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Common-Edge Sigraphs*

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Abstract

A *signed graph* (or, *sigraph* in short) is a graph G in which each edge x carries a value $s(x) \in \{-1, +1\}$ called its *sign*. Given a sigraph S, a new sigraph $C_E(S)$, called the *common-edge sigraph* of S, is that sigraph whose vertex-set is the set of pairs of adjacent edges in S and two vertices of $C_E(S)$ are adjacent if the pairs of adjacent edges of S have exactly one edge in common, with the same sign as that of the common edge. In this paper, we study common-edge sigraphs and characterize sigraphs whose common-edge sigraphs and second iterated line sigraphs are switching equivalent. Also, we determine the sigraphs S for which their common-edge sigraphs are balanced, S-consistent and S-cycle-compatible, respectively.

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1. Introduction

For all terminology and notations in *graph theory*, except for those that are specifically defined here, the reader is referred to West [31]. The *graphs* considered here are finite, undirected, without self-loops or multiple edges. By a (p,q) *-graph* G we mean a graph having p vertices and q edges; p is called the *order* and q is called the *size* of G.

In the spirit of a study of *graph-valued functions*, obtaining the *line graph* $L(G)$ of a given graph $G=(V,E)$ may be treated as a mapping L that *operates* on G to give rise to $L(G)$ as the graph whose vertices are the edges of G with two of these vertices *joined to each other* (or, *adjacent*) whenever the edges of G they represent have a common vertex in G, or equivalently the two edges form a P_3 in G. This observation motivated Broersma and Hoede [12] to define in general *path graphs* $P_k(G)$ of G for any positive integer k as follows: $P_k(G)$ has for its vertex-set the set $\mathcal{P}_k(G)$ of all distinct paths in G having k vertices, and two vertices in $P_k(G)$ are adjacent if they represent two paths $P,Q \in \mathcal{P}_k(G)$ whose union forms either a path P_{k+1} or a cycle C_k in G. Some improvement of their paper was subsequently given by Li and Lin [27].

Much earlier, making independently the same observation as above on the formation of a line graph $L(G)$ of a given graph G, Kulli [26] had defined the *common-edge graph* $C_E(G)$ of G as the *intersection graph* of the family $\mathcal{P}_3(G)$ of 2 *-paths* (*i.e.*, paths of length two) each member of which is treated as a set of edges of the corresponding 2-path. It is easy to verify that for any graph G, $P_3(G)$ is a spanning subgraph of $C_E(G)$. Furthermore, it is not hard to establish the following two results.

Lemma 1.1. *For any graph G of order at least three, $P_3(G)$ is triangle-free if and only if G is triangle-free.*

Proposition 1. *For any connected graph G of order at least three, $C_E(G) \cong P_3(G)$ if and only if G is isomorphic to either a path or a cycle.*

Further, Kulli [26] has established

$$C_E(G) \cong L^2(G) \tag{1}$$

for any isolate-free graph G, where $L(G) := L^1(G)$ and $L^t(G)$ denotes the t *-th iterated line graph* of G for any integer $t \geq 2$. In fact, he has explicitly shown (1) by describing the natural isomorphism $\varphi : L^2(G) \to C_E(G)$ which associates each 2-path (e_re_s, e_se_t) in G with the vertex labelled (e_re_s, e_se_t) in $C_E(G)$ so that the 2-paths (e_re_s, e_se_t) and (e_ue_v, e_ve_w) as vertices in $C_E(G)$ are adjacent whenever one of the edges in each of the 2-paths is common.

The *degree* of a vertex $v \in V(G)$ in G is denoted $d_G(v)$ or just $d(v)$ if G is clear from the context, and for any edge $e = uv$ its *edge-degree* is defined as the number $d_1(e) = d(u) + d(v) - 2$. The following result has been proved by Kulli.

Theorem 1.2. [26] *If* G *is a* (p,q)*-graph, then* $C_E(G)$ *has,* $\frac{1}{2}\sum_{i=1}^{p} d(v_i)^2 - q$ *vertices and* $\frac{1}{2}\sum_{j=1}^{q} d_1(e_j)^2 - \frac{1}{2}\sum_{i=1}^{p} d(v_i)^2 + q$ *edges.*

We now extend the notion of $C_E(G)$ to the realm of signed graphs. As in Harary *et al.* [20] (*also see*, Chartrand [14]) by a *signed graph* S (or, *sigraph* in short) we mean a graph $G=(V,E)$, called the *underlying graph* of S and denoted by S^u, in which each edge x carries a value $s(x) \in \{-1,+1\}$ called the *sign* of x; an edge x is *positive* or *negative* according to $s(x)=+1$ or $s(x)=-1$. The set of positive edges of S is denoted by $E^+(S)$ and $E^-(S)=E(G)-E^+(S)$ is the set of negative edges. We regard graphs themselves as sigraphs in which every edge is positive. Given a graph G, let $\psi(G)$ denote the set of all sigraphs whose underlying graph is G. In general, a subgraph S' of a sigraph S is said to be *all-positive* (*all-negative*) if all the edges of S' are positive (negative). A sigraph is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise.

By a *negative section* (Gill and Patwardhan [18]) of a subsigraph S' of a sigraph S we mean a maximal edge-induced connected subsigraph in S' consisting of only negative edges of S; in particular, a negative section in a heterogeneous cycle of S is essentially a maximal all-negative path in the cycle. A cycle in a sigraph S is said to be *positive* if the product of the signs of its edges is positive or, equivalently, if the number of negative edges in it is even. A cycle which is not positive is said to be *negative.* Given S, a function $\mu : V(S) \to \{-1,+1\}$ is called a *marking* of S. We shall denote by $\mathcal{M}_S$ the set of all markings of S. A sigraph S together with one of its markings μ is denoted by S_μ.

Given a marking μ of S, by *switching* S with respect to μ we mean changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs in S_μ. The sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called the μ*-switched sigraph* or just *switched sigraph* when the marking is clear from the context (Gill and Patwardhan [18]).

We say that sigraph S_1 *switches to sigraph* S_2 (or that they are *switching equivalent* to each other), written as $S_1 \sim S_2$, whenever there exists $\mu \in \mathcal{M}_{S_1}$ such that $\mathcal{S}_\mu(S_1) \cong S_2$, where "$\cong$" denotes the isomorphism between any two sigraphs in the standard sense. Note that $S_1 \sim S_2$ implies that $(S_1)^u \cong (S_2)^u$.

Two sigraphs S_1 and S_2 are said to be *weakly isomorphic* (*e.g., see* Sozánski [30]) or *cycle isomorphic* (*e.g., see* Zaslavsky [32]) if there exists an isomorphism $f:(S_1)^u \to (S_2)^u$ such that the sign of every cycle Z in S_1 equals the sign of $f(Z)$ in S_2 (*i.e.,* f preserves both vertex adjacencies and the signs of the cycles of S_1 in S_2). The following theorem will also be useful in our further investigation.

Theorem 1.3. [30, 32] *Given a graph* G, *any two sigraphs in* $\psi(G)$ *are switching equivalent if and only if they are cycle isomorphic.*

Given a sigraph S, its *common-edge sigraph* $C_E(S)$ is that sigraph whose vertex-set is the set of pairs of adjacent edges in S and two vertices of $C_E(S)$ are adjacent if the corresponding pairs of adjacent edges of S have exactly one edge in common, with the same sign as that of their common edge.

In this paper we initiate a study of the common-edge sigraph of a given sigraph and solve some important sigraph equations and equivalences involving it. We begin by finding some new results on common-edge graphs which will be helpful towards this end.

2. Some new properties of common-edge graphs

We shall first characterize the graph G whose common-edge graph $C_E(G)$ is isomorphic to G. We need the following well known result for the purpose.

Lemma 2.1. [28] *For any positive integer* k *and for any graph* G, $G \cong L^k(G)$ *if and only if* G *is* 2*-regular.*

Theorem 2.2. *For a simple connected graph* G, $G \cong C_E(G)$ *if and only if* G *is a cycle.*

Proof. The *sufficiency* part of the proof being obvious, we will prove only its *necessity part.*

Hence, let $G \cong C_E(G)$. By (1), we get $G \cong C_E(G) \cong L^2(G)$ whence by Lemma 2.1 it follows that G must be a cycle. ☐

In the following lemma, a tree is called an *n-spider* if it has exactly one vertex of degree n and all other vertices of degree at most $n-1$.

Lemma 2.3. *For any simple connected* (p,q)*-graph* G *with* $p \geq 2$,

$$q = \sum_{i=1}^{p} \binom{d(v_i)}{2}$$

if and only if G *is a cycle or a 3-spider.*

Proof. Since $\sum_{i=1}^{p} \binom{d(v_i)}{2}$ is the number of edges in the line graph of G the result follows from an earlier one due to Acharya [2]. ☐

Theorem 2.4. *For a connected* (p,q)*-graph* G, $L(G) \cong C_E(G)$ *if and only if* G *is a cycle or* $K_{1,3}$.

Proof. *Sufficiency* is obvious from the definition of $L(G)$ and $C_E(G)$, so we will prove only the *necessity* part. Let $C_E(G) \cong L(G)$. The isomorphism implies that the number of vertices in $C_E(G)$ is equal to the number of vertices in $L(G)$, whence by Lemma 2.3 the graph must be either a cycle or a 3-spider.

Suppose G is not a cycle. Then, G is a 3-spider. We need to show that $G \cong K_{1,3}$. Towards this end, suppose this is false. Then, there exists a vertex u_1 of degree two in

G adjacent to the center c of the 3-spider G. Let u_2 be the other vertex adjacent to u_1. Let v_1 and w_1 be the other two vertices adjacent to c in G. Then, we get a 4-cycle $(\{u_2, u_1, c\}, \{u_1, c, v_1\}, \{v_1, c, w_1\}, \{w_1, c, u_1\})$ in $C_E(G)$ whereas there is no 4-cycle in $L(G)$, a contradiction to our hypothesis that $L(G) \cong C_E(G)$. Thus, $G \cong K_{1,3}$ and the proof is complete. □

3. Properties of Common-edge Sigraphs

We begin with an extension of Theorem 2.2 to the realm of sigraphs.

Theorem 3.1. *For any sigraph S, $S \cong C_E(S)$ if and only if S^u is 2-regular.*

Proof. The proof follows from the definition of $C_E(S)$, Lemma 2.1 and Theorem 2.2. □

Now, we recall the following definition of *line sigraph* $L(S)$ of a sigraph S introduced by Behzad and Chartrand [9]: the vertices of $L(S)$ correspond one-to-one with the edges of S, $e_ie_j \in E(L(S)) \Leftrightarrow$ the edges e_i and e_j have a common vertex in S and an edge e_ie_j in $L(S)$ is negative if both e_i and e_j are negative edges in S. A given sigraph H is called a *line sigraph* if there exists a sigraph S such that $H \cong L(S)$. A structural characterization of line sigraphs was recently given by these authors [8]. Our next result is an extension of Theorem 2.4 to the class of sigraphs, where $\ell(N_i)$ denotes the length of the negative section N_i. We have the following result.

Theorem 3.2. *For any connected sigraph S, $L(S) \sim C_E(S)$ if and only if*

(a) S^u is a cycle or $K_{1,3}$, and

(b) S is homogeneous, or S is a cycle containing an even number of negative sections.

Proof. The *sufficiency* part of the proof is straightforward. Therefore, we prove only the *necessity* part. Hence, let S be a connected sigraph satisfying $L(S) \sim C_E(S)$. This implies that $L(S^u) \cong C_E(S^u)$. Thus $S^u \cong C_n$, $n \geq 3$ or $S^u \cong K_{1,3}$ by Theorem 2.4. Hence (a) follows. To prove (b) suppose that S is heterogeneous. Since $L(S) \sim C_E(S)$, it follows that S is a cycle, say $S^u \cong C_n$, $n \geq 3$. Let $N_1, N_2, N_3, ..., N_r$ be the negative sections in S. Since $L(S) \sim C_E(S)$ we must have

$$\sum_{i=1}^{r} \{\ell(N_i)\} + \sum_{i=1}^{r} \{\ell(N_i) - 1\} \equiv 0 (mod\ 2).$$

Therefore, $r \equiv 0(mod\ 2)$ and (b) follows. □

In the next theorem we give the analogous properties of the relation between $L^2(S)$ and $C_E(S)$. We need the nonstandard notation $K_{1,r}^{t(k)}$ to define what is called a *wounded spider* for the graph obtained from the *star* $K_{1,r}$ by subdividing t of its edges k times each. Now if $d^+(v)$ and $d^-(v)$ represent respectively the positive degree and the negative degree of the vertex v in the sigraph S we have the following theorem, where the constraint of connectedness in the theorem is imposed due to the fact that without the condition the theorem is false as the following example shows: S^u has two components each one being C_6. One component has a negative edge and the other has two negative sections, one of length one and the other of length three.

Theorem 3.3. *For any connected sigraph* S, $L^2(S) \sim C_E(S)$ *if and only if* S *is homogeneous or it satisfies the following conditions:*

(a) for every cycle Z *in* S, *the number of negative sections of length one is even, and*

(b) for any vertex v *of degree greater than or equal to three,*

(i) either $d^-(v) = 0$ *or* $d^+(v) \leq 1$, *and*

(ii) if uv *is a negative edge then* $d^+(u) = 0$ *or* $d^+(v) = 0$.

Proof. ***Necessity***: Suppose $L^2(S)$ is switching equivalent to $C_E(S)$. This implies that $L^2(S^u) \cong C_E(S^u)$. Now, since they are switching equivalent they are cycle isomorphic by Theorem 1.3. So, the result is trivially true for homogeneous sigraphs. Hence, suppose that S is a heterogeneous sigraph and let Z be a heterogeneous cycle in S. Let the number of negative sections of length one in Z be n and $N_1, N_2, N_3, ..., N_r$ be the negative sections of length greater than one in Z. The length ℓ of any negative section of Z remains same in $C_E(Z)$ but reduces by two in $L^2(Z)$ except for a negative section of length one, whose length reduces by one. Since $C_E(Z) \sim L^2(Z)$,

$$n + \sum_{i=1}^{r}\{\ell(N_i)\} + \sum_{i=1}^{r}\{\ell(N_i) - 2\} \equiv 0(mod\ 2).$$

Therefore, $n \equiv 0(mod\ 2)$, proving (a).

On the other hand, suppose not all the vertices of a cycle $Z_C^{'}$ in $C_E(S)$ correspond to the adjacent edges of a cycle in S. Then, there must exist a vertex ee' in $Z_C^{'}$ which corresponds to the adjacent edges e and e' in S with a common vertex v of degree at least three. Since, $L^2(S) \sim C_E(S)$ it is easy to see (through the natural isomorphism between $L^2(G)$ and $C_E(G)$ pointed out by Kulli [26]) that either $d^-(v) = 0$ or $d^+(v) = 0$. Now, let at least one positive and at least one negative edge of S be incident at v and the number of positive edges incident at v be $r \geq 2$. Clearly, if $2 \leq r \leq m$, where $m \geq 3$ is the degree of v, then any two positive edges with one negative edge incident at v will create a negative triangle in $C_E(S)$ and an all-positive triangle in $L(S)$ and $L^2(S)$, thus contradicting the hypothesis. Therefore, $d^+(v) = r \leq 1$ and hence (b)(i) follows.

Next, suppose $uv = e_0$ is a negative edge of S and there exist positive edges e_1 and e_2 of S incident at u and v respectively. Let e_3 be another edge incident at v. Now, the triangle $Z'_C = ((e_1, uv), (uv, e_2), (uv, e_3))$, which is due to the negative edge uv in S being common to all the three 2-paths $(e_1, uv), (uv, e_2), (uv, e_3)$ represented as vertices in Z'_C, is clearly all-negative by the definition of $C_E(S)$. However, under the natural isomorphism the corresponding triangle Z'_{L^2} in $L^2(S)$ must be all-positive due to the definition of $L(S)$. This contradicts the hypothesis. Hence (b)(ii) follows.

Sufficiency: Suppose the conditions in the statement of the theorem hold for a given sigraph S. We shall show that $L^2(S) \sim C_E(S)$. If S is homogeneous then $L^2(S) \sim C_E(S)$ is obvious due to (1). Hence, suppose that S is a heterogeneous sigraph and let $L^2(S) \not\sim C_E(S)$. However, $L^2(S^u) \cong C_E(S^u)$ due to the universal fact (1) for graphs; therefore, under the natural isomorphism φ (say), there exists a cycle Z'_C in $C_E(S)$ such that, $Z'_C \not\sim Z'_{L^2}$ where Z'_{L^2} is the corresponding cycle in $L^2(S)$. Without loss of generality, we may assume that Z'_C has the least possible length. Now, if the cycle Z'_{L^2} in $L^2(S)$ and the corresponding cycle Z'_C in $C_E(S)$ are due to the edges of a single cycle Z in S, then, with n as the number of negative sections of length one and $N_1, N_2, N_3, ..., N_r$ as the negative sections of length greater than one in Z, we get

$$n + \sum_{i=1}^{r}\{\ell(N_i)\} + \sum_{i=1}^{r}\{\ell(N_i) - 2\} \not\equiv 0(mod\ 2),$$

implying that n is odd. This contradicts (a).

Therefore, the cycle Z'_{L^2} in $L^2(S)$ and the corresponding cycle $\varphi(Z'_{L^2}) = Z'_C$ in $C_E(S)$ are not due to the edges of a single cycle Z in S and hence Z'_C must contain a vertex v'_i which corresponds to adjacent edges e_i and e_{i-1} in S at least one of which, say e_i, does not lie on a single cycle in S but incident to a vertex v of degree greater than or equal to three in S. Let $Z'_C = (v'_1, v'_2, v'_3, ..., v'_i, ..., v'_k, v'_1)$, $k \geq 3$. Now, if none of the vertices of the cycle Z'_C corresponds to an edge of any cycle in S then either all the edges in S are incident to v or they form an edge-induced subsigraph containing v whose underlying graph is isomorphic to the wounded spider $K_{1,3}^{-1(1)}$. Suppose Z'_C is of length three. Since the degree of the vertex v is equal to three, we have by condition (b), Z'_C cycle-isomorphic to the corresponding cycle Z'_{L^2} in $L^2(S)$ contrary to our assumption. Thus, Z'_C must be of length greater than or equal to four. Now, the condition (b) and the definition of $C_E(S)$ imply together that the triangles formed in $C_E(S)$ due to the edges incident at v are cycle-isomorphic to the corresponding triangles in $L^2(S)$. Thus, Z'_C is the symmetric difference of the edge sets of these triangles and that of the cycles in $C_E(S)$ each of which is cycle-isomorphic to the corresponding cycle in $L^2(S)$ symmetric difference of which forms Z'_{L^2}. This implies that $Z'_C \sim Z'_{L^2}$, contrary to our assumption. Hence the proof is complete by contraposition. □

Remark 3.4. *For the case when S is disconnected, the problem will be treated elsewhere. Henceforth, in the rest of this paper all sigraphs will be assumed to be connected.*

4. Balanced Common-Edge Sigraphs

A sigraph is said to be *balanced* if every cycle in it is positive (Harary [19], Cartwright and Harary [13], Acharya and Acharya [5]. We need the following lemma:

Lemma 4.1. [33] *A sigraph in which every chordless cycle is positive is balanced.*

The following result gives a characterization of sigraphs whose common-edge sigraphs are balanced.

Theorem 4.2. *For any connected sigraph S, $C_E(S)$ is balanced if and only if S is a balanced sigraph such that for every vertex $v \in V(S)$ with $d(v) \geq 3$,*

(i) if $d(v) > 3$ then $d^-(v) = 0$;

(ii) if $d(v) = 3$ then $d^-(v) = 0$ or $d^-(v) = 2$; and

(iii) for every path $P_4 = (x, v, w, y)$ of length three, vw is a positive edge in S.

Proof. ***Necessity***: Let $C_E(S)$ be balanced. This implies, by the definition of balance, that every cycle Z' in $C_E(S)$ has an even number of negative edges. Since the sequence of adjacent edges of every cycle in S creates a cycle of the same sign in $C_E(S)$, by the definition of $C_E(S)$, the hypothesis implies that S must be balanced.

Next, let v be a vertex of S having degree at least three. If $d^-(v) \geq 3$, then any of the three negative edges incident at v would form a negative cycle in $C_E(S)$, a contradiction to our assumption. Therefore, $d^-(v) < 3$. If $d(v) > 3$, then $d^-(v)$ being equal to one or two would again contradict the assumption. Thus, (i) follows. Hence, $d(v) = 3$. Since $C_E(S)$ is balanced, it is clear from the definition of $C_E(S)$ that either $d^-(v) = 0$ or $d^-(v) = 2$. Thus (ii) follows. Now, suppose there exists a path $P_4 = (x, v, w, y)$ of length three through v such that vw is negative in S and e_1 is the edge incident at v but not on P_4. Then the triangle $Z' = ((e_1, vw), (vw, wy), (xv, vw), (e_1, vw))$ is all-negative and hence a contradiction to the assumption. Thus, vw must be positive. Therefore, (iii) follows.

Sufficiency: To achieve this part of the proof, suppose conditions in the statement of the theorem hold for a sigraph S. We shall show that $C_E(S)$ is balanced. Suppose that $C_E(S)$ is not balanced. Then, there exists a negative cycle in $C_E(S)$. Let Z' be a negative cycle of least possible length in $C_E(S)$. If all the vertices of Z' correspond to the adjacent edges of a single cycle in S, it would imply by our assumption and by the definition of $C_E(S)$ that such a cycle in S would also be negative, a contradiction to the assumption that S is balanced. Thus, Z' must contain a vertex $v_i{}'$ which corresponds to a pair of adjacent edges say e_i and e_{i-1} at least one of which, say

e_i, does not lie on a single cycle in S but incident to a vertex v with $d(v) \geq 3$ in S. Let $Z' = (v_1', v_2', v_3', \ldots, v_i', \ldots, v_k', v_1')$, $k \geq 3$. Now, if none of the vertices $v_1', v_2', v_3', \ldots, v_i', \ldots, v_k'$ corresponds to an edge of a cycle in S then either all the edges in S are incident to v or they form an edge-induced subsigraph containing v whose underlying graph is isomorphic to the wounded spider $K_{1,3}^{1(1)}$. Suppose Z' is of length three then by condition (ii) and (iii) Z' is positive and hence a contradiction to our assumption. Therefore, $\ell(Z') \geq 4$. Now, the hypothesis together with the definition of $C_E(S)$ implies that all the cycles formed in $C_E(S)$ due to the edges incident at v are positive triangles and also by the condition of balance in the hypothesis the cycle of $C_E(S)$ formed due to the edges of the cycle Z in S is positive. Next, Z' being the symmetric difference of the edge sets of these positive triangles and positive cycles in $C_E(S)$ is also positive due to minimality of $\ell(Z')$ and hence due to Lemma 4.1, a contradiction to our assumption. Thus, by contraposition it follows that $C_E(S)$ must be balanced and that the proof is complete. □

5. S-Consistent Common-Edge Sigraphs

A marked graph G_μ is *consistent* if every cycle in the graph possesses an even number of negative vertices (*c.f.*, Beineke and Harary [10, 11]). The *mark* $\mu(G')$ of a nonempty subgraph G' of G_μ is defined as the product of the marks of vertices in G'. A cycle Z in G_μ is said to be *consistent* if $\mu(Z) = +1$; otherwise, it is *inconsistent*. Thus, G_μ is consistent if every cycle in it is consistent. We will need the following result.

Theorem 5.1. [21] *A marked graph G_μ is consistent if and only if for any spanning tree T of G all fundamental cycles with respect to T are consistent and all common paths of pairs of those fundamental cycles have end vertices carrying the same marks.*

We can now extend the above notion for graphs in a natural way to the class of sigraphs with an idea that the signs of the edges in a sigraph S can interplay in some prespecified manner with the marks of its vertices; in fact, in the above definition one can replace G by S to mean the same definitions for sigraphs. We can indeed go a step further by bringing in the notion of assigning marks, which themselves are signs specified by the elements of an arbitrary signed discrete structure such as the edges of a given sigraph S. In particular, a given sigraph $\Gamma = (H, \xi)$ is $(S, \mathcal{R})$*-marked* if there exists a sigraph $S = (G, \sigma)$, a bijection $\varphi : E(S) \to V(H)$, a binary relation $\mathcal{R}$ on $E(S)$ and a marking $\mu : V(H) \to \{-, +\}$ of H satisfying the following *compatibility conditions*,

(CC1): $\mu(u) = \sigma(\varphi^{-1}(u)) \; \forall \; u \in V(\Gamma)$
(CC2): $uv \in E(H) \Leftrightarrow \{\sigma(\varphi^{-1}(u)), \; \sigma(\varphi^{-1}(v))\} \in \mathcal{R}$

The case when $\mathcal{R}$ is defined by the condition that $\varphi^{-1}(u) \cap \varphi^{-1}(v) \neq \phi$ has been dealt in Sinha [29] in respect of sigraph equations involving line sigraphs.

The following result gives a characterization of S-consistent common-edge sigraphs.

Theorem 5.2. *For any connected sigraph* S, $C_E(S)$ *is* S*-consistent if and only if the following conditions are satisfied by* S,

(a) *for every cycle* Z *in* S,

(i) *if* Z *is all-negative, then it is of even length;*

(ii) *if* Z *is heterogeneous then the total number of negative edges in the negative sections of length greater than one in* Z *and the number of negative sections of length greater than one in* Z *are of the same parity; and*

(b) for every vertex $v \in V(S)$ *with* $d(v) \geq 3$,

(i) $d^-(v) \leq 1$;

(ii) *size of any negative section containing* v *is at most one.*

Proof. ***Necessity***: Let $C_E(S)$ be S-consistent. Then by definition, there exists an S-marking

$$\mu : V(C_E(S)) \to \{+, -\}$$

such that for every, $v\ (= e_ie_j,\ e_i, e_j \in E(S)) \in V(C_E(S))$,

$$\mu(v) = -\ \ if\ \ \sigma(e_i) = \sigma(e_j) = -$$

$$= +\ \ otherwise.$$

and every cycle in $(C_E(S))_\mu$ is consistent. Further, by the definition of S-consistency, every cycle Z' in $(C_E(S))_\mu$ must have an even number of negative vertices. Therefore, since adjacent pairs of edges of every cycle in S create a cycle in $C_E(S)$ it follows by the definition of $C_E(S)$ that every cycle in S must be positive and hence (a)(i) follows. Hence, suppose the cycle Z in S is heterogeneous and let $N_1, N_2, N_3, ..., N_r$ be the negative sections of lengths greater than one in S. Then, the definition of $C_E(S)$ implies that the number of negative marks in the negative section N'_i in $(C_E(S))_\mu$ corresponding to N_i in S is precisely one less than that in N_i, for each $i \in \{1, 2, \ldots, r\}$, we have,

$$\sum_{i=1}^{r}\{\ell(N_i) - 1\} = \sum_{i=1}^{r}\{\ell(N_i)\} - r \equiv 0 (mod\ 2).$$

Hence (a)(ii) follows.

Let v be a vertex of S with $d(v) \geq 3$. If $d^-(v) > 1$ then, any two of the negative edges incident at v together with one other edge incident at v would form an inconsistent cycle in $(C_E(S))_\mu$, a contradiction to our assumption. Therefore, (b)(i) follows. Next, suppose that the size of a negative section containing v is greater than one. Then, due to the definition of a negative section in S (see Section 1), (b)(i) implies that there must exist a negative edge wz where vz is a negative edge. Since $d(v) \geq 3$, (b)(i) also implies existence of two vertices x and y such that vx and vy are positive edges in S. These edges form an edge-induced subsigraph H of S whose underlying graph is isomorphic to the wounded spider $K_{1,3}^{1(1)}$ which creates the triangle Z' in $(C_E(S))_\mu$ formed due to the common negative edge vz in H. However, it is easy to see that Z' is inconsistent, contrary to our assumption. Thus, (b)(ii) follows.

Sufficiency: For sufficiency, suppose conditions (a) and (b) hold for a given sigraph S. We shall show that $C_E(S)$ is S-consistent. Suppose, on the contrary, $C_E(S)$ is S-inconsistent. Then for any S-marking $\mu : V(C_E(S)) \rightarrow \{+,-\}$ there exists an inconsistent cycle in $(C_E(S))_\mu$. Let Z' be an inconsistent cycle of least possible length in $(C_E(S))_\mu$. If all the vertices of Z' correspond to the adjacent edges of a single cycle Z in S, it would imply by our assumption and by the definition of $C_E(S)$ that such a cycle in S if all-negative is of odd length or if Z is heterogeneous then the number of negative sections of length greater than one and the total number of negative edges in them are not of the same parity, contrary to (a). Thus, Z' must contain a vertex v_i' which corresponds to a pair of adjacent edges in S, say e_i and e_{i-1} at least one of which say e_i does not lie on a single cycle in S but incident to a vertex v with $d(v) \geq 3$ in S. Let $Z' = (v_1^{'}, v_2^{'}, v_3^{'}, ..., v_i^{'}, ..., v_k^{'}, v_1^{'})$, $k \geq 3$. If none of the vertices $v_1^{'}, v_2^{'}, v_3^{'}, ..., v_i^{'}, ..., v_k^{'}$ corresponds to a pair of consecutive edges of a cycle in S then either all the edges in S are incident to the vertex v or they form an edge-induced subsigraph H of S whose underlying graph is isomorphic to the wounded spider $K_{1,3}^{1(1)}$ containing v. Suppose Z' is of length three then by condition (b)(i) and (b)(ii) Z' is S-consistent, a contradiction to our assumption. Thus Z' is of length greater than or equal to four. Then, condition (b) and the definition of $C_E(S)$ together imply that all the cycles formed in $C_E(S)$ due to the edges incident at v in S are S-consistent triangles in $(C_E(S))_\mu$ and, also, by condition (a) the cycle of $(C_E(S))_\mu$ due to the edges of the cycle Z in S is S-consistent. Then, Z' being the symmetric difference of the edge sets of the S-consistent triangles and that of S-consistent cycles in $(C_E(S))_\mu$ is also S-consistent due to Theorem 5.1, a contradiction to our assumption that Z' is S-inconsistent. Thus, by contraposition, it follows that $(C_E(S))_\mu$ must be S-consistent and the proof is seen to be complete. □

6. S-Cycle-Compatible Common-Edge Sigraphs

Next, we study the concept of S-cycle-compatibility introduced by Acharya *et al.* [7] for common-edge sigraphs. A marked sigraph S_μ is *cycle-compatible* if for every cycle Z in S_μ the product of the signs of its vertices equals the product of the signs of its edges; it is *cycle-incompatible* otherwise. Given sigraphs $S = (G, \sigma)$, $\Gamma = (H, \xi)$ and a symmetric binary relation $\mathcal{R}$ on $E(S)$, the sigraph Γ is $(S, \mathcal{R})$*-cycle-compatible* if there exists an S*-marking* μ of Γ and a bijection $\varphi : E(S) \rightarrow V(\Gamma)$ such that Γ becomes cycle-compatible with respect to φ in the sense that the following conditions are satisfied:

(i) $\mu(u) = \sigma(\varphi^{-1}(u)) \ \forall \ u \in V(\Gamma)$
(ii) $uv \in E(H) \Leftrightarrow \{\sigma(\varphi^{-1}(u)), \ \sigma(\varphi^{-1}(v))\} \in \mathcal{R}$

(iii) $\prod_{e\in E(Z)} \xi(e) = \prod_{v\in V(Z)} \mu(v) \ \ \forall \ \ Z \in \mathcal{C}_\Gamma$

where $\mathcal{C}_\Gamma$ denotes the set of all cycles in Γ. In this section, we give a characterization of sigraphs S whose common-edge sigraphs $C_E(S)$ are S-*cycle-compatible* in the sense that

$$\{\varphi^{-1}(u), \varphi^{-1}(v)\} \in \mathcal{R} \Leftrightarrow \varphi^{-1}(u) \cap \varphi^{-1}(v) \neq \phi \qquad (2)$$

We need the following notion for the purpose: A sigraph S is *skew-balanced* (see Gill and Patwardhan [18], Acharya [3]) if for every cycle Z in S either Z is homogeneous or the number of negative sections in Z is even.

Theorem 6.1. *For any connected sigraph* $S = (G, \sigma)$, $C_E(S)$ *is* S-cycle-compatible *if and only if* S *is skew-balanced and for any* $v \in V(S)$ *with* $d(v) \geq 3$ *either* $d^-(v) = 0$ *or* $d^+(v) = 0$ *and for every path* $P_4 = (x, v, w, y)$ *if* $d^+(v) = 0$ *then* $d^+(w) = 0$.

Proof. ***Necessity***: Let $C_E(S)$ be S-cycle-compatible. Then there exists a marking μ of $C_E(S)$ such that for every cycle Z' in $(C_E(S))_\mu$,

$$\prod_{(e_ie_j, e_je_k)\in E(Z')} \sigma_1(e_ie_j, e_je_k) = \prod_{u'\in V(Z')} \mu(u') \qquad (A)$$

where σ_1 is the signing defined for $E(C_E(S))$ and μ is the marking defined on $C_E(S)$ for given S with respect to the bijection $\varphi : E(S) \to V(C_E(S))$ satisfying the S-cycle compatibility conditions stated above. Suppose now, for any cycle Z in S, n is the number of negative sections of length one, $N_1, N_2, ..., N_m$ be the negative sections of even lengths greater than one and $P_1, P_2, ..., P_r$ be the negative sections of odd lengths greater than one in Z. Thus, (A) implies that

$$n + \sum_i^m \ell(N_i) + \sum_j^r \ell(P_j) + \sum_i^m \{\ell(N_i) - 1\} + \sum_j^r \{\ell(P_j) - 1\} \equiv 0 (mod\ 2)$$

$$\Rightarrow n - m - r \equiv 0 (mod\ 2),$$

which implies that $C_E(S)$ is skew-balanced.

Next, let $v \in V(S)$ be such that $d(v) \geq 3$ and suppose that neither $d^+(v) = 0$ nor $d^-(v) = 0$. That means, $d^-(v) \geq 1$ and $d^+(v) \geq 1$. Let $e_1 = vx$ be a negative edge at v and $e_2 = vy$ be a positive edge at v. Since $d(v) \geq 3$, there must exist another edge $e_3 = vz$ at v. If $\sigma(e_3) = +1$, then the cycle $Z' = (e_1e_3, e_1e_2, e_2e_3, e_1e_3)$ in $(C_E(S))_\mu$ does not satisfy (A) by the definition of $C_E(S)$. On the other hand, if $\sigma(e_3) = -1$, then again Z' does not satisfy (A) by the definition of $C_E(S)$. Thus, we have a contradiction to (A) in either case. Hence, all the edges incident at v are of the same sign. Also, if there exists an $x - y$ path $P_4 = (x, v, w, y)$ such that wy is positive and if $e_1 = vz$ is the negative edge incident at v then the triangle $Z' = ((e_1, vw), (vw, wy), (xv, vw), (e_1, vw))$ due to the negative edge vw common to all the three vertices of Z' is all-negative. But, for one of the vertices in the triangle, $\mu((vw, wy)) = +$, a contradiction to (A). Thus, necessity of the conditions is seen to hold.

Sufficiency: To achieve this part of the proof, suppose the conditions stated in the theorem hold for S. We shall show that $C_E(S)$ is S-cycle-compatible. If S is homogeneous then the result is obvious. So, we shall suppose that S is heterogeneous and $C_E(S)$ is not S-cycle-compatible. Then, for the S-marking μ of $C_E(S)$ such that for some cycle Z' in $(C_E(S))_\mu$,

$$\prod_{(e_ie_j,e_je_k)\in E(Z')} \sigma_1(e_ie_j, e_je_k) \neq \prod_{u'\in V(Z')} \mu(u') \qquad (B)$$

Without loss of generality, let Z' be of minimum possible length with the property. Let n be the number of negative sections of length one, $N_1, N_2, ..., N_m$ be the negative sections of even lengths greater than one and $P_1, P_2, ..., P_r$ be the negative sections of odd lengths greater than one in Z in S. Now, if all the vertices of Z' in $(C_E(S))_\mu$ correspond to the pairs of consecutive edges of a cycle in S, the definition of $C_E(S)$ and (B) would together imply that the number of negative edges and the number of vertices of Z' are of opposite parity so that $\sum_{i=1}^{m}[\ell(N_i)-1]+\sum_{j=1}^{r}[\ell(P_j)-1]$ and $\sum_{i=1}^{m}\ell(N_i)+\sum_{j=1}^{r}\ell(P_j)+n$ are not of the same parity whence we get,

$$2\sum_{i=1}^{m}\ell(N_i)+2\sum_{j=1}^{r}\ell(P_j)+n-m-r\not\equiv 0(mod\ 2).$$

This contradicts the hypothesis that S is skew-balanced. Thus, Z' must contain a vertex v_i' which corresponds to a pair of adjacent edges say e_i and e_{i-1} at least one of which say e_i does not lie on a cycle in S but incident to a vertex v of $d(v) \geq 3$ in S. Let $Z' = (v_1', v_2', v_3', ..., v_i', ..., v_k', v_1')$, $k \geq 3$. Now, if none of the vertices $v_1', v_2', v_3', ..., v_i', ..., v_k'$ corresponds to an edge of a cycle in S then either all these edges in S are incident to v, or they form an edge-induced subsigraph H of S whose underlying graph is isomorphic to the wounded spider $K_{1,3}^{1(1)}$ containing v. If Z' is of length three then by the conditions in the hypothesis it is not difficult to verify that Z' is S-cycle-compatible in either case, a contradiction to our assumption. Therefore, $k \geq 4$.

Now, the hypothesis on the vertex v and the definition of $C_E(S)$ together imply that all the cycles formed in $C_E(S)$ due to the edges incident at v are homogeneous triangles and also by the hypothesis of skew-balance of S the cycle Z' of $(C_E(S))_\mu$ due to consecutive edges of the cycle Z in S is S-cycle-compatible. Thus Z' being the symmetric difference of the edge sets of the cycle-compatible triangles and that of cycle-compatible cycles due to skew-balance of S is also cycle-compatible, a contradiction to our assumption that Z' is S-cycle-incompatible. Thus, by contraposition, it follows that $(C_E(S))_\mu$ must be S-cycle-compatible. Thus, the proof is seen to be complete. □

7. Conclusion and scope

In this paper, we have introduced the notion of common-edge sigraph $C_E(S)$ of a given sigraph S and studied its properties, especially with regard to its states of balance, $S-$consistency and $S-$cycle-compatibility; since its underlying graph $C_E(G)$ is the second iterated line graph $L^2(G)$ of the underlying graph G of the given sigraph S, the conditions in S under which $C_E(S)$ and $L^2(S)$ are switching equivalent were also determined. One of the outstanding problems is to characterize a *common-edge sigraph* defined as a sigraph H for which there exists a sigraph S such that $H \cong C_E(S)$. This problem is important, at least from the socio-psychological point of view that it may help understand how interacting dyads in a social network tend to change the social structure in a way prescribed by the structure of common-edge sigraph of the initial social network; structural evolution of social networks has been a topic of current research interest (*e.g., see*, Holland and Lienhardt, [22], Acharya [1, 4], Acharya and Acharya [5, 6], Doreian [15, 16, 17], Kovchegov [23, 24, 25]).

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