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A survey on graphs which have equal domination and closed neighborhood packing numbers

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Abstract

Vizing's conjecture on the Cartesian product $G\Box H$ of two graphs, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$, will hold for all graphs G if the domination and (closed neighborhood) packing number of G are equal. By using the theory of integer programming, we state known sufficient conditions for when the packing number and the domination number of a graph are equal. We also give a sufficient condition using the theory of efficient domination. For regular graphs we find a necessary and sufficient condition for equlity to hold.

Keywords: domination, packing, efficient domination, strongly chordal, linear programming, balanced matrices, totally balanced matrices.

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1. Introduction

This paper was motivated by Doug Rall's presentation of "Domination and Packing in Graph Products" at the 20th Clemson mini-conference and is an extension of two independent surveys contained in Domke, Fricke, Laskar, and Majumdar [7] and

Section 7.4 of Scheinerman and Ullman [29]. Our notation follows that of Haynes, Hedetniemi and Slater [15, 16]. In this paper, graphs $G = (V, E)$ are finite with $|V(G)| = n$ vertices and have no multiple edges or loops. The *open neighborhood* of a vertex $v \in V(G)$ is defined as $N_G(v) = \{u \in V(G) \mid uv \in E\}$, the set of all vertices adjacent to v. The *closed neighborhood* of a vertex $v \in V(G)$ is defined as $N_G[v] = \{v\} \cup N_G(v)$. When the graph G is clear from context, $N_G(v)$ and $N_G[v]$ will be denoted by $N(v)$ and $N[v]$, respectively. For a set $S \subseteq V(G)$, let $N(S) = \bigcup_{u \in S} N(u)$ and let $N[S] = \bigcup_{u \in S} N[u]$. The distance between any two vertices $u, v \in V(G)$, denoted by $dist(u, v)$, is the length of a shortest path from u to v. The *diameter* of a connected graph G, denoted by $diam(G)$ is the maximum distance between any two vertices of $V(G)$.

We say that a vertex "dominates" itself and all of its neighbors. A set of vertices $S \subseteq V(G)$ is called a *dominating set* if every vertex $v \in V(G)$ is either an element of S or is adjacent to some element of S. That is, a set S is a dominating set if $|N[v] \cap S| \geq 1$ for all vertices $v \in V(G)$. The domination number $\gamma(G)$ is the size of a smallest dominating set.

A subset $S \subseteq V(G)$ is called a k-packing, if for any two distinct vertices u, v in S, we have $dist(u, v) > k$. In particular, a set $S \subseteq V(G)$ is a 2*-packing* if the minimum distance between any distinct vertices u, v of S is at least three. A set S is a *closed neighborhood packing* if for each $u, v \in S$, $u \neq v$ we have $N[u] \cap N[v] = \emptyset$. Alternatively, a set S is a closed neighborhood packing if $|N[v] \cap S| \leq 1$ for all vertices $v \in V(G)$. A set S is a closed neighborhood packing if and only if S is a 2-packing, since vertices $u \neq v$ are in a closed neighborhood packing S if and only if $dist(u, v) > 2$. The packing number $\rho(G)$ is the size of a largest closed neighborhood packing. For all graphs G, $1 \leq \rho(G) \leq n$. The only graphs with $\rho(G) = n$ are graphs with no edges.

Observation 1.1. *For all graphs G, $\rho(G) \leq \gamma(G)$.*

Proof. Let S be a maximum packing of G, and let D be a dominating set. For any $v, w \in S$ we have $N[v] \cap N[w] = \emptyset$ and $N[v] \cap D \neq \emptyset$. It follows that $|D| \geq |S| = \rho(G)$, hence $\gamma(G) \geq \rho(G)$. □

Below we give an example of a graph which has $\rho(G) < \gamma(G)$ and one which has $\rho(G) = \gamma(G)$.

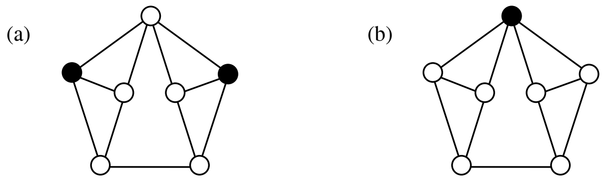


Figure 1: (a) A minimum dominating set and (b) a maximum packing.

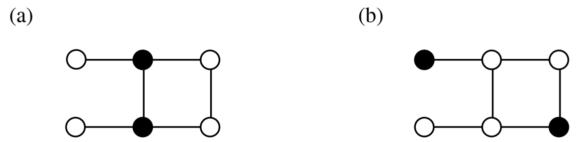


Figure 2: (a) A minimum dominating set and (b) a maximum packing.

2. Trees

The first result on which graphs have equal domination and packing numbers comes from Meir and Moon [22] in the language of k-coverings and k-packings. (A 1-covering is a dominating set.) Their proof is by induction, and it should be noted that this result predates all three sufficient conditions given in Theorems 3.1, 6.2, and 6.4. We take the proof from Scheinerman and Ullman [29].

Theorem 2.1. [22] *For all trees T, $\rho(T) = \gamma(T)$.*

Proof. The proof is by induction on the order of T, $|V(T)|$. If T has no path of length 3, then $\rho(T) = \gamma(T) = 1$. For the induction step, let T be any tree and let P be a longest path in T (also called a diametrical path) with the vertices of P in order $v_1, v_2, v_3, \ldots, v_r$. Note that $T - v_2$ can have only one component with more than one vertex, since otherwise we violate the maximality of P. We consider two cases, depending on whether v_3 has degree 2 (with neighbors v_2 and v_4) or degree greater than 2.

Suppose v_3 has degree 2. Let T' be the component of $T - \{v_2, v_3\}$ containing v_4. Apply the induction hypothesis to T' to obtain a dominating set D' and a (closed neighborhood) packing S' with $|S'| = \rho(T') = \gamma(T') = |D'|$. Let $D = D' \cup \{v_2\}$ and $S = S' \cup \{v_2\}$. It is clear that D is a dominating set for T and S is a packing of T, and that $|D| = |D'| + 1 = |S'| + 1 = |S|$.

Now consider the alternative case, when v_3 has degree greater than 2 in T. Let T' be the component of $T - v_2$ containing v_3. Applying the induction hypothesis to T' yields as before a dominating set D' and a packing S' with $|S'| = \rho(T') = \gamma(T') = |D'|$. The set $D = D' \cup \{v_2\}$ is clearly a dominating set for T. We now produce a packing of T of the same size. If $v_3 \notin S'$, then put $S = S' \cup \{v_1\}$. If $v_3 \in S'$, let u be a vertex adjacent to v_3 but not equal to v_2 or v_4. The maximality of P implies that the neighborhood of u contains, other than v_3, only leaves of T. Since $v_3 \in S'$, neither u nor any leaves v adjacent to u are in S'. Set $S = S' \cup \{v_1, u\} - v_3$. In either case we have $|S| = |D|$, and hence $\rho(T) = \gamma(T)$. □

2.1. Block graphs

For any connected graph G a vertex $x \in V(G)$ is called a *cutvertex* of G if $G - x$ is no longer connected. A connected subgraph B of G is called a *block*, if B has no cutvertex and every subgraph $B' \subseteq G$ with $B \subseteq B'$ and $B \neq B'$ has at least one cutvertex. A graph G is called a *block graph* if every block in G is complete.

Theorem 2.2. [1] *If G is a connected block graph, then $\rho(G) = \gamma(G)$*

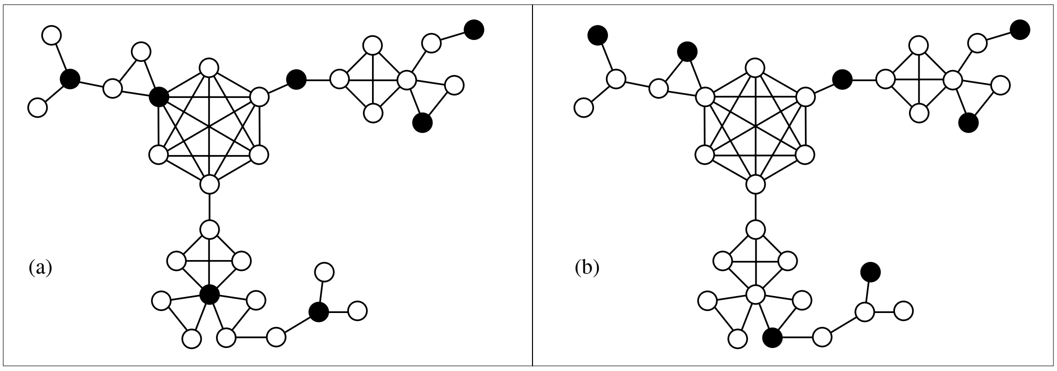


Figure 3: A minimum dominating set (a) and a maximum packing (b) of a block graph.

3. Efficient dominating sets

Bange, Barkauskas, and Slater [2] define a set S to be an *efficient dominating set* if $|N[v] \cap S| = 1$ for all vertices $v \in V(G)$, and they also introduced the following efficiency measure for a graph G. The *efficient domination number* of a graph, denoted $F(G)$, is the maximum number of vertices that can be dominated by a set S that dominates each vertex at most once. A graph G of order $n = |V(G)|$ has an efficient dominating set if and only if $F(G) = n$.

A graph is efficient if and only if there exists an efficient dominating set. Alternatively, a graph is efficient if and only if there exists a set S which is both dominating and a closed neighborhood packing.

3.1. A sufficient condition for $\rho(G) = \gamma(G)$

A partition of the vertex set into closed neighborhoods exists if we can find vertices $v_1, \ldots, v_\rho$ so that $V = N[v_1] \cup N[v_2] \cup \cdots \cup N[v_\rho]$ and $N[v_i] \cap N[v_j] = \emptyset$ for all $i \neq j$ (see Figure 4). If such a partition exists, then the set of veritces $\{v_1, \ldots, v_\rho\}$ is a dominating set and a packing, thus $\rho(G) = \gamma(G)$. In fact, this set is an efficient dominating set.

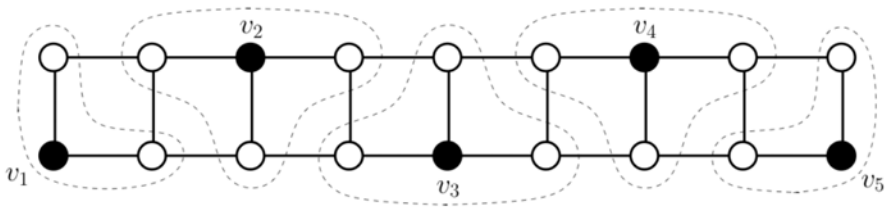


Figure 4: An efficient dominating set of $P_2 \square P_9$.

Theorem 3.1. *If a graph G is efficient, then $\rho(G) = \gamma(G)$.*

Proof. Let S be an efficient dominating set. We have that $|N[v] \cap S| = 1$ for all vertices $v \in V(G)$, thus S is a dominating set, and $|S| \geq \gamma(G)$; S is also a packing, and we have $|S| \leq \rho(G)$. Thus, $\rho(G) \geq |S| \geq \gamma(G)$, which implies $\rho(G) = \gamma(G)$ (since $\rho(G)$ can never exceed $\gamma(G)$). Alternatively, any set S which is both dominating and packing is necessarily minimum dominating and maximum packing, thus $\rho(G) = |S| = \gamma(G)$. □

Another sufficient condition for equality to hold in $\rho(G) \leq \gamma(G)$ is the existence of a vertex of degree $n-1$, in which case, $\rho = \gamma = 1$. Note that this is a special case of a closed neighborhood partition of $V(G)$, with just one closed neighborhood, by choosing a vertex u of degree $n-1$, we get $V(G) = N[u]$.

Equality can still hold in $\rho(G) \leq \gamma(G)$ if the graph is not efficient, for instance the graph G in Figure 2 has $F(G) = 5 < 6$. Note that determining whether or not a graph has an efficient dominating set is NP-complete (see [2]).

4. Integer and linear programming

A *linear program* is an optimization problem where we are maximizing or minimizing a function subject to some constraints. Let M be a real k by m matrix and $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ be real column vectors of the appropriate sizes. For our purposes, linear programs (or LPs) can be expressed in the following two forms:

$$\text{maximize } \boldsymbol{c}^T\boldsymbol{x} \text{ subject to } M\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0} \tag{1}$$

$$\text{minimize } \boldsymbol{b}^T\boldsymbol{y} \text{ subject to } M^T\boldsymbol{y} \geq \boldsymbol{c},\ \boldsymbol{y} \geq \boldsymbol{0} \tag{2}$$

The linear program in (2) is called the (*linear programming*) *dual* of the linear program in (1). The expression $\boldsymbol{c}^T\boldsymbol{x}$ in (1) is called the *objective function* and any vector $\boldsymbol{x}$ satisfying the constraints $M\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is called a *feasible solution.* The maximum (respectively, minimum) value of the objective function taken over all feasible solutions is called the "value" of the LP (denoted by $|\,\mathrm{LP}\,|$). Any feasible solution to the LP on which the objective function attains the value is called an *optimal solution.*

If we require, in addition, that the optimal solutions be *integer* valued, then the above two linear programs are called *integer programs* (or IPs). When we start with an integer program and then remove the constraint that the optimal solutions need to be integer valued, we obtain the *linear relaxation* of the IP. We now state a few fundamental theorems from linear programming (see [23]).

Theorem 4.1. [11] *A linear program and its dual have the same value.*

In the next few theorems, we find conditions on the matrix M in the integer program for which the value of the IP is equal to the value of its linear relaxation (LP). A $\{0,1\}$ matrix M is called *totally unimodular* if and only if $\det S \in \{-1,0,1\}$ for all square submatrices S (not necessarily formed from contiguous rows or columns).

The family of k by k, $\{0,1\}$ matrices with exactly two ones in every row and column is denoted by $\mathcal{M}_k$. A $\{0,1\}$ matrix M is called *totally balanced* provided it contains no $\mathcal{M}_k$ for $k \geq 3$ as a submatrix. A $\{0,1\}$ matrix M is called *balanced* provided it contains no $\mathcal{M}_k$ for $k \geq 3$, k odd, as a submatrix.

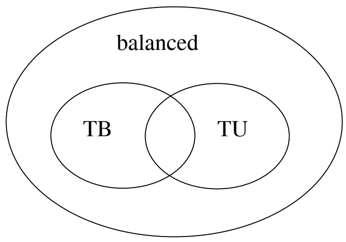


Figure 5: Balanced, totally balanced, and totally unimodular $\{0,1\}$ matrices.

Theorem 4.2. [18] *If M, $\boldsymbol{b}$, $\boldsymbol{c}$ all have integer entries and M is totally unimodular, then the value of the integer program: maximize $\boldsymbol{c}^T\boldsymbol{x}$ subject to $M\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq 0$, and its linear relaxation have the same value.*

Theorem 4.3. [10] *If M, $\boldsymbol{b}$, $\boldsymbol{c}$ all have integer entries, at least one of $\boldsymbol{b}$ or $\boldsymbol{c}$ is a constant vector, and M is balanced, then the value of the integer program: maximize $\boldsymbol{c}^T\boldsymbol{x}$ subject to $M\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq 0$, and its linear relaxation have the same value.*

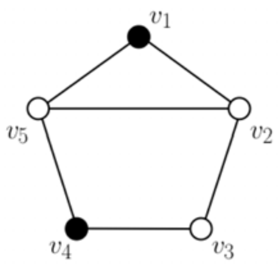
Theorem 4.4. [17] *If M, $\boldsymbol{b}$, $\boldsymbol{c}$ all have integer entries, at least one of $\boldsymbol{b}$ or $\boldsymbol{c}$ is a constant vector, and M is totally balanced, then the integer program: maximize $\boldsymbol{c}^T\boldsymbol{x}$ subject to $M\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq 0$, can be solved in polynomial time.*

Since every $\{0,1\}$ totally balanced matrix is, in fact, a balanced matrix, any integer program with a totally balanced matrix, integer vectors (at least one of them constant) and its linear relaxation will have the same value. The importance of Theorem 4.4 lies in the speed of solving an IP with a totally balanced matrix.

If a $\{0,1\}$ matrix M is balanced, totally balanced or totally unimodular, respectively, then M^T is balanced, totally balanced or totally unimodular, respectively. Thus, each of Theorems 4.2, 4.3, 4.4 could be applied twice to achieve a duality gap of zero. That is, if the matrix M in an IP is totally unimodular, totally balanced, or balanced and the vectors satisfy the integrality and constant restrictions then all four of IP, LP, dual LP, and its un-relaxation have the same value.

5. Dominating and packing functions

The adjacency matrix of a graph is denoted by A and its closed neighborhood matrix $A+I_n$ by N (where I_n is the $n\times n$ identity matrix). Recall that vectors are denoted by bold variables, so the expression $N\boldsymbol{x}$ denotes the usual matrix multiplication of the $n\times n$ matrix N and the $n\times 1$ vector $\boldsymbol{x}$. Two vectors satisfy $\boldsymbol{x} > \boldsymbol{y}$ if and only if $x_i > y_i$ for all i. Likewise, $\boldsymbol{x} < \boldsymbol{y}$ if and only if $x_i < y_i$ for all i; $\boldsymbol{x} \le \boldsymbol{y}$ and $\boldsymbol{x} \ge \boldsymbol{y}$ are defined similarly. Every function $\varphi : V(G) \to \mathbb{R}$ has a column vector representation $\boldsymbol{\varphi} = \langle \varphi(v_1), ..., \varphi(v_n)\rangle^T$ for any fixed ordering $v_1, ..., v_n$ of the vertices of G. For example, the set $\{v_1, v_4\}$ of vertices of the graph in Figure 6 can be represented by the function $x(v_1) = x(v_4) = 1, x(v_2) = x(v_3) = x(v_5) = 0,$ or as a vector $\boldsymbol{x} = \langle 1,0,0,1,0\rangle^T$.



$$N\boldsymbol{x} = \begin{bmatrix} 1&1&0&0&1\\ 1&1&1&0&1\\ 0&1&1&1&0\\ 0&0&1&1&1\\ 1&1&0&1&1 \end{bmatrix} \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} \ge \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$$

Figure 6: A graph and the matrix inequality $N\boldsymbol{x} \ge \mathbf{1}$.

A *dominating function* on a graph G is a function $g : V(G) \to \{0,1\}$ such that $g(N[v]) = \sum_{u\in N[v]} g(u) \ge 1$ for all vertices $v \in V(G)$. The *characteristic function* φ_S defined by $\varphi_S(v) = 1$ when $v \in S$ and 0 otherwise is a dominating function iff S is a dominating set. A *minimum* dominating function on a graph G is a dominating function g which attains the minimum value of $|g| = \sum_{v\in V(G)} g(v)$. Recall, this minimum is denoted by $\gamma(G)$, the *domination number* of G. The $\{0,1\}$-vector $\boldsymbol{x} =< x(v_1), x(v_2), \ldots, x(v_n) >^T$ of any dominating function x satisfies the matrix inequality $N\boldsymbol{x} \ge \mathbf{1}$,since the matrix product $N\boldsymbol{x}$ is the column vector $\langle x(N[v_1]), x(N[v_2]), \ldots, x(N[v_n])\rangle^T$. Figure 6 depicts the dominating function $\boldsymbol{x}$, and the verification that $N\boldsymbol{x} \ge \mathbf{1}$.

If we relax the condition that a dominating function need be $\{0,1\}$ valued, and instead allow the function to take on any non-negative real value then we have the following (there is no need for the function to have values greater than 1, and as we will see later, optimal functions always have rational values): a *fractional dominating function* is a function $g : V(G) \to [0,1]$ such that $g(N[v]) = \sum_{u\in N[v]} g(u) \geq 1$ for all vertices $v \in V(G)$.

A *minimum fractional dominating function* is a fractional dominating function g such that the value $|g| = \sum_{v\in V(G)} g(v)$ is as small as possible. This minimum value is the *fractional domination number* of G, denoted by $\gamma_f(G)$. The vector $\boldsymbol{x}$ of any fractional dominating function x satisfies the matrix inequality $N\boldsymbol{x} \geq \mathbf{1}$.

Recall, a set $S \subseteq V(G)$ is a (*closed neighborhood*) *packing* if for any vertex $x \in G$, we have $|S \cap N[x]| \leq 1$. A *packing function* is a function $h : V(G) \to \{0,1\}$ such that $h(N[v]) = \sum_{u\in N[v]} h(u) \leq 1$ for all vertices $v \in V(G)$. A maximum packing function on a graph G is a packing function h which attains the maximum value of $|h| = \sum_{v\in V(G)} h(v)$, denoted by $\rho(G)$, the *packing number* of G. The $\{0,1\}$-vector $\boldsymbol{x}$ of any packing function x satisfies the matrix inequality $N\boldsymbol{x} \leq \mathbf{1}$.

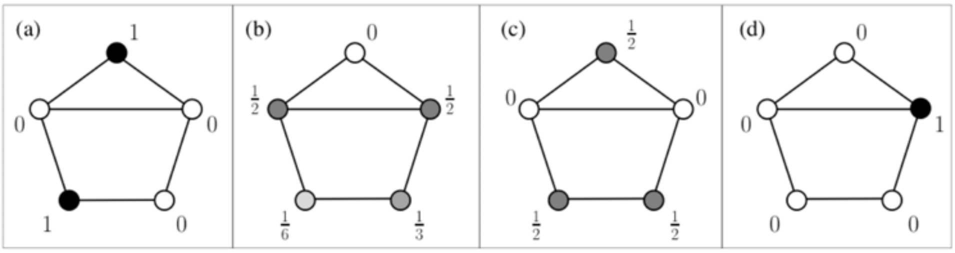


Figure 7: (a) A minimum dominating function, (b) minimum fractional dominating function, (c) maximum fractional packing function, and (d) maximum packing of C_5 with a chord.

A function $h : V(G) \to [0,1]$ is a *fractional packing function* provided that $h(N[v]) = \sum_{u\in N[v]} h(u) \leq 1$ for all $v \in V(G)$. A *maximum fractional packing function* is a fractional packing function h such that the value attained by $|h| = \sum_{v\in V(G)} h(v)$ is as large as possible. This maximum is the *fractional* (*closed neighborhood*) *packing number* of G and is denoted by $\rho_f(G)$. The vector $\boldsymbol{x}$ of any packing function x satisfies the matrix inequality $N\boldsymbol{x} \leq \mathbf{1}$.

5.1. Domination as an integer program

The problem of determining the domination number can be formulated as an integer program using the neighborhood matrix $N = A + I_n$; $\gamma(G)$ is the value of the integer program (3). From this, we can define the fractional domination number as the value of the linear programming relaxation of the integer program (3); γ_f is the value of the linear program (4). Determining $\rho_f(G)$ can be likewise formulated in LP terms, by taking the linear programming dual of (4). Determining the packing number can be formulated in IP terms, by adding the additional constraint to (5) that the optimal solutions need to be integer valued; $\rho(G)$ is the value of the integer program (6).

$$\text{minimize } \mathbf{1}^T\boldsymbol{y} \text{ subject to: } N\boldsymbol{y} \geq \mathbf{1},\ \boldsymbol{y} \geq \mathbf{0},\ y_i \in \mathbb{Z} \tag{3}$$

$$\text{minimize } \mathbf{1}^T\boldsymbol{y} \text{ subject to: } N\boldsymbol{y} \geq \mathbf{1},\ \boldsymbol{y} \geq \mathbf{0} \tag{4}$$

$$\text{maximize } \mathbf{1}^T\boldsymbol{x} \text{ subject to: } N\boldsymbol{x} \leq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0} \tag{5}$$

$$\text{maximize } \mathbf{1}^T\boldsymbol{x} \text{ subject to: } N\boldsymbol{x} \leq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0},\ x_i \in \mathbb{Z} \tag{6}$$

By the theory of linear relaxations, $\rho(G) \leq \rho_f(G)$ and $\gamma_f(G) \leq \gamma(G)$. Putting these inequalities together, and using Theorem 4.1 we get the well-known string of inequalities for all graphs G

$$\rho(G) \leq \rho_f(G) = \gamma_f(G) \leq \gamma(G) \tag{7}$$

Note that since the matrix and vectors of the LP (4) all have integral entries, the value of (4), γ_f will be rational, hence, the reason the term "fractional" instead of real in (4) (see [23]).

6. Chordal graphs

A graph is *chordal* if there are no induced cycles on 4 or more vertices. Note that all trees are chordal, since trees have no induced cycles. A *perfect elimination ordering* of a graph G is an ordering $v_1, v_2, \ldots, v_n$ of $V(G)$ with the property that for each i, j, l with $i < j$, $j < l$, and $v_l, v_j \in N[v_i]$, then $v_l \in N[v_j]$. In [28] Rose showed that a graph is chordal if and only if it admits a perfect elimination ordering. Note that the Hajós graph H in Figure 8(b) is chordal, however, $\rho(H) = 1 < 2 = \gamma(H)$. Something more is needed to guarantee equality in $\rho(G) \leq \gamma(G)$.

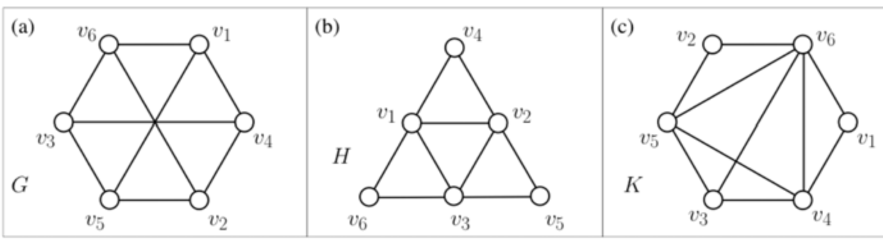


Figure 8: (a) A graph G which is *not* chordal, (b) a chordal (not strongly chordal) graph H and (c) a strongly chordal graph K.

6.1. Strongly chordal graphs

A *strong elimination ordering* of a graph G is an ordering $v_1, v_2, \dots, v_n$ of $V(G)$ with the property that for each i, j, l, k with $i < j$, $k < l$, $v_k, v_l \in N[v_i]$, and $v_k \in N[v_j]$ then $v_l \in N[v_j]$. In [9] Farber defines a graph to be strongly chordal if and only if it admits a strong elimination ordering. Chordal graphs are defined in terms of forbidden induced subgraphs, namely cycles of length greater than or equal to four. A k-sun (also called a trampoline) on $2k$ vertices is a construction starting with a Hamiltonian graph H of order k, with Hamilton cycle $v_1, v_2, \dots, v_k$, next k new vertices $u_1, u_2, \dots, u_k$ are added so that each u_i is adjacent to v_i and v_{i+1} (where $v_{k+1} = v_1$). In Figure 8(b), H is a 3-sun (or trampoline on 6 vertices). Farber in [9] showed that a graph is strongly chordal if and only if it is chordal and contains no k-suns as induced subgraphs.

Farber [9] and independently Iijima and Shibata [20] proved that the closed neighborhood matrix N of a graph G is totally balanced if and only if G is strongly chordal.

Theorem 6.1. [9, 20] *A graph G is strongly chordal if and only if N is a totally balanced matrix.*

Proof. ([9], $\Rightarrow$) Let M be a submatrix of N in the family $\mathcal{M}_k$ (with exactly two ones in every row and column), then M bay be viewed as the vertex-edge incidence matrix of a cycle on k vertices. Next, it observed that the ordering $v_1, v_2, \dots, v_n$ is a strong elimination ordering if and only if $F = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is not a submatrix of N, where N was formed with this ordering (if F is a submatrix of N, from columns $i < j$ and rows $k < l$ of N, then we would have $v_k, v_l \in N[v_i]$ and $v_k \in N[v_j]$ but $v_l \notin N[v_j]$). Next, since the class of totally balanced matrices is closed under arbitrary row and column permutations, F is a submatrix of the vertex-edge incidence matrix of every cycle of length at least three. Hence, for sufficiency it suffices to show that if G is a k-sun or a C_l ($l \geq 4$) then N contains the vertex-edge incidence matrix of C_l ($l \geq 3$) as a submatrix. The proof of any cycle containing the the vertex-edge incidence matrix of C_l ($l \geq 3$) as a submatrix is straightforward and omitted. Suppose G is a k-sun on $2k$ vertices. Then the submatrix fromed from the k rows corresponding to the vertices on the Hamiltonian cycle and the k columns corresponding to the vertices of degree 2 is exactly the vertex-edge incidence matrix of a C_k. □

Thus, Theorem 4.4 gives $\gamma_f(G) = \gamma(G)$ for all strongly chordal graphs G. More is true, since N is symmetric, we have $\rho(G) = \rho_f(G)$ for all strongly chordal graphs G by Theorem 4.4 (and therefore $\rho(G) = \rho_f(G) = \gamma_f(G) = \gamma(G)$).

Theorem 6.2. *If a graph G is strongly chordal, then $\rho(G) = \gamma(G)$*

This result is a generalization of Theorem 2.1 of Meir and Moon [22], since all trees are strongly chordal. This is also a generalization of Theorem 2.2 [6] since block graphs are strongly chordal. In fact, the class of strongly chordal graphs contains the class of trees, interval graphs, split graphs, and directed path graphs.

As an example, if we inspect the closed neighborhood matrix of $G = K_{3,3}$ in Figure 8(a), we find submatrices in the family $\mathcal{M}_3$, thus $N(G)$ is not totally balanced.

$$N(G) = \begin{bmatrix} \mathbf{1} & 0 & \mathbf{0} & 1 & \mathbf{1} & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ \mathbf{0} & 0 & \mathbf{1} & 1 & \mathbf{1} & 1 \\ \mathbf{1} & 1 & \mathbf{1} & 1 & \mathbf{0} & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

If we inspect the closed neighborhood matrix of the Hajós graph H in Figure 8(b), we will find a submatrix in the family $\mathcal{M}_3$, thus $N(H)$ is not totally balanced. We also inspect the closed neighborhood matrix of a strongly chordal graph K in Figure 8(c). There are no submatrices in the family $\mathcal{M}_k$, for $3 \le k \le 6$, thus $N(K)$ is balanced. Further we can see that $v_1, v_2 \ldots, v_6$ (with vertex labels as in Figure 8(c)) is a strong elimination ordering of $V(K)$, since $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is not a submatrix of $N(K)$.

$$N(H) = \begin{bmatrix} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ 1 & 1 & 1 & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad N(K) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

It is possible for a graph G to satisfy $\rho(G) = \gamma(G)$ where G is neither strongly chordal nor efficient. The graph T in Figure 9(a) is neither strongly chordal (T is a 4-sun and has two induced 3-suns), nor does it have an efficient dominating set ($F(T) = 6 < 8$); however $\rho(T) = \gamma(T) = 2$. Thus, neither condition is necessary for equality in $\rho(G) \le \gamma(G)$. If we inspect the closed neighborhood matrix of T, we find submatrices in the families $\mathcal{M}_3$, $\mathcal{M}_4$, thus $N(T)$ is not totally balanced (or balanced).

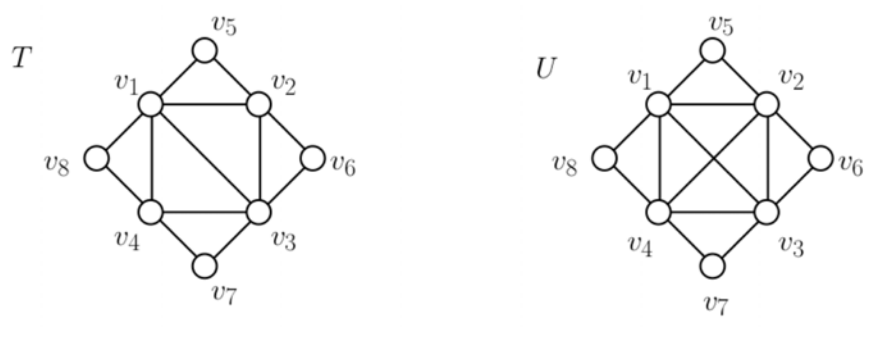


Figure 9: A 4-sun T with $\rho(T) = \gamma(T) = 2$ and a complete 4-sun U with $\rho(U) = \gamma(U) = 2$.

$$N(T) = \begin{bmatrix} 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ 1 & 1 & 1 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 1 & 1 & 1 & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & 0 & 1 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad N(U) = \begin{bmatrix} 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 1 & 1 & 1 & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ 1 & 1 & 1 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

6.2. Odd-sun-free chordal

Recall that strongly chordal graphs are graphs which are chordal and contain no induced suns. A graph G is *odd-sun-free chordal* if G is chordal and contains no induced odd suns. The class of odd-sun-free chordal contains the class of chordal graphs, since strongly chordal graphs are sun-free chordal (even or odd). The complete 4-sun U, depicted in Figure 9(b) is odd-sun-free chordal but not strongly chordal. While the 4-sun T , depicted in Figure 9(a), is neither strongly chordal nor odd-sun-free chordal.

A *hypergraph* $\mathcal{H} = (V, \mathcal{X})$ is a collection of $|V(\mathcal{H})| = n$ vertices, together with a collection $\mathcal{X}$ of subsets of $V(\mathcal{H})$. A (simple) graph is then a hypergraph where $\mathcal{X}$ contains only subsets of two distinct elements of $V(\mathcal{H})$. Fix $G = (V, E)$, then form the hypergraph $\mathcal{N}(G) = (V, \mathcal{X})$ as follows. For every $v \in V(G)$, let $N_G[v] \in \mathcal{X}$ be a hyperedge in $\mathcal{N}(G)$. A hypergraph is balanced if its vertex hyperedge incidence matrix is balanced (see Berge [3]). The vertex hyperedge incidence matrix of $\mathcal{N}(G)$, $B(\mathcal{N}(G))$ is just the closed neighborhood matrix N in disguise, that is, $B(\mathcal{N}(G)) = N$. Thus, the hypergraph $\mathcal{N}(G)$ is balanced if and only if the closed neighborhood matrix $N(G)$ is a balanced matrix.

Theorem 6.3. [5] *A graph G is odd-sun-free chordal if and only if $\mathcal{N}(G)$ is balanced.*

This together with Theorem 4.3 (Fulkerson et al [10]), gives equality in $\rho \leq \gamma$ when $N = A + I_n$ is a balanced matrix, that is, the matrix N contains no odd sized square submatrix with exactly two ones in every row or column.

Theorem 6.4. *If a graph G is odd-sun-free chordal, then $\rho(G) = \gamma(G)$.*

As an example, if we inspect the closed neighborhood matrix of the complete 4-sun U (as in Figure 9(b)), we will find a submatrix in the family $\mathcal{M}_4$ (but no submatrix in the families $\mathcal{M}_3$, $\mathcal{M}_5$ or $\mathcal{M}_7$), thus $N(U)$ is balanced but not totally balanced. Thus, $\rho(U) = \gamma(U)$.

7. Another sufficient condition

As we saw in Figures 2 and 4, chordality is not a necessary condition for equality in $\rho(G)$ and $\gamma(G)$.

Theorem 7.1. *If a graph G is any even-sun, then $\rho(G) = \gamma(G)$.*

Proof. Recall, a k-sun on $2k$ vertices is a graph obtained by a construction starting with a Hamiltonian graph H of order k, with Hamilton cycle $v_1, v_2, \ldots, v_k$, next k new vertices $u_1, u_2, \ldots, u_k$ are added so that each u_i is adjacent to v_i and v_{i+1} (where $v_{k+1} = v_1$). Let D be the set $\{v_2, v_4, \ldots, v_n\}$ of H. D is a dominating set of size $k/2$. Let P be the set $\{u_2, u_4, \ldots, u_n\}$. P is a packing of size $k/2$, and thus $\rho(G) = \gamma(G)$. □

The class of odd-sun-free graphs are defined by forbidden induced subgraphs (in particular, cycles on four or more vertices and odd suns). Although we have equality in ρ and γ for all even suns, if a graph G contains a non chordal even sun H as an induced subgraph, we can not guarantee equality in $\rho(G)$ and $\gamma(H)$.

8. The search for a necessary condition

So far, we only have sufficient conditions for equality in $\rho(G) \leq \gamma(G)$. No necessary condition for equality in $\rho(G) \leq \gamma(G)$ can be of the form of forbidding particular induced subgraphs. For, if there were a forbidden induced subgraph H, then form the graph G by adding a new vertex v adjacent to all vertices of H. Then $\Delta_G(v) = n - 1$, and $\rho(G) = \gamma(G) = 1$.

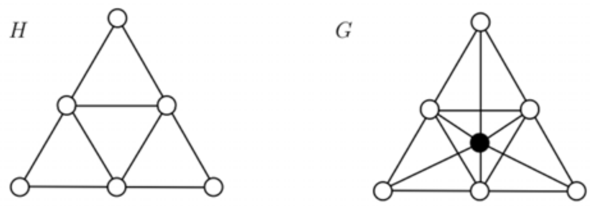


Figure 10: An induced subgraph H of G, with $\rho(G) = \gamma(G) = 1$.

Theorem 8.1. *If $\rho(G) = \gamma(G) > 1$ and G is connected, then $diam(G) > 2$.*

Proof. Let G be a connected graph with diameter at most two. Then $\rho(G) = 1$, a contradiction. □

Theorem 8.2. *If $\rho(G) = \gamma(G)$, then $\gamma_f(G)$ must be an integer.*

Proof. Since we have $\rho(G) \leq \gamma_f(G) \leq \gamma(G)$ for all graphs G, if $\gamma_f(G)$ is not an integer, then $\gamma_f(G) < \gamma(G)$ and the duality gap $\gamma(G) - \rho(G) \geq 1$. Alternatively, suppose $\rho(G) = \gamma(G)$, then $\rho(G) = \gamma_f(G) = \gamma(G)$, and thus $\gamma_f(G)$ is an integer. □

This necessary condition is not sufficient, see Figure 11 where γ_f is an integer (as shown in [29]).

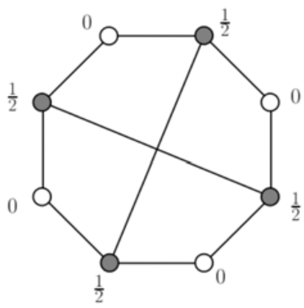


Figure 11: A graph with $\rho = \gamma_f = 2 < \gamma = 3$.

9. Regular graphs

Here, we find necessary and sufficient conditions for regular graphs G of degree r on n vertices satisfying $\rho(G) = \gamma(G)$.

Theorem 9.1. *If G is regular of degree r on n vertices, then $\rho(G) = \gamma(G)$ if and only if G is efficient (that is, there exists an efficient dominating set).*

Proof. Let G be a regular graph of degree r on n vertices. Since each closed neighborhood has size $r+1$, we have $\rho(G) \leq \lfloor \frac{n}{r+1} \rfloor$. Since each vertex v_i in a minimum dominating set S dominates exactly $r+1$ vertices and (each $v_i \in S$) can dominate at most $r+1$ distinct vertices, we have $\gamma(G) \geq \lceil \frac{n}{r+1} \rceil$. Thus we have $\rho(G) \leq \lfloor \frac{n}{r+1} \rfloor \leq \lceil \frac{n}{r+1} \rceil \leq \gamma(G)$. If $\rho(G) = \gamma(G)$, then $\lfloor \frac{n}{r+1} \rfloor = \lceil \frac{n}{r+1} \rceil$, and thus, $\frac{n}{r+1}$ is an integer and $r+1$ divides n. Let S be a γ-set. Then since $\gamma(G) = \frac{n}{r+1}$, and each $u \in S$ dominates exactly $r+1$ vertices, no vertex can be dominated more than once, thus each vertex $v \in V(G)$ is dominated by exactly one vertex in S, thus S is an efficient dominating set, and the graph G is efficient. □

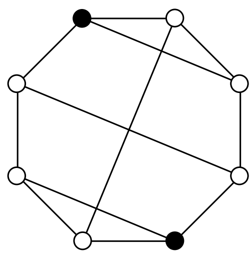


Figure 12: An efficient 3-regular graph on $2(3+1)$ vertices with $\rho = \gamma = 2$.

Note that a necessary condition for any regular graph G of degree r on n vertices to have $\rho(G) = \gamma(G)$ is that n must be an integral multiple of $r+1$. Upper and lower bounds on the fractional domination number were found independently in [6] and [13]. For any graph on n vertices, $\frac{n}{\Delta(G)+1} \leq \gamma_f(G) \leq \frac{n}{\delta(G)+1}$. Thus, this necessary condition for r-regular graphs can also be observed from Theorem 8.2, since $\gamma_f(G) = \frac{n}{r+1}$ is an integer only if $r+1$ divides n. For arbitrary r-regular graphs on $k \cdot (r+1)$ vertices $\rho(G)$ can still be strictly less than $\gamma(G)$ (see Figure 13). Thus, the necessary condition is not sufficient, unless the graph has an efficient dominating set.

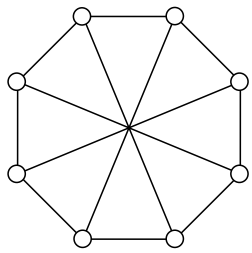


Figure 13: A 3-regular graph on $2(3+1)$ vertices with $\rho = 1 < \gamma_f = 2 < \gamma = 3$.

10. The duality gap and graph products

This paper is concerned with the question of when the duality gap $\gamma(G) - \rho(G)$ is zero, that is when $\rho(G) = \gamma(G)$. As we will see, when the gap is at most one, there still can be extremely beneficial results.

The *Cartesian product* of G and H is denoted by $G \square H$; the vertices are the ordered pairs $\{(x,y) | x \in V(G), y \in V(H)\}$, and two vertices (u,v) and (x,y) are adjacent if and only if one of the following is true: $u = x$ and v is adjacent to y in H; or $v = y$ and u is adjacent to x in G. The *direct product* of G and H is denoted by $G \times H$. The vertices are the ordered pairs $\{(x,y) | x \in V(G), y \in V(H)\}$, and two distinct vertices (u,v) and (x,y) are adjacent if and only if $u \in N_G(x)$ and $v \in N_H(y)$. The *strong direct product* of G and H is denoted by $G \boxtimes H$. The vertices are the ordered pairs $\{(x,y) | x \in V(G), y \in V(H)\}$, and two distinct vertices (u,v) and (x,y) are adjacent if and only if $u \in N_G[x]$ and $v \in N_H[y]$. (See Figure 14.)

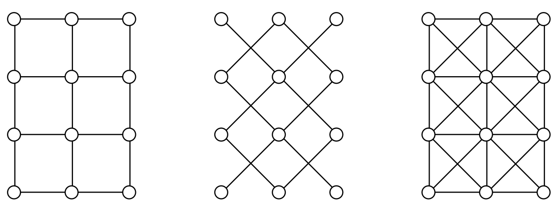


Figure 14: Graph products: the (a) Cartesian product $P_3 \square P_4$, (b) direct product $P_3 \times P_4$, and (c) strong direct product $P_3 \boxtimes P_4$.

10.1. Vizing's conjecture

In 1968, Vizing [31] conjectured that for all graphs G and H, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$. Jacobson and Kinch [19] proved $\gamma(G\Box H) \geq \rho(G)\gamma(H)$. In Doug Rall's presentation of "Domination and Packing in Graph Products" at the 20th Clemson mini-conference, Rall stated a corollary of this result: if $\rho(G) = \gamma(G)$, then for all graphs H, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$, that is, Vizing's conjecture [31] holds for G. This sufficient condition is not necessary, as Hartnell and Rall [14] showed that Vizing's conjecture will still hold for G if $\rho(G) + 1 = \gamma(G)$. For instance any cycle C_n satisfies $\gamma(C_n) \leq \rho(C_n) + 1$ (with equality when $n \equiv 0 \pmod 3$).

10.2. Other graph products

Nowakowski and Rall [24] explored how domination parameters (as well as several others) behave with respect to several associative graph products.

Theorem 10.1. [24] *If* $\rho(G) = \gamma(G)$*, then for any non-trivial graph* H, $\gamma(G \times H) \geq \gamma(G)\gamma(H)$ *and* $\gamma(G \boxtimes H) = \gamma(G)\gamma(H)$.

11. Random graphs

For connected graphs on four or fewer vertices there is only one graph which has $\rho < \gamma$, the four cycle. For connected graphs of order 5, there are only three which have $\rho < \gamma$ and of the 112 connected graphs of order 6, only 26 have $\rho < \gamma$. However, as $n = |V(G)|$ grows large, a smaller percentage of graphs will have equal domination and packing numbers.

A random graph $G_{n,p}$ is a graph on n vertices and any pair of the $\binom{n}{2}$ vertices are adjacent with fixed probability p with $0 < p < 1$. It is known that a large random graph can be relied upon to have diameter exactly two, thus $\rho(G(n,p)) = 1$ with probability approaching one as $n \to \infty$. Dreyer [8], showed that for p fixed and any $\epsilon > 0$, any fixed set of cardinality $(1+\epsilon)\log_{1/1(1-p)}(n)$ is a dominating set of $G(n,p)$ with probability approaching one as $n \to \infty$, and that sets of size $(1-\epsilon)\log_{1/1(1-p)}(n)$ dominate with probability approaching zero as $n \to \infty$. (See also Godbole and Weiland [12].) Thus, the probability of having $\gamma(G(n,p)) = \rho(G(n,p))$ approaches zero as n approaches infinity.

12. Summary / future work

12.1. Families of graphs

We depict the results thus far in Figures 15. Each region of the venn diagram can be shown to be nonempty. Note that the graph G in Figure 2, has $\rho(G) = \gamma(G) = 2$, yet is not efficient, not chordal, and not an even sun, and thus, is in none of the families depicted in Figure 15.

12.2. Possible avenues for extending results

The results of [9], [20] on strongly chordal graphs and later [5] on odd-sun-free chordal graphs rest on theorems in linear programming which guarantee integral objective function solutions to arbitrary linear programs satisfying certain matrix and vector conditions. These matrix conditions were based on viewing the matrix in the linear program as a vertex-edge incidence matrix of a hypergraph, and then excluding certain submatrices in order to guarantee integral solutions to the linear program. Since the matrix in the IP/LP pair of domination and fractional domination is square and the vectors are both the constant ones vector, it might be possible to extend the results of [10] beyond balanced matrices, for our particular IP/LP pair of domination and fractional domination.

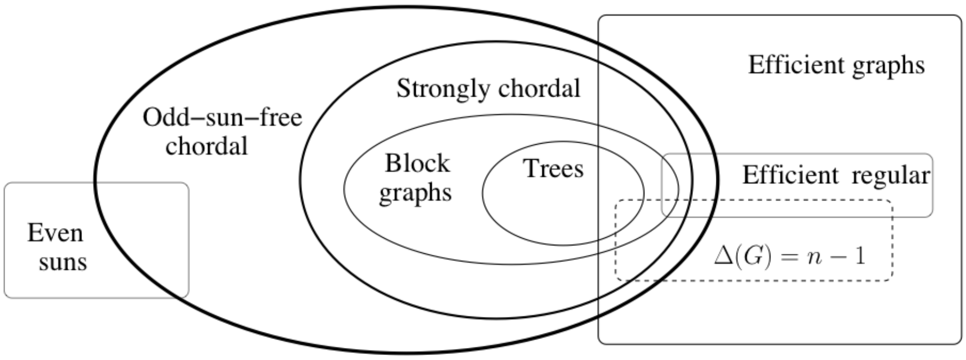


Figure 15: Families of graphs with $\rho = \gamma$.

Apart from exploring closed neighborhood matrix conditions to find new families of graphs with equal domination and packing numbers, we could explore combinatorial avenues, analogous to Rall [26]. A set S is an *open (neighborhood) packing* if for each $u, v \in S$, we have $N(u) \cap N(v) = \emptyset$. The open packing number $\rho^{\circ}(G)$ is the size of a largest open (neighborhood) packing. Rall [26] proved that all trees have equal total domination and open packing numbers, using (in part) the following. For a graph G without isolates, $N_{\circ}(G)$ is the *open neighborhood intersection graph* of G . That is, $N_{\circ}(G)$ is the graph with vertex set $V(G)$ and $xy \in E(N_{\circ}(G))$ if and only if $N_G(x) \cap N_G(y) \neq \emptyset$. An independent set of maximum size in $N_{\circ}(G)$ will correspond to an open packing of maximum size, and we have $\rho^{\circ}(G) = \beta(N_{\circ}(G))$.

For a graph G , consider the *closed neighborhood intersection graph* of G . That is, the graph with vertex set $V(G)$ and xy is an edge if and only if $N_G[x] \cap N_G[y] \neq \emptyset$. This is nothing more than the square of G , denoted by G^2 , the graph with the same vertex set of G and xy is an edge if and only if $dist_G(x, y) \leq 2$. An independent set S of maximum size in G^2 will correspond to a packing S of maximum size in G , and we have $\rho(G) = \beta(G^2)$. (See Figure 16.)

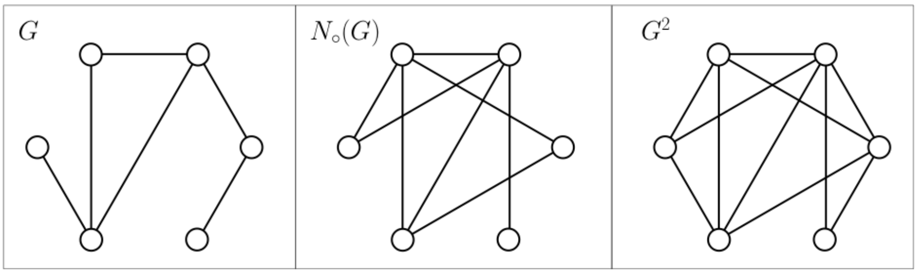


Figure 16: A graph and its open and closed neighborhood intersection graphs.

We end with a depiction of all connected graphs with $n \leq 6$ with $\rho(G) < \gamma(G)$, obtained with the help of an atlas by Read and Wilson [27]. Each graph G in Figure 17 satisfies $\rho(G) = 1 < 2 = \gamma(G)$.

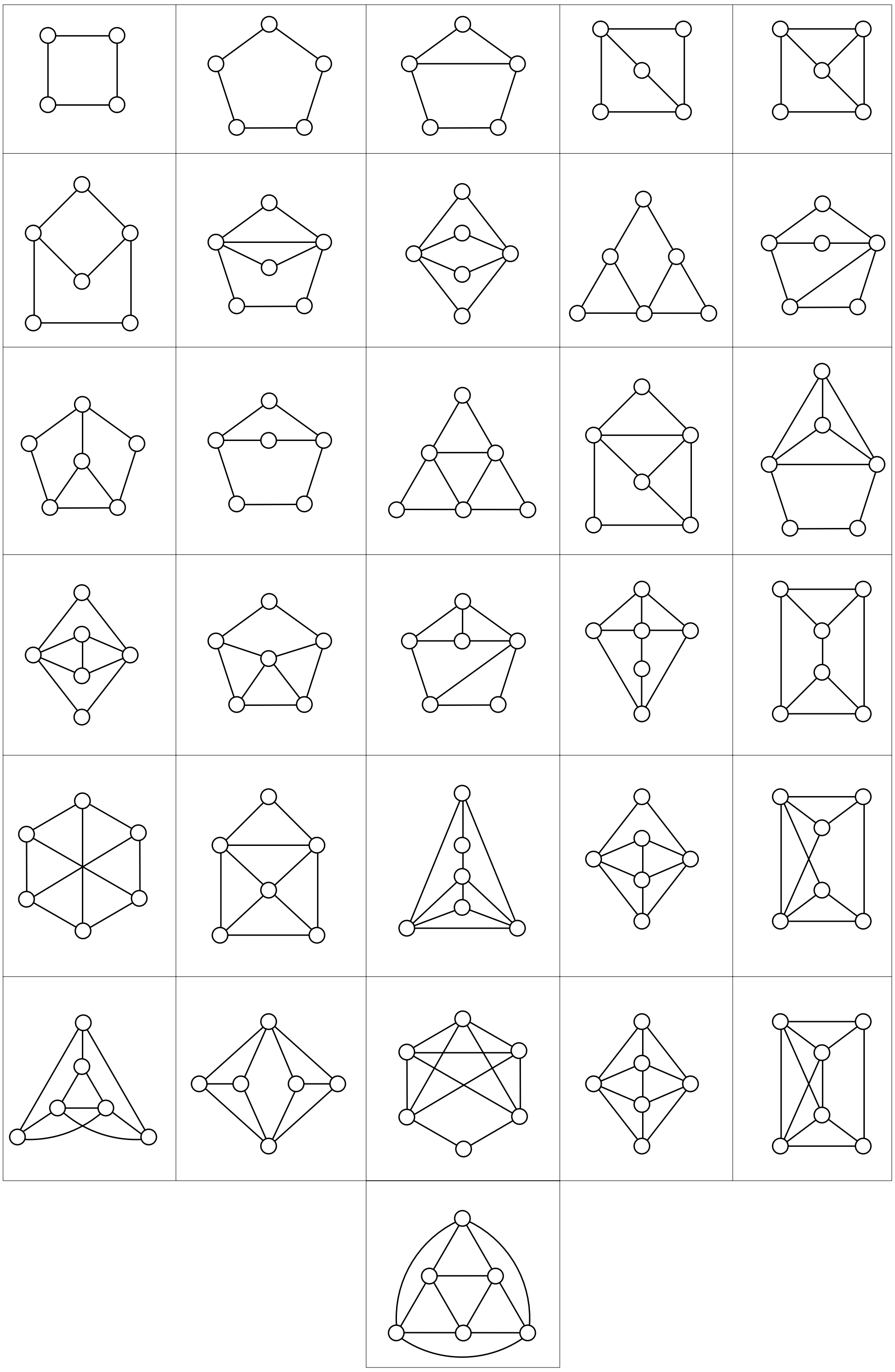


Figure 17: All connected graphs on six or fewer vertices with $\rho(G) < \gamma(G)$.

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