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Graphs with Minimum Flow Number 3

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Abstract

Given a graph G, let $\Lambda = \Lambda(G)$ denote the smallest integer Λ for which G has a nowhere-zero Λ-flow. Shahmohamad determined the exact value $\Lambda(G)$ for $G \in \{K_n, K_{m,n}, B_n\}$, where B_n is the join of a cycle on $n-2$ vertices and K_2. In this paper, we further determine $\Lambda(G)$ for a set of graphs containing $\{K_{2n} : n \geq 3\} \cup \{K_{2n+1,2n+1} : n \geq 1\} \cup \{B_{2n+2} : n \geq 2\}$.

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1. Introduction

An *orientaion* of a graph G is a digraph D obtained from G by assigning a direction to each of its edge. Given a graph G with vertex set $V(G)$ and edge set $E(G)$, where multiple edges are allowed, let (D, f) be an ordered pair where D is an orientation of G and $f : E(G) \to \mathbb{Z}$ be an integer-valued function. An oriented edge of G is called an *arc*. For a vertex $v \in V(G)$, let $N_D^+(v)$ and $N_D^-(v)$ denote, respectively, the set of all arcs of D with their tail at v and the set of all all arcs of D with their head at v.

A *nowhere-zero* k*-flow* of a graph G is an ordered pair (D, f) such that for every edge $e \in E(G)$, $f(e) \in \{1, 2, \ldots, k-1\}$ and for every vertex $v \in V(G)$,

$$(\textstyle\sum_{e \in N_D^+(v)} f(e)) \equiv (\textstyle\sum_{e \in N_D^-(v)} f(e)) \,(mod\ k).$$

The *join* $G \vee H$ of disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. For $r \geq 5$, let $B_r = C_{r-2} \vee K_2$.

In [1], Shahmohamad considered the following probelm about flows. Given a graph G, what is the smallest integer k for which G has a nowhere-zero k-flow. Let $\Lambda(G)$ denote this minimum k. The results in [1] are:
1. A connected graph G is eulerian if and only if $\Lambda(G) = 2$.
2. For $n \geq 3$, $\Lambda(K_{2n}) = 3$.
3. If m and n are not both even integers, then $\Lambda(K_{m,n}) = 3$ $(m, n \geq 2)$.
4. For $n \geq 2$, $\Lambda(B_{2n+2}) = 3$.

In this paper, we extend the set $\{K_{2n} : n \geq 3\} \cup \{K_{2n+1,2n+1} : n \geq 1\} \cup \{B_{2n+2} : n \geq 2\}$ of graphs with $\Lambda = 3$.

2. Results

Let r be a positive integer and let L be a subset of $\{1, 2, \ldots, \lfloor \frac{r}{2} \rfloor\}$. A *circulant* $X(r; L)$ is a simple graph with vertex set $V(X(r; L)) = \mathbb{Z}_r$ and edge set $E(X(r; L)) = \{\{i, i + \ell\} : i \in \mathbb{Z}_r, \ell \in L\}$, where $\mathbb{Z}_r$ is the set of integers *modulo* r. We need the following Lemma.

Lemma (Tutte [2]). *A cubic graph G admits a nowhere-zero 3-flow if and only if G is bipartite.*

The circulant graph $X(4n + 2; \{1, 2n + 1\})$ is a cubic bipartite graph with bipartition (X, Y), where $X = \{0, 2, 4, \ldots, 4n\}$ and $Y = \{1, 3, 5, \ldots, 4n+1\}$, and hence, by Tutte's Lemma, $X(4n + 2; \{1, 2n + 1\})$ admits a nowhere-zero 3-flow.

Theorem 2.1. *If $1, 2n + 1 \in L$, and if $n \geq 1$, then $\Lambda(X(4n + 2; L)) = 3$.*

Proof. As $X(4n+2; L)$ is $(2|L|-1)$-regular, $\Lambda(X(4n+2; L)) \neq 2$, and as $X(4n+2; L)$ can be written as an edge-disjoint union of a nowhere-zero 3-flow graph $X(4n+2; \{1, 2n+1\})$ and an even graph, $\Lambda(X(4n + 2; L)) \leq 3$. Hence $\Lambda(X(4n + 2; L)) = 3$. □

Corollary 2.2. *For $n \geq 1$, $\Lambda(K_{4n+2}) = 3$.*

Proof. Since $K_{4n+2} = X(4n + 2; \{1, 2, \ldots, 2n + 1\})$. □

Corollary 2.3. *For $n \geq 1$, $\Lambda(K_{2n+1,2n+1}) = 3$.*

Proof. Since $K_{2n+1,2n+1} = X(4n + 2; \{1, 3, 5, 7, \ldots, 2n + 1\})$. □

Lemma 2.4. *For an even positive integer n, $\Lambda(K_n^c \vee K_2) = 3$.*

Proof. Let $V(K_n^c) = \{v_1, v_2, ..., v_n\}$ and $V(K_2) = \{x, y\}$. We consider three cases.

Case 1. $n \equiv 0$ $(mod\ 6)$.

Orient $y \to x$ and for $j \in \{1, 2, \ldots, n - 1\}$, orient $x \to v_j$ and $v_j \to y$; for $j = n$, orient $v_n \to x$ and $y \to v_n$. Now assign 1 to each arc. Then $inflow\ at\ x = 2$,

$outflow\,at\,x = n-1 \equiv 2\,(mod\,3)$, $inflow\,at\,y = n-1 \equiv 2\,(mod\,3)$, $outflow\,at\,y = 2$, $inflow\,at\,v_i = 1$, $outflow\,at\,v_i = 1$.

Case 2. $n \equiv 2$ $(mod\ 6)$.

Orient $x \to y$ and for $i \in \{1, 2, \ldots, n\}$, orient $x \to v_i$ and $v_i \to y$. Now assign 1 to each arc. Then $inflow\,at\,x = 0$, $outflow\,at\,x = n+1 \equiv 0\,(mod\,3)$, $inflow\,at\,y = n+1 \equiv 0\,(mod\,3)$, $outflow\,at\,y = 0$, $inflow\,at\,v_i = 1$, $outflow\,at\,v_i = 1$.

Case 3. $n \equiv 4$ $(mod\ 6)$.

Orient $y \to x$ and for $i \in \{1, 2, \ldots, n\}$, orient $x \to v_i$ and $v_i \to y$. Now assign 1 to each arc. Then $inflow\,at\,x = 1$, $outflow\,at\,x = n \equiv 1\,(mod\,3)$, $inflow\,at\,y = n \equiv 1\,(mod\,3)$, $outflow\,at\,y = 1$, $inflow\,at\,v_i = 1$, $outflow\,at\,v_i = 1$.

Thus we have constructed nowhere-zero 3-flows in all the three cases and hence $\Lambda(K_n^c \vee K_2) \leq 3$. As $K_n^c \vee K_2$ is not eulerian, $\Lambda(K_n^c \vee K_2) \neq 2$ and hence $\Lambda(K_n^c \vee K_2) = 3$. □

A graph is *even* if it has no vertices of odd degree.

Theorem 2.5. *If G is an even graph with even number of vertices, then* $\Lambda(G \vee K_2) = 3$.

Proof. Since $G \vee K_2$ is an edge-disjoint union of an even graph G and a nowhere-zero 3-flow graph $K_n^c \vee K_2$. □

Corollary 2.6. *For* $n \geq 2$, $\Lambda(B_{2n+2}) = 3$.

Theorem 2.7. *If G admits a nowhere-zero 3-flow and if the number of vertices of G is even, then* $\Lambda(G \vee K_2) = 3$.

Proof. Since $G \vee K_2$ is an edge-disjoint union of two nowhere-zero 3-flow graphs G and $K_n^c \vee K_2$. □

Corollary 2.8. *If G is a cubic bipartite graph, then* $\Lambda(G \vee K_2) = 3$.

Let G be a cubic bipartite spanning subgraph of the complete bipartite graph $K_{2n,2n}$, $n \geq 2$. Since $K_{2n,2n}$ is 1-factorable, any three 1-factors in any 1-factorization of $K_{2n,2n}$ will form a G. The graph $K_{4n} - E(G)$ is $4(n-1)$-regular and so it is 2-factorable. Let $\mathbb{F} = \{F_1, F_2, \ldots, F_{2(n-1)}\}$ be any 2-factorization of $K_{4n} - E(G)$. Further, let $\mathbb{H} = \{H : H = G \cup F_{i_1} \cup F_{i_2} \cup \ldots \cup F_{i_r}, r \geq 0\}$. It is clear that, if $H \in \mathbb{H}$, then $\Lambda(H) = 3$. Again, as $K_{4n} \in \mathbb{H}$, $\Lambda(K_{4n}) = 3$. Thus, we have:

Theorem 2.9. *If* $H \in \mathbb{H}$, *then* $\Lambda(H) = 3$.

Corollary 2.10. *For* $n \geq 2$, $\Lambda(K_{4n}) = 3$.

References

[1] H. Shahmohamad, Minimum flow number of graphs, *Bulletin of the ICA,* **35** (2002) 26-36.

[2] W.T.Tutte, On the imbedding of linear graphs in surfaces, *Proc. Lond. Math. Soc. Ser. 2,* **51** (1949) 474-483.

[3] C.-Q. Zhang, *Integer flows and cycle covers of graphs,* Marcel Dekker, Inc., 1997.