J'(3)| = |J(3)|$ or $|J(3)| + 1$. Note that if $|J'(3)| = |J(3)| + 1$, then $K \neq \emptyset$. Hence $|(V(C_i) \cup V(C_j)) - (V(C_i') \cup V(C_j'))| \geq |J'(3)| - |J(3)| + 1$ by the conditions (i) and (iv) in the definition of admissible paths. Fix $|J'(3)| - |J(3)| + 1$ distinct vertices $b_\alpha \in (V(C_i) \cup V(C_j)) - (V(C_i') \cup V(C_j'))$ $(1 \leq \alpha \leq |J'(3)| - |J(3)| + 1)$. We can now argue as in the second paragraph of the proof of Claims 4.2 and 4.3, to obtain a contradiction to the minimality of $|J(3)|$. □

5. Proof of Theorem 1.4

We continue with the notation of the preceding section, and complete the proof of Theorem 1.4. By the assumption of Theorem 1.4,

$$l \geq \frac{399}{100}k. \tag{8}$$

Hence

$$s \geq \frac{199}{200}k \tag{9}$$

by Claim 3.21. In view of Theorem 1.3, we may assume $k \geq 100$. Recall $J(3) = \{i \mid i \in J, |V(C_i)| = 3\}$ and $J(4) = \{i \mid i \in J, |V(C_i)| = 4\}$. Let $t' = |J(3)|$.

Claim 5.1. $t' < \frac{3}{4}s$.

Proof. At the cost of relabeling, we may assume $|V(C_k)| = \max\{|V(C_i)|\,|\,1 \le i \le k\}$. Let

$$\begin{aligned} I(3) &= \{i\,|\,1 \le i \le k-1, |V(C_i)| = 3\}, \\ I(\ge 4) &= \{i\,|\,1 \le i \le k-1, |V(C_i)| \ge 4\}, \end{aligned}$$

and let $t = |I(3)|$. Now suppose that $t' \ge \frac{3}{4}s$. Then $t \ge \frac{3}{4}s$, and hence

$$|V(C_k)| \ge \frac{l-3t}{k-t} \ge \frac{\frac{399}{100}k - \frac{9}{4}s}{k - \frac{3}{4}s} = \frac{399}{100} + \frac{\frac{297}{400}s}{k-\frac{3}{4}s} = \frac{399}{100} + \frac{\frac{297}{400}\frac{199}{200}k}{k - \frac{3}{4}\frac{199}{200}k} > 6 \tag{10}$$

by (9). Define C'_k by letting $C'_k = C_k$ if $k \in \overline{J}$, and $C'_k = C_k - \{u_k\}$ if $k \in J$. By Claims 4.1, 4.3 and 4.5,

$$\begin{aligned} \sum_{x \in V(C'_k)} d_L(x) &\le \sum_{j \in I(3)} (2|V(C'_k)| + 2) + \sum_{j \in I(\ge 4)} (3(|V(C'_k)| + |V(C_j)|) - 10) + 4|V(C'_k)| \\ &= (2t + 3|I(\ge 4)| + 4)|V(C'_k)| + 2t - 10|I(\ge 4)| + \sum_{j \in I(\ge 4)} 3|V(C_j)| \\ &= (3k - t + 1)|V(C'_k)| + 2t - 10|I(\ge 4)| + 3(l - 3t - |V(C_k)|) \\ &= (l-k+1)|V(C'_k)| + (-l + 4k - t)|V(C'_k)| - 7t - 10|I(\ge 4)| \\ &\quad + 3l - 3|V(C_k)| \\ &= (l-k+1)|V(C'_k)| - (|V(C'_k)| - 3)l + (4k - t)|V(C'_k)| - 10(k-1) \\ &\quad + 3t - 3|V(C_k)| \\ &\le (l-k+1)|V(C'_k)| - (|V(C'_k)| - 3)l + (4k - t)|V(C'_k)| - 10(k-1) \\ &\quad + 3t - 3|V(C'_k)| \\ &= (l-k+1)|V(C'_k)| - (|V(C'_k)| - 3)l + (4|V(C'_k)| - 10)k \\ &\quad -(|V(C'_k)| - 3)t - 3|V(C'_k)| + 10 \\ &\le (l-k+1)|V(C'_k)| - (|V(C'_k)| - 3)\frac{399}{100}k + (4|V(C'_k)| - 10)k \\ &\quad - (|V(C'_k)| - 3)\frac{3}{4}k - 3|V(C'_k)| + 10 \\ &= (l-k+1)|V(C'_k)| - \Big(\Big(\frac{399}{100} - 4 + \frac{3}{4}\Big)k + 3\Big)|V(C'_k)| \\ &\quad + \Big(\frac{1197}{100} - 10 + \frac{9}{4}\Big)k + 10. \end{aligned}$$

Since $|V(C'_k)| \ge 7$ by (10), we now obtain

$$\begin{aligned} \sum_{x \in V(C'_k)} d_L(x) &\le (l-k+1)|V(C'_k)| - \Big(\frac{37}{50}k + 3\Big)7 + \frac{211}{50}k + 10 \\ &= (l-k+1)|V(C'_k)| - \frac{49}{50}k - 11. \end{aligned}$$

On the other hand, by Claim 3.7, we have $\sum_{x\in V(C'_k)} d_L(x) \geq (l-k+1)|V(C'_k)|$, a contradiction. □

Claim 5.2. $\sum_{i\in J}|V(C_i)| > \frac{15}{4}s$.

Proof. If $t'+|J(4)| \leq \frac{1}{2}s$, then $\sum_{i\in J}|V(C_i)| \geq 3t'+4|J(4)|+5(s-t'-|J(4)|) = 5s-2t'-|J(4)| \geq 5s-2(t'+|J(4)|) \geq 4s > \frac{15}{4}s$. Thus we may assume $t'+|J(4)| > \frac{1}{2}s$. If $t' < \frac{1}{4}s$, then $\sum_{i\in J}|V(C_i)| \geq 3t'+4(s-t') = 4s-t' > \frac{15}{4}s$. Thus we may assume $t' \geq \frac{1}{4}s$ If $s=k$, then we clearly have $\sum_{i\in J}|V(C_i)| = l \geq \frac{399}{100}s > \frac{15}{4}s$. Thus we may also assume $s<k$, i.e., $\overline{J} \neq \emptyset$. At the cost of relabeling, we may assume that $k\in\overline{J}$ and $|V(C_k)| = \max\{|V(C_i)| \,|\, i\in\overline{J}\}$. Suppose that $\sum_{i\in J}|V(C_i)| \leq \frac{15}{4}s$. Then

$$|V(C_k)| \geq \frac{l-\sum_{i\in J}|V(C_i)|}{k-s} \geq \frac{\frac{399}{100}k-\frac{15}{4}s}{k-s} = \frac{399}{100}+\frac{\frac{6}{25}s}{k-s} > 51 \tag{11}$$

by (9).

We now argue as in the proof of Claim 5.1 using Claims 4.2 and 4.4 in place of Claims 4.3 and 4.5. Then we get

$$\begin{aligned}\sum_{x\in V(C_k)} d_L(x) \;&\leq\; (l-k+1)|V(C_k)| - (|V(C_k)|-3)l + (4|V(C_k)|-10)k \\ &\quad - (|V(C_k)|-3)t' - 3|V(C_k)| + 10.\end{aligned}$$

Since $t' \geq \frac{1}{4}s$ this implies

$$\begin{aligned}\sum_{x\in V(C_k)} d_L(x) \;&\leq\; (l-k+1)|V(C_k)| - (|V(C_k)|-3)\frac{399}{100}k + (4|V(C_k)|-10)k \\ &\quad - (|V(C_k)|-3)\frac{1}{4}s - 3|V(C_k)| + 10 \\ &\leq\; (l-k+1)|V(C_k)| - (|V(C_k)|-3)\frac{399}{100}k + (4|V(C_k)|-10)k \\ &\quad - (|V(C_k)|-3)\frac{199}{800}k - 3|V(C_k)| + 10 \\ &=\; (l-k+1)|V(C_k)| - \Big(\Big(\frac{399}{100}-4+\frac{199}{800}\Big)k+3\Big)|V(C_k)| \\ &\quad + \Big(\frac{1197}{100}-10+\frac{597}{800}\Big)k + 10.\end{aligned}$$

Since $|V(C_k)| \geq 52$ by (11), we now obtain

$$\begin{aligned}\sum_{x\in V(C_k)} d_L(x) \;&<\; (l-k+1)|V(C_k)| - \Big(\frac{191}{800}k+3\Big)52 + \frac{2173}{800}k + 10 \\ &=\; (l-k+1)|V(C_k)| - \frac{7759}{800}k - 146.\end{aligned}$$

On the other hand, by Claim 3.6, we have $\sum_{x\in V(C_k)} d_L(x) \geq (l-k+1)|V(C_k)|$, a contradiction. □

We are now in a position to complete the proof of Theorem 1.4. Let $L_1 = \bigcup_{i\in J}(V(C_i) - \{u_i, v_i\})$. Choose $w_1 \in L_1$ so that $|N_G(w_1) \cap \{v_j \,|\, j \in J - J(3)\}|$ is maximum. By Claim 3.4, we have $d_{L_1}(v_j) \geq |L_1| - ((s-1)-1) = \sum_{i\in J}|V(C_i)| - 3s - 2$ for each $j \in J$. Since $|J - J(3)| = s - t'$, this together with the maximality of $|N_G(w_1) \cap \{v_j \,|\, j \in J - J(3)\}|$ implies

$$\begin{aligned}
|N_G(w_1) \cap \{v_j \,|\, j \in J - J(3)\}| &\geq \frac{\sum_{j\in J-J(3)} d_{L_1}(v_j)}{|L_1|} \\
&\geq \frac{\sum_{j\in J-J(3)}(\sum_{i\in J}|V(C_i)| - 3s + 2)}{\sum_{i\in J}|V(C_i)| - 2s} \\
&= (s-t')\Big(1 - \frac{s-2}{\sum_{i\in J}|V(C_i)| - 2s}\Big) \\
&> (s-t')\Big(1 - \frac{s}{\sum_{i\in J}|V(C_i)| - 2s}\Big).
\end{aligned}$$

Hence by Claim 5.2,

$$|N_G(w_1) \cap \{v_j \,|\, j \in J - J(3)\}| \;\geq\; (s-t')\Big(1 - \frac{4}{7}\Big) > \frac{3}{7}(s-t'). \tag{12}$$

At the cost of relabeling, we may assume $w_1 \in V(C_1)$. By reversing the direction of C_1 if necessary, we may also assume $w_1 \in V(C_1(v_1, u_1))$. By (2), there exists $j \in J - \{1\}$ such that $u_j \in N_G(v_1)$. We may assume $j = 2$. Take $w_2 \in V(C_2) - \{u_2, v_2\}$, and redefine the direction of C_2 so that $w_2 \in V(C_2(v_2, u_2))$. For convenience, let $J' = J - \{1, 2\}$. We define J_1 and J_2 as follows :

$$\begin{aligned}
J_1 &= \{i \,|\, i \in J', w_1v_i \in E(G), |V(C_i)| \geq 4\}, \\
J_2 &= \{i \,|\, i \in J', w_1v_i \notin E(G), |V(C_i)| \geq 4\}.
\end{aligned}$$

Then

$$J' = (J' \cap J(3)) \cup J_1 \cup J_2 \qquad \text{(disjoint union)}$$

by definition, and

$$|J_1| > \frac{3}{7}(s-t') - 2 \tag{13}$$

by (12). Further we define I_1, I_2, I_3, I_4, I_5, I_6 and I_7 as follows :

$$\begin{aligned} I_1 &= \{i \mid i \in J_1, d_{C_i}(v_2) \geq 2\}, \\ I_2 &= \{i \mid i \in J_2, d_{C_i}(v_2) = |V(C_i)| \geq 6\}, \\ I_3 &= \{i \mid i \in J_2 - I_2, |V(C_i)| \geq 5, d_{C_i}(v_2) \geq 5\}, \\ I_4 &= J' - I_1 - I_2 - I_3, \\ I_5 &= \{i \mid i \in \overline{J}, d_{C_i}(v_2) = |V(C_i)| \geq 6\}, \\ I_6 &= \{i \mid i \in \overline{J} - I_5, |V(C_i)| \geq 4, d_{C_i}(v_2) \geq 4\}, \\ I_7 &= \overline{J} - I_5 - I_6. \end{aligned}$$

Thus

$$I_4 = (J' \cap J(3)) \cup (J_1 - I_1) \cup (J_2 - I_2 - I_3) \qquad \text{(disjoint union).}$$

By Claims 3.11, 3.15, 3.18, 3.19, 3.16 and 3.20,

$$\begin{aligned} & d_L(w_1) + d_L(v_2) + d_L(w_2) \\ \leq\ & 2|V(C_1)| + 1 + 2|V(C_2)| + \sum_{i \in I_4}(2|V(C_i)| + 1) + \sum_{i \in I_1}(2|V(C_i)| + 4 - |V(C_i)|) \\ & + \sum_{i \in I_3}(2|V(C_i)| + 5 - d_{C_i}(v_2)) + \sum_{i \in I_2}(2|V(C_i)| + 6 - d_{C_i}(v_2)) \\ & + \sum_{i \in I_7}(2|V(C_i)| + 2) + \sum_{i \in I_6}(2|V(C_i)| + 5 - d_{C_i}(v_2)) \\ & + \sum_{i \in I_5}(2|V(C_i)| + 6 - d_{C_i}(v_2)) \\ =\ & \sum_{i \in J}(2|V(C_i)| + 1) - 1 + \sum_{i \in \overline{J}}(2|V(C_i)| + 2) - \sum_{i \in I_1}(|V(C_i)| - 3) \\ & - \sum_{i \in I_3}(d_{C_i}(v_2) - 4) - \sum_{i \in I_2}(d_{C_i}(v_2) - 5) - \sum_{i \in I_6}(d_{C_i}(v_2) - 3) \\ & - \sum_{i \in I_5}(d_{C_i}(v_2) - 4). \end{aligned} \tag{14}$$

We have

$$\begin{aligned} |V(C_i)| - 3 \geq \frac{1}{3}(|V(C_i)| - 3) \geq \frac{1}{3}(d_{C_i}(v_2) - 3) &\quad \text{for each } i \in I_1, \\ d_{C_i}(v_2) - 5 \geq \frac{1}{2}(d_{C_i}(v_2) - 4) &\quad \text{for each } i \in I_2, \text{ and} \\ d_{C_i}(v_2) - 4 \geq \frac{2}{3}(d_{C_i}(v_2) - 3) &\quad \text{for each } i \in I_5. \end{aligned}$$

Since

$$\sum_{i\in J}(2|V(C_i)|+1)+\sum_{i\in \overline{J}}(2|V(C_i)|+2)=2l+s+2(k-s),$$

it follows from (14) that

$$\begin{aligned} &d_L(w_1)+d_L(v_2)+d_L(w_2)\\ &\le\ 2l+2k-s-1-\frac{1}{3}\sum_{i\in I_1}(d_{C_i}(v_2)-1)-\frac{1}{2}\sum_{i\in I_2\cup I_3}(d_{C_i}(v_2)-4)\\ &\qquad -\frac{2}{3}\sum_{i\in I_5\cup I_6}(d_{C_i}(v_2)-3))\\ &\le\ 2l+2k-s-1-\frac{1}{3}\Big(\sum_{i\in I_1}(d_{C_i}(v_2)-1)+\sum_{i\in I_2\cup I_3}(d_{C_i}(v_2)-4)+\sum_{i\in I_5\cup I_6}(d_{C_i}(v_2)-3)\Big).\end{aligned}$$

By the definition of I_1 through I_7, we have $d_{C_i}(v_2)\le 1$ for each $i\in J_1-I_1$, $d_{C_i}(v_2)\le 4$ for each $i\in J_1-I_2-I_3$, and $d_{C_i}(v_2)\le 3$ for each $i\in I_7$. We also have $d_{C_i}(v_2)\le 3$ for each $i\in J'\cap J(3)$. Consequently,

$$\begin{aligned} &d_L(w_1)+d_L(v_2)+d_L(w_2)\\ &\le\ 2l+2k-s-1-\frac{1}{3}\Big(\sum_{i\in J_1}(d_{C_i}(v_2)-1)+\sum_{i\in J_2}(d_{C_i}(v_2)-4)\\ &\qquad +\sum_{i\in (J'\cap J(3))\cup\overline{J}}(d_{C_i}(v_2)-3)\Big)\\ &=\ 2l+2k-s-1-\frac{1}{3}\sum_{3\le i\le k}d_{C_i}(v_2)+\frac{1}{3}(|J_1|+4|J_2|+3|J'\cap J(3)|+3|\overline{J}|).\end{aligned}$$

Therefore by Claims 3.12 and 3.14,

$$\begin{aligned} &d_L(w_1)+d_L(v_2)+d_L(w_2)\\ &\le\ 2l+2k-s+\frac{4}{3}-\frac{1}{3}d_L(v_2)+\frac{1}{3}(|J_1|+4|J_2|+3|J'\cap J(3)|+3|\overline{J}|),\end{aligned}$$

i.e.,

$$\begin{aligned} &d_L(w_1)+\frac{4}{3}d_L(v_2)+d_L(w_2)\\ &\le\ 2l+2k-s+\frac{4}{3}+\frac{1}{3}(|J_1|+4|J_2|+3|J'\cap J(3)|+3|\overline{J}|). \qquad (15)\end{aligned}$$

On the other hand, it follows from (12) that

$$|J_1|+4|J_2|=4|J_1\cup J_2|-3|J_1|<4(s-t')-3\Big(\frac{3}{7}(s-t')-2\Big)\le\frac{19}{7}(s-t')+6.$$

Since $|J' \cap J(3)| \leq t'$ and $|\overline{J}| = k - s$, this together with Claim 5.1 implies

$$\begin{aligned} |J_1| + 4|J_2| + 3|J' \cap J(3)| + 3|\overline{J}| &< \frac{19}{7}(s - t') + 6 + 3t' + 3(k - s) \\ &= 3k - \frac{2}{7}s + \frac{2}{7}t' + 6 \\ &< 3k - \frac{1}{14}s + 6. \end{aligned}$$

Consequently it follows from (13) that

$$d_L(w_1) + \frac{4}{3}d_L(v_2) + d_L(w_2) < 2l + 3k - \frac{43}{42}s + \frac{10}{3}.$$

Since

$$d_L(w_1) + \frac{4}{3}d_L(v_2) + d_L(w_2) \geq \frac{10}{3}(l - s + 1)$$

by Claim 3.4, we now obtain

$$\frac{10}{3}(l - s + 1) < 2l + 3k - \frac{43}{42}s + \frac{10}{3}.$$

Therefore

$$\frac{4}{3}l < 3k + \frac{97}{42}s < \frac{223}{42}k,$$

which implies $l < \frac{223}{56}k$. But this contradicts (8).

This completes the proof of Theorem 1.4.

References

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