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Score Sets in Oriented k-partite Graphs

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Abstract

Let $D(U_1, U_2, ..., U_k)$ be an oriented k-partite graph with $|U_i| = n_i, 1 \leq i \leq k$. Then, the score of a vertex u_i in U_i is defined by a_{u_i} (or simply a_i) $= \sum_{t=1,\ t\neq i}^{k} n_t + d_{u_i}^+ - d_{u_i}^-$, where $d_{u_i}^+$ and $d_{u_i}^-$ are respectively the outdegree and indegree of u_i. The set A of distinct scores of the vertices of $D(U_1, U_2, ..., U_k)$ is called its score set. In this paper, we prove that if a_1 is a non-negative integer, a_i $(2 \leq i \leq n-1)$ are even positive integers and a_n is any positive integer, then for every $n \geq k \geq 3$, there exists an oriented k-partite graph with score set $A = \left\{a_1, \sum_{i=1}^{2} a_i, ..., \sum_{i=1}^{n} a_i\right\}$, except when $a_1 = 0,\ a_k = 1,\ n = k \geq 3$.

Keywords: Oriented graph, tournament, score sequence, score set, oriented k-partite graph

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1. Introduction

An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops. Let D be an oriented graph with vertex set $V = \{v_1, v_2, ..., v_p\}$, and let d_v^+ and d_v^- denote the outdegree and indegree respectively of vertex v. Avery [1] defined a_{v_i} (or simply a_i) $= p - 1 + d_{v_i}^+ - d_{v_i}^-$, the score of v_i, so $0 \leq a_{v_i} \leq 2p-2$. Then, the sequence $A_1 = [a_1, a_2, ..., a_p]$ in non-decreasing order is called the score sequence of D. The set A of distinct scores of the vertices of D is called its score set.

Avery obtained the following criterion for score sequences in oriented graphs.

Theorem 1.1. [1] *A non-decreasing sequence of non-negative integers* $A_1 = [a_1, a_2, ..., a_p]$ *is the score sequence of an oriented graph if and only if*

$$\sum_{i=1}^{k} a_i \geq k(k-1), \quad for\ 1 \leq k \leq p,$$

with equality when $k = p$.

Pirzada and Naikoo [7] obtained the following results for score sets in oriented graphs.

Theorem 1.2. [7] *Let $A = \{a, ad, ad^2, ..., ad^n\}$, where a and d are positive integers with $d > 1$. Then, there exists an oriented graph with score set A, except for $a = 1, d = 2, n > 0$ and $a = 1, d = 3, n > 0$.*

Theorem 1.3. [7] *If $a_1, a_2, ..., a_n$ are non-negative integers with $a_1 < a_2 < ... < a_n$. Then, there exists an oriented graph with score set $A = \{a_1^/, a_2^/, ..., a_n^/\}$, where*

$$a_i^/ = \begin{cases} a_{i-1} + a_i + 1 & \text{for } i > 1 \\ a_i & \text{for } i = 1. \end{cases}$$

An oriented k-partite graph ($k \geq 2$) is the result of assigning a direction to each edge of a simple k-partite graph. Thus, it has no loops and parallel arcs. For $k = 2, 3$ we get respectively oriented bipartite and oriented 3-partite graphs. Further, if $D(U_1, U_2, ..., U_k)$ is an oriented k-partite graph with $|U_i| = n_i, 1 \leq i \leq k$, then the score of a vertex u_i in U_i is defined by a_{u_i} (or simply a_i) $= \sum_{t=1,\ t\neq i}^{k} n_t + d_{u_i}^+ - d_{u_i}^-$, where $d_{u_i}^+$ and $d_{u_i}^-$ are respevtively the outdegree and indegree of u_i. Clearly, $0 \leq a_{u_i} \leq 2\sum_{t=1,\ t\neq i}^{k} n_t$. Let $D(U_1, U_2, ..., U_k)$ be an oriented k-partite graph with $U_i = \{u_1^i, u_2^i, ..., u_{n_i}^i\}, 1 \leq i \leq k$ and let $a_1^i, a_2^i, ..., a_{n_i}^i$ be the respective scores of $u_1^i, u_2^i, ..., u_{n_i}^i$. Then, the sequences $A_i = [a_1^i, a_2^i, ..., a_{n_i}^i], 1 \leq i \leq k$, in non-decreasing order are called the score sequences of $D(U_1, U_2, ..., U_k)$. The set A of distinct scores of the vertices of $D(U_1, U_2, ..., U_k)$ is called its score set.

The following result is the bipartite version of Theorem 1.1 given by Pirzada, Merajuddin and Yin [8].

Theorem 1.4. [8] *Two non-decreasing sequences $A_1 = [a_1^1, a_2^1, ..., a_{n_1}^1]$ and $A_2 = [a_1^2, a_2^2, ..., a_{n_2}^2]$ of non-negative integers are the score sequences of some oriented bipartite graph if and only if*

$$\sum_{i=1}^{p} a_i^1 + \sum_{j=1}^{q} a_j^2 \geq 2pq, \text{ for } 1 \leq p \leq n_1 \text{ and } 1 \leq q \leq n_2,$$

with equality when $p = n_1$ and $q = n_2$.

In [9], Pirzada, Naikoo and Chishti proved that every set A of positive integers is the score set of an oriented bipartite graph when $|A| = 1, 2, 3$, or when A is a geometric or arithmetic progression.

The next result is the multipartite version of Theorem 1.1 obtained by Pirzada and Merajuddin [4].

Theorem 1.5. [4] *Let $n_1, n_2, ..., n_k$ be k positive integers. The k non-decreasing sequences of integers $A_i = [a_1^i, a_2^i, ..., a_{n_i}^i], 1 \leq i \leq k$, form the score sequences of some oriented k-partite graph of order $n = n_1 + n_2 + ... + n_k$ if and only if*

$$\sum_{i=1}^{k}\sum_{j=1}^{m_i} a_j^i \geq 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j,$$

for all sets of k *integers* m_i *satisfying* $1 \leq m_i \leq n_i$, *with equality when* $m_i = n_i$ *for all* $i, 1 \leq i \leq k$.

Various results regarding of score sets in complete oriented graphs (tournaments), complete oriented bipartite graphs (bipartite tournaments) and complete oriented k-partite graphs (k-partite tournaments) can be found in [2, 3, 5, 6, 10, 11, 12, 13].

For any nonempty vertex sets X and Y , $X \to Y$ means that each vertex of X dominates every vertex of Y. Also, for any two vertices x and y , $x \to y$ means that there is an arc from x to y , and $x \sim y$ or $y \sim x$ means that neither $x \to y$ nor $y \to x$.

2. Result

Now, we obtain the following result.

Theorem 2.1. *Let* a_1 *be a non-negative integer,* a_i $(2 \leq i \leq n-1)$ *are even positive integers and* a_n *be any positive integer. Then, for every* $n \geq k \geq 3$, *there exists an oriented* k*-partite graph with score set* $A = \left\{a_1, \sum_{i=1}^{2} a_i, ..., \sum_{i=1}^{n} a_i\right\}$, *except when* $a_1 = 0,\ a_k = 1,\ n = k \geq 3$.

Proof. For $2 \leq i \leq n-1$, let $a_i = 2r_i$ where $r_i \geq 1$.

Case 1. $a_1 = 0,\ n = k \geq 3$.

(a) $a_2 = a_3 = ... = a_k = 2$. Consider an oriented k-partite graph $D(U_1, U_2, ..., U_k)$ with $|U_i| = 1$, and $U_i \to U_j$ whenever $i > j$. Then, the scores of the vertices of $D(U_1, U_2, ..., U_k)$ are

$a_{u_1} = \sum_{i=2}^{k} |U_i| + 0 - \sum_{i=2}^{k} |U_i| = 0 = a_1$, for $u_1 \in U_1$, and for $2 \leq i \leq k$

$$\begin{aligned} a_{u_i} &= \sum_{j=1,\ j\neq i}^{k} |U_j| + \sum_{j=1}^{i-1} |U_j| - \sum_{j=i+1}^{k} |U_j| \\ &= \sum_{j=1,\ j\neq i}^{k} 1 + \sum_{j=1}^{i-1} 1 - \sum_{j=i+1}^{k} 1 \\ &= k - 1 + (i-1) - (k-i) = 2i - 2 = 2(i-1) \\ &= a_1 + \sum_{j=2}^{i} 2 = a_1 + \sum_{j=2}^{i} a_i, \text{ for all } u_i \in U_i. \end{aligned}$$

Therefore, score set of $D(U_1, U_2, ..., U_k)$ is $A = \left\{a_1, \sum_{i=1}^{2} a_i, ..., \sum_{i=1}^{k} a_i\right\}$.

(b). $a_2, a_3, ..., a_{k-1} \geq 2$, $a_k > 2$. Construct an oriented k-partite graph $D(U_1, U_2, ..., U_k)$ as follows.

Let $U_1 = X_1$,

$U_2 = X_2,$

$\vdots$

$U_{k-2} = X_{k-2},$

$U_{k-1} = X_{k-1} \cup Y_1,$

$U_k = X_k,$

with $X_{k-1} \cap Y_1 = \phi$, $|X_i| = r_{i+1}$ for all i where $1 \leq i \leq k-2$, $|X_{k-1}| = 1$, $|Y_1| = a_k - 2$, $|X_k| = a_k$. Let $X_i \to X_j$ whenever $i > j$, and $Y_1 \to X_1, X_2, ..., X_{k-2}$, so that we get the oriented k-partite graph $D(U_1, U_2, ..., U_k)$ with

$|U_i| = |X_i| = r_{i+1}$ for all i where $1 \leq i \leq k-2$,

$|U_{k-1}| = |X_{k-1}| + |Y_1| = 1 + a_k - 2 = a_k - 1,$

$|U_k| = |X_k| = a_k,$

and the scores of vertices

$a_{x_1} = \sum_{i=2}^{k} |U_i| + 0 - \sum_{i=2}^{k} |U_i| = 0 = a_1,$ for all $x_1 \in X_1$,

for $2 \leq i \leq k-1$

$$a_{x_i} = \sum_{j=1,\ j\neq i}^{k} |U_j| + \sum_{j=1}^{i-1} |U_j| - \sum_{j=i+1}^{k} |U_j|$$

$$= \sum_{j=1,\ j\neq i}^{k-2} r_{j+1} + |U_{k-1}| + |U_k| + \sum_{j=1}^{i-1} r_{j+1} - \left(\sum_{j=i+1}^{k-2} r_{j+1} + |U_{k-1}| + |U_k|\right)$$

$$= (r_2 + r_3 + ... + r_i + r_{i+2} + ... + r_{k-1}) + (r_2 + r_3 + ... + r_i) - (r_{i+2} + r_{i+3} + ... + r_{k-1})$$

$$= 2r_2 + 2r_3 + ... + 2r_i = a_1 + a_2 + a_3 + ... + a_i, \text{ for all } x_i \in X_i,$$

$$a_{y_1} = \sum_{i=1,\ i\neq k-1}^{k} |U_i| + \sum_{i=1}^{k-2} |U_i| - 0$$

$$= \sum_{i=1}^{k-2} r_{i+1} + |U_k| + \sum_{i=1}^{k-2} r_{i+1} = 2\sum_{i=1}^{k-2} r_{i+1} + a_k$$

$$= a_1 + a_2 + a_3 + ... + a_{k-1} + a_k, \text{ for all } y_1 \in Y_1, \text{ and}$$

$$a_{x_k} = \sum_{i=1}^{k-1} |U_i| + \sum_{i=1}^{k-1} |X_i| - 0$$

$$= \sum_{i=1}^{k-2} r_{i+1} + |U_{k-1}| + \sum_{i=1}^{k-2} r_{i+1} + |X_{k-1}|$$

$$= 2\sum_{i=1}^{k-2} r_{i+1} + a_k - 1 + 1$$

$$= a_1 + a_2 + a_3 + ... + a_{k-1} + a_k, \text{ for all } x_k \in X_k.$$

Therefore, score set of $D(U_1, U_2, ..., U_k)$ is $A = \left\{a_1, \sum_{i=1}^{2} a_i, ..., \sum_{i=1}^{k} a_i\right\}$.

Case 2. $a_1 = 0$, $n > k \geq 3$. Then, $n = k + m$ where $m \geq 1$. Construct an oriented k-partite graph $D(U_1, U_2, ..., U_k)$ as follows.

Let $U_1 = X_1 \cup Y_{k+1} \cup Y_{k+2} \cup ... \cup Y_{k+m-1} \cup Y_{k+m},$

$$U_2 = X_2,$$
$$\vdots$$
$$U_{k-1} = X_{k-1},$$
$$U_k = X_k \cup Z_{k+1} \cup Z_{k+2} \cup ... \cup Z_{k+m-1} \cup Z_{k+m},$$

with $X_1 \cap Y_i = \phi$, $Y_i \cap Y_j = \phi$, $X_k \cap Z_i = \phi$, $Z_i \cap Z_j = \phi$ $(i \neq j)$, $|X_i| = r_{i+1}$ for all i where $1 \leq i \leq k-1$, $|X_k| = r_2$, $|Y_{k+i}| = |Z_{k+i}| = r_{k+i}$ for all i where $1 \leq i \leq m-1$, $|Y_{k+m}| = |Z_{k+m}| = a_n$. Let $X_i \rightarrow X_j$ whenever $i > j; Y_{k+i} \rightarrow X_2, X_3, ..., X_k, Z_{k+1}, Z_{k+2}, ..., Z_{k+i}$ for all i where $1 \leq i \leq m-1$; $Y_{k+m} \rightarrow X_2, X_3, ..., X_k, Z_{k+1}, Z_{k+2}, ..., Z_{k+m-1}$; $Z_{k+i} \rightarrow X_1, X_2, ..., X_{k-1}, Y_{k+1}, Y_{k+2}, ..., Y_{k+i-1}$ for all i where $1 \leq i \leq m,$ so that we get the oriented k-partite graph $D(U_1, U_2, ..., U_k)$ with

$$|U_1| = |X_1| + \sum_{i=1}^{m} |Y_{k+i}| = |X_k| + \sum_{i=1}^{m} |Z_{k+i}| = |U_k|$$
$$= r_2 + \sum_{i=1}^{m-1} r_{k+i} + |Y_{k+m}| = r_2 + \sum_{i=1}^{m-1} r_{k+i} + a_n,$$

$|U_i| = |X_i| = r_{i+1}$ for all i where $2 \leq i \leq k-1$, and the scores of vertices

$$a_{x_1} = \sum_{i=2}^{k} |U_i| + 0 - \left(\sum_{i=2}^{k} |X_i| + \sum_{i=1}^{m} |Z_{k+i}|\right)$$
$$= \sum_{i=2}^{k-1} r_{i+1} + |U_k| - \left(\sum_{i=2}^{k-1} r_{i+1} + |X_k| + \sum_{i=1}^{m-1} r_{k+i} + |Z_{k+m}|\right)$$
$$= r_2 + \sum_{i=1}^{m-1} r_{k+i} + a_n - \left(r_2 + \sum_{i=1}^{m-1} r_{k+i} + a_n\right)$$
$$= 0 \quad = a_1, \text{ for all } x_1 \in X_1,$$

for $1 \leq i \leq m-1$

$$a_{y_{k+i}} = \sum_{i=2}^{k} |U_i| + \sum_{i=2}^{k} |X_i| + \sum_{j=1}^{i} |Z_{k+j}| - \sum_{j=i+1}^{m} |Z_{k+j}|$$
$$= \sum_{i=2}^{k-1} r_{i+1} + |U_k| + \sum_{i=2}^{k-1} r_{i+1} + |X_k| + \sum_{j=1}^{i} r_{k+j} - \left(\sum_{j=i+1}^{m-1} r_{k+j} + |Z_{k+m}|\right)$$
$$= 2\sum_{i=2}^{k-1} r_{i+1} + r_2 + \sum_{i=1}^{m-1} r_{k+i} + a_n + r_2$$
$$+(r_{k+1} + r_{k+2} + ... + r_{k+i}) - (r_{k+i+1} + r_{k+i+2} + ... + r_{k+m-1} + a_n)$$
$$= 2r_2 + 2\sum_{i=2}^{k-1} r_{i+1} + (r_{k+1} + r_{k+2} + ... + r_{k+i} + r_{k+i+1} + ... + r_{k+m-1})$$
$$+(r_{k+1} + r_{k+2} + ... + r_{k+i}) - (r_{k+i+1} + r_{k+i+2} + ... + r_{k+m-1})$$
$$= 2r_2 + 2\sum_{i=2}^{k-1} r_{i+1} + 2r_{k+1} + 2r_{k+2} + ... + 2r_{k+i}$$
$$= a_1 + a_2 + a_3 + a_4 + ... + a_k + a_{k+1} + a_{k+2} + ... + a_{k+i}, \text{ for all } y_{k+i} \in Y_{k+i},$$

$$a_{y_{k+m}} = \sum_{i=2}^{k} |U_i| + \sum_{i=2}^{k} |X_i| + \sum_{i=1}^{m-1} |Z_{k+i}| - 0$$
$$= \sum_{i=2}^{k-1} r_{i+1} + |U_k| + \sum_{i=2}^{k-1} r_{i+1} + |X_k| + \sum_{i=1}^{m-1} r_{k+i}$$
$$= 2\sum_{i=2}^{k-1} r_{i+1} + r_2 + \sum_{i=1}^{m-1} r_{k+i} + a_n + r_2 + \sum_{i=1}^{m-1} r_{k+i}$$

$$= 2r_2 + 2\sum_{i=2}^{k-1} r_{i+1} + 2\sum_{i=1}^{m-1} r_{k+i} + a_n$$

$$= a_1 + a_2 + a_3 + a_4 + ... + a_k + a_{k+1} + a_{k+2} + ... + a_{k+m-1} + a_n,$$

for all $y_{k+m} \in Y_{k+m}$,

for $2 \leq i \leq k-1$

$$a_{x_i} = \sum_{j=1,\ j\neq i}^{k} |U_j| + \sum_{j=1}^{i-1} |X_j| - \left(\sum_{j=i+1}^{k} |X_j| + \sum_{i=1}^{m} |Y_{k+i}| + \sum_{i=1}^{m} |Z_{k+i}|\right)$$

$$= |U_1| + \sum_{j=2}^{k-1} r_{j+1} + |U_k| - |U_i| + \sum_{j=1}^{i-1} r_{j+1}$$

$$- \left(\sum_{j=i+1}^{k-1} r_{j+1} + |X_k| + \sum_{i=1}^{m-1} r_{k+i} + |Y_{k+m}| + \sum_{i=1}^{m-1} r_{k+i} + |Z_{k+m}|\right)$$

$$= r_2 + \sum_{i=1}^{m-1} r_{k+i} + a_n + \sum_{j=2}^{k-1} r_{j+1} + r_2 + \sum_{i=1}^{m-1} r_{k+i} + a_n - r_{i+1}$$

$$+ \sum_{j=1}^{i-1} r_{j+1} - \left(\sum_{j=i+1}^{k-1} r_{j+1} + r_2 + \sum_{i=1}^{m-1} r_{k+i} + a_n + \sum_{i=1}^{m-1} r_{k+i} + a_n\right)$$

$$= r_2 + (r_3 + r_4 + ... + r_i + r_{i+1} + r_{i+2} + ... + r_k) - r_{i+1}$$

$$+ (r_2 + r_3 + ... + r_i) - (r_{i+2} + r_{i+3} + ... + r_k)$$

$$= 2r_2 + 2r_3 + 2r_4 + ... + 2r_i = a_1 + a_2 + a_3 + ... + a_i, \text{ for all } x_i \in X_i,$$

$$a_{x_k} = \sum_{i=1}^{k-1} |U_i| + \sum_{i=1}^{k-1} |X_i| - \sum_{i=1}^{m} |Y_{k+i}|$$

$$= |U_1| + \sum_{i=2}^{k-1} r_{i+1} + \sum_{i=1}^{k-1} r_{i+1} - \left(\sum_{i=1}^{m-1} |Y_{k+i}| + |Y_{k+m}|\right)$$

$$= r_2 + \sum_{i=1}^{m-1} r_{k+i} + a_n + r_2 + 2\sum_{i=2}^{k-1} r_{i+1} - \left(\sum_{i=1}^{m-1} r_{k+i} + a_n\right)$$

$$= 2r_2 + 2\sum_{i=2}^{k-1} r_{i+1} = a_1 + a_2 + a_3 + a_4 + ... + a_k, \text{ for all } x_k \in X_k,$$

for $1 \leq i \leq m-1$

$$a_{z_{k+i}} = \sum_{j=1}^{k-1} |U_j| + \sum_{j=1}^{k-1} |X_j| + \sum_{j=1}^{i-1} |Y_{k+j}| - \sum_{j=i}^{m} |Y_{k+j}|$$

$$= |U_1| + \sum_{j=2}^{k-1} r_{j+1} + \sum_{j=1}^{k-1} r_{j+1} + \sum_{j=1}^{i-1} r_{k+j} - \left(\sum_{j=i}^{m-1} r_{k+j} + |Y_{k+m}|\right)$$

$$= r_2 + \sum_{j=1}^{m-1} r_{k+j} + a_n + r_2 + 2\sum_{j=2}^{k-1} r_{j+1} + \sum_{j=1}^{i-1} r_{k+j}$$

$$- \left(\sum_{j=i}^{m-1} r_{k+j} + a_n\right)$$

$$= 2r_2 + 2\sum_{j=2}^{k-1} r_{j+1} + (r_{k+1} + r_{k+2} + ... + r_{k+i-1} + r_{k+i} + ... + r_{k+m-1})$$

$$+ (r_{k+1} + r_{k+2} + ... + r_{k+i-1}) - (r_{k+i} + r_{k+i+1} + ... + r_{k+m-1})$$

$$= 2r_2 + 2\sum_{j=2}^{k-1} r_{j+1} + 2r_{k+1} + 2r_{k+2} + ... + 2r_{k+i-1}$$

$$= a_1 + a_2 + a_3 + a_4 + ... + a_k + a_{k+1} + a_{k+2} + ... + a_{k+i-1}, \text{ for all } z_{k+i} \in Z_{k+i},$$

and

$a_{z_{k+m}} = \sum_{i=1}^{k-1}|U_i| + \sum_{i=1}^{k-1}|X_i| + \sum_{i=1}^{m-1}|Y_{k+i}| - 0$

$= |U_1| + \sum_{i=2}^{k-1} r_{i+1} + \sum_{i=1}^{k-1} r_{i+1} + \sum_{i=1}^{m-1} r_{k+i}$

$= r_2 + \sum_{i=1}^{m-1} r_{k+i} + a_n + r_2 + 2\sum_{i=2}^{k-1} r_{i+1} + \sum_{i=1}^{m-1} r_{k+i}$

$= 2r_2 + 2\sum_{i=2}^{k-1} r_{i+1} + 2\sum_{i=1}^{m-1} r_{k+i} + a_n$

$= a_1 + a_2 + a_3 + a_4 + ... + a_k + a_{k+1} + a_{k+2} + ... + a_{k+m-1} + a_n$, for all $z_{k+m} \in Z_{k+m}$.

Therefore, score set of $D(U_1, U_2, ..., U_k)$ is $A = \left\{a_1, \sum_{i=1}^{2} a_i, ..., \sum_{i=1}^{n} a_i\right\}$.

Case 3. $a_1 > 0$. Since $n \geq k$, therefore $n = k + m$ for some $m \geq 0$. Construct an oriented k-partite graph $D(U_1, U_2, ..., U_k)$ as follows.

Let $U_1 = X_1$,

$U_2 = X_2 \cup X_{k-1}^{/} \cup Y_k \cup Y_{k+1} \cup ... \cup Y_{k+m-1}$,

$U_3 = X_3$,

$\vdots$

$U_{k-1} = X_{k-1}$,

$U_k = X_k \cup Z_{k+1} \cup Z_{k+2} \cup ... \cup Z_{k+m}$,

with $X_2 \cap X_{k-1}^{/} = \phi$, $X_2 \cap Y_i = \phi$, $X_{k-1}^{/} \cap Y_i = \phi$, $Y_i \cap Y_j = \phi$, $X_k \cap Z_i = \phi$, $Z_i \cap Z_j = \phi$ $(i \neq j)$, $|X_1| = a_1$, $|X_i| = r_i$ for all i where $2 \leq i \leq k-1$, $|X_k| = r_2$, $\left|X_{k-1}^{/}\right| = a_1 + r_3 + r_4 + ... + r_{k-1} + a_n$, $|Y_{k+i}| = r_{k+i}$ for all i where $0 \leq i \leq m-1$, $|Z_{k+i}| = r_{k+i-1}$ for all i where $1 \leq i \leq m$. Let $X_i \rightarrow X_j$ whenever $i > j > 1$; $X_{k-1}^{/} \rightarrow X_3, X_4, ..., X_k, Z_{k+1}, Z_{k+2}, ..., Z_{k+m}$; $Y_{k+i} \rightarrow X_3, X_4, ..., X_k, Z_{k+1}, Z_{k+2}, ..., Z_{k+i}$ for all i where $0 \leq i \leq m-1$; $Z_{k+i} \rightarrow X_2, X_3, ..., X_{k-1}, Y_k, Y_{k+1}, ..., Y_{k+i-1}$ for all i where $1 \leq i \leq m$, so that we get the oriented k-partite graph $D(U_1, U_2, ..., U_k)$ with

$|U_1| = |X_1| = a_1$,

$|U_2| = |X_2| + \left|X_{k-1}^{/}\right| + \sum_{i=0}^{m-1}|Y_{k+i}|$

$= r_2 + a_1 + r_3 + r_4 + ... + r_{k-1} + a_n + \sum_{i=0}^{m-1} r_{k+i}$

$= a_1 + r_2 + r_3 + ... + r_{k-1} + r_k + r_{k+1} + ... + r_{k+m-1} + a_n$

$= a_1 + \sum_{i=2}^{k+m-1} r_i + a_n$,

$|U_i| = |X_i| = r_i$ for all i where $3 \leq i \leq k-1$,

$|U_k| = |X_k| + \sum_{i=1}^{m}|Z_{k+i}| = r_2 + \sum_{i=1}^{m} r_{k+i-1}$, and the scores of vertices

$a_{x_1} = \sum_{i=2}^{k}|U_i| + 0 - 0 = |U_2| + \sum_{i=3}^{k-1} r_i + |U_k|$

$$= a_1 + \sum_{i=2}^{k+m-1} r_i + a_n + \sum_{i=3}^{k-1} r_i + r_2 + \sum_{i=1}^{m} r_{k+i-1}$$

$$= a_1 + (r_2 + r_3 + ... + r_{k-1} + r_k + ... + r_{k+m-1}) + a_n$$

$$+(r_3 + r_4 + ... + r_{k-1}) + r_2 + (r_k + r_{k+1} + ... + r_{k+m-1})$$

$$= a_1 + 2r_2 + 2r_3 + ... + 2r_{k-1} + 2r_k + 2r_{k+1} + ... + 2r_{k+m-1} + a_n$$

$$= a_1 + a_2 + a_3 + ... + a_{k+m-1} + a_n, \text{ for all } x_1 \in X_1,$$

$$a_{x_2} = \sum_{i=1,\ i\neq 2}^{k} |U_i| + 0 - \left(\sum_{i=3}^{k} |X_i| + \sum_{i=1}^{m} |Z_{k+i}|\right)$$

$$= |U_1| + \sum_{i=3}^{k-1} r_i + |U_k| - \left(\sum_{i=3}^{k-1} r_i + |X_k| + \sum_{i=1}^{m} r_{k+i-1}\right)$$

$$= a_1 + r_2 + \sum_{i=1}^{m} r_{k+i-1} - (r_2 + \sum_{i=1}^{m} r_{k+i-1}) = a_1, \text{ for all } x_2 \in X_2,$$

for $3 \leq i \leq k-1$

$$a_{x_i} = \sum_{j=1,\ j\neq i}^{k} |U_j| + \sum_{j=2}^{i-1} |X_j|$$

$$-\left(\left|X_{k-1}^{/}\right| + \sum_{j=0}^{m-1} |Y_{k+j}| + \sum_{j=i+1}^{k} |X_j| + \sum_{j=1}^{m} |Z_{k+j}|\right)$$

$$= |U_1| + |U_2| + \sum_{j=3}^{k-1} r_j + |U_k| - |U_i| + \sum_{j=2}^{i-1} r_j$$

$$-\Big((a_1 + r_3 + r_4 + ... + r_{k-1} + a_n) + \sum_{j=0}^{m-1} r_{k+j}$$

$$+ \sum_{j=i+1}^{k-1} r_j + |X_k| + \sum_{j=1}^{m} r_{k+j-1}\Big)$$

$$= a_1 + a_1 + \sum_{i=2}^{k+m-1} r_i + a_n + \sum_{j=3}^{k-1} r_j + r_2 + \sum_{i=1}^{m} r_{k+i-1} - r_i$$

$$+ \sum_{j=2}^{i-1} r_j - \Big((a_1 + r_3 + r_4 + ... + r_{k-1} + a_n) + \sum_{j=0}^{m-1} r_{k+j}$$

$$+ \sum_{j=i+1}^{k-1} r_j + r_2 + \sum_{j=1}^{m} r_{k+j-1}\Big)$$

$$= a_1 + (r_2 + r_3 + ... + r_{k+m-1}) + (r_3 + r_4 + ... + r_{i-1} + r_i + r_{i+1}$$

$$+... + r_{k-1}) - r_i + (r_2 + r_3 + ... + r_{i-1}) - ((r_3 + r_4 + ... + r_{k-1})$$

$$+(r_{i+1} + r_{i+2} + ... + r_{k-1}) + (r_k + r_{k+1} + ... + r_{k+m-1}))$$

$$= a_1 + 2r_2 + 2r_3 + 2r_4 + ... + 2r_{i-1}$$

$$= a_1 + a_2 + a_3 + ... + a_{i-1}, \text{ for all } x_i \in X_i,$$

$$a_{x_{k-1}^{/}} = \sum_{i=1,\ i\neq 2}^{k} |U_i| + \left(\sum_{i=3}^{k} |X_i| + \sum_{i=1}^{m} |Z_{k+i}|\right) - 0$$

$$= |U_1| + \sum_{i=3}^{k-1} r_i + |U_k| + \sum_{i=3}^{k-1} r_i + |X_k| + \sum_{i=1}^{m} r_{k+i-1}$$

$$= a_1 + 2\sum_{i=3}^{k-1} r_i + r_2 + \sum_{i=1}^{m} r_{k+i-1} + r_2 + \sum_{i=1}^{m} r_{k+i-1}$$

$$= a_1 + 2r_2 + 2\sum_{i=3}^{k-1} r_i + 2\sum_{i=1}^{m} r_{k+i-1}$$

$$= a_1 + a_2 + a_3 + a_4 + ... + a_{k-1} + a_k + a_{k+1} + ... + a_{k+m-1}, \text{ for all } x^{/}_{k-1} \in X^{/}_{k-1},$$

$$a_{y_k} = \sum_{i=1,\ i\neq 2}^{k} |U_i| + \sum_{i=3}^{k} |X_i| - \sum_{i=1}^{m} |Z_{k+i}|$$

$$= |U_1| + \sum_{i=3}^{k-1} r_i + |U_k| + \sum_{i=3}^{k-1} r_i + |X_k| - \sum_{i=1}^{m} r_{k+i-1}$$

$$= a_1 + 2\sum_{i=3}^{k-1} r_i + r_2 + \sum_{i=1}^{m} r_{k+i-1} + r_2 - \sum_{i=1}^{m} r_{k+i-1}$$

$$= a_1 + 2r_2 + 2\sum_{i=3}^{k-1} r_i$$

$$= a_1 + a_2 + a_3 + a_4 + ... + a_{k-1}, \text{ for all } y_k \in Y_k,$$

for $1 \leq i \leq m-1$

$$a_{y_{k+i}} = \sum_{j=1,\ j\neq 2}^{k} |U_j| + \left(\sum_{j=3}^{k} |X_j| + \sum_{j=1}^{i} |Z_{k+j}|\right) - \sum_{j=i+1}^{m} |Z_{k+j}|$$

$$= |U_1| + \sum_{j=3}^{k-1} r_j + |U_k| + \sum_{j=3}^{k-1} r_j + |X_k| + \sum_{j=1}^{i} r_{k+j-1} - \sum_{j=i+1}^{m} r_{k+j-1}$$

$$= a_1 + 2\sum_{j=3}^{k-1} r_j + r_2 + \sum_{i=1}^{m} r_{k+i-1} + r_2 + \sum_{j=1}^{i} r_{k+j-1} - \sum_{j=i+1}^{m} r_{k+j-1}$$

$$= a_1 + 2r_2 + 2\sum_{j=3}^{k-1} r_j + (r_k + r_{k+1} + ... + r_{k+i-1} + r_{k+i} + ... + r_{k+m-1})$$

$$+ \sum_{j=1}^{i} r_{k+j-1} - (r_{k+i} + r_{k+i+1} + ... + r_{k+m-1})$$

$$= a_1 + 2r_2 + 2\sum_{j=3}^{k-1} r_j + (r_k + r_{k+1} + ... + r_{k+i-1})$$

$$+(r_k + r_{k+1} + ... + r_{k+i-1})$$

$$= a_1 + 2r_2 + 2\sum_{j=3}^{k-1} r_j + 2\sum_{j=k}^{k+i-1} r_j$$

$$= a_1 + a_2 + a_3 + a_4 + a_5 + ... + a_{k-1} + a_k + a_{k+1} + ... + a_{k+i-1}, \text{ for all } y_{k+i} \in Y_{k+i},$$

$$a_{x_k} = \sum_{i=1}^{k-1} |U_i| + \sum_{i=2}^{k-1} |X_i| - \left(\left|X^{/}_{k-1}\right| + \sum_{i=0}^{m-1} |Y_{k+i}|\right)$$

$$= |U_1| + |U_2| + \sum_{i=3}^{k-1} r_i + \sum_{i=2}^{k-1} r_i$$

$$- \left(a_1 + r_3 + r_4 + ... + r_{k-1} + a_n + \sum_{i=0}^{m-1} r_{k+i}\right)$$

$$= a_1 + a_1 + \sum_{i=2}^{k+m-1} r_i + a_n + r_2 + 2\sum_{i=3}^{k-1} r_i$$

$$- \left(a_1 + r_3 + r_4 + ... + r_{k-1} + a_n + \sum_{i=0}^{m-1} r_{k+i}\right)$$

$$= a_1 + r_2 + 2\sum_{i=3}^{k-1} r_i + (r_2 + r_3 + ... + r_{k+m-1})$$

$$-(r_3 + r_4 + ... + r_{k-1} + r_k + r_{k+1} + ... + r_{k+m-1})$$

$$= a_1 + 2r_2 + 2\sum_{i=3}^{k-1} r_i$$

$$= a_1 + a_2 + a_3 + a_4 + ... + a_{k-1}, \text{ for all } x_k \in X_k \text{ and for } 1 \le i \le j$$

$$a_{z_{k+i}} = \sum_{j=1}^{k-1} |U_j| + \left(\sum_{j=2}^{k-1} |X_j| + \sum_{j=0}^{i-1} |Y_{k+j}|\right) - \left(\left|X_{k-1}^{/}\right| + \sum_{j=i}^{m-1} |Y_{k+j}|\right)$$

$$= |U_1| + |U_2| + \sum_{j=3}^{k-1} r_j + \sum_{j=2}^{k-1} r_j + \sum_{j=0}^{i-1} r_{k+j}$$

$$- \left(a_1 + r_3 + r_4 + ... + r_{k-1} + a_n + \sum_{j=i}^{m-1} r_{k+j}\right)$$

$$= a_1 + a_1 + \sum_{i=2}^{k+m-1} r_i + a_n + 2\sum_{j=3}^{k-1} r_j + r_2$$

$$+(r_k + r_{k+1} + ... + r_{k+i-1}) - (a_1 + r_3 + r_4 + ... + r_{k-1}$$

$$+a_n + r_{k+i} + r_{k+i+1} + ... + r_{k+m-1})$$

$$= a_1 + (r_2 + r_3 + ... + r_{k-1} + r_k + r_{k+1} + ... + r_{k+i-1} + r_{k+i}$$

$$+r_{k+i+1} + ... + r_{k+m-1}) + 2\sum_{j=3}^{k-1} r_j + r_2 + (r_k + r_{k+1} + ... + r_{k+i-1})$$

$$-(r_3 + r_4 + ... + r_{k-1} + r_{k+i} + r_{k+i+1} + ... + r_{k+m-1})$$

$$= a_1 + 2r_2 + 2\sum_{j=3}^{k-1} r_j + 2r_k + 2r_{k+1} + ... + 2r_{k+i-1}$$

$$= a_1 + a_2 + a_3 + a_4 + ... + a_{k-1} + a_k + a_{k+1} + ... + a_{k+i-1}, \text{ for all } z_{k+i} \in Z_{k+i}.$$

Therefore, score set of $D(U_1, U_2, ..., U_k)$ is $A = \left\{a_1, \sum_{i=1}^{2} a_i, ..., \sum_{i=1}^{n} a_i\right\}$. □

Remark 2.2. *If $n < k$ is taken in Theorem 2.1 then, in general, there is no oriented k-partite graph with score set $A = \left\{a_1, \sum_{i=1}^{2} a_i, ..., \sum_{i=1}^{n} a_i\right\}$. For example, let $n = 3, k = 5$, and let $A = \{0, 2, 4\}$ be the score set of some oriented 5-partite graph $D(U_1, U_2, U_3, U_4, U_5)$. Then, there exists at least one vertex u_1 in one partite set, say U_1, with $a_{u_1} = 0$ and at least four vertices u_2, u_3, u_4 and u_5 in other four partite sets U_2, U_3, U_4 and U_5 respectively such that $u_i \to u_1$ for all i where $2 \le i \le 5$. Clearly, $a_{u_i} = 2$ or 4 for all i where $2 \le i \le 5$. If $a_{u_2} = 2$, then $u_i \to u_2$ for all i where $3 \le i \le 5$. For $3 \le i, j \le 5$, we have either $u_i \sim u_j$, or $u_i \to u_j$, or $u_j \to u_i$, so that $a_{u_i} \ge 5$ and $a_{u_j} \ge 5$, or $a_{u_i} \ge 6$, or $a_{u_j} \ge 6$, which is a contradiction. Now, if $a_{u_2} = 4$, then there exists at least one vertex in $\{u_3, u_4, u_5\}$, say u_3, such that $u_3 \to u_2$ and either $u_4 \sim u_2$ and $u_5 \sim u_2$, or $u_2 \to u_4$ and $u_5 \to u_2$, or $u_2 \to u_5$ and $u_4 \to u_2$, or $u_4 \sim u_2$ and $u_5 \to u_2$, or $u_5 \sim u_2$ and $u_4 \to u_2$, or $u_4 \to u_2$ and $u_5 \to u_2$. Since $a_{u_3} \le 4$, therefore $u_4 \to u_3$ and $u_5 \to u_3$. But then either $a_{u_4} \ge 5$, or $a_{u_5} \ge 5$, or $a_{u_4} \ge 5$ and $a_{u_5} \ge 5$, which is again a contradiction.*

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