AKCE J. Graphs. Combin., 3, No. 2 (2006), pp. 147-150

A note on the double domination number in trees

Mustapha Chellali
Department of Mathematics
University of Blida.
B.P. 270, Blida, Algeria.
E-mail: *m_chellali@yahoo.com*

Communicated by: T.W. Haynes
Received 27 March 2006; accepted 21 July 2006

Abstract

In a graph G, a vertex dominates itself and its neighbors. A subset S of vertices of G is a double dominating set if every vertex in G is dominated at least twice by the vertices of S. The minimum cardinality of a double dominating set of G is the double domination number $\gamma_{\times 2}(G)$. We show that for a nontrivial tree T of order n, with ℓ leaves and s support vertices, $\gamma_{\times 2}(T) \geqslant (2n+\ell-s+2)/3$, and we characterize the trees attaining this lower bound.

Keywords: double domination number, trees.

2000 Mathematics Subject Classification: 05C69

1. Introduction

In a graph $G=(V,E)$, the *degree* of a vertex v denoted by $\deg_G(v)$ is the number of vertices adjacent to v. A *leaf* of a tree T is a vertex of degree one, while a *support vertex* of T is a vertex adjacent to a leaf. A *strong support vertex* is adjacent to at least two leaves. We denote the number of leaves of a tree T by ℓ, and the number of support vertices by s. If $T=P_2$ then $\ell = s = 2$.

A subset $S \subseteq V$ is a *double dominating set* of G, abbreviated *DDS*, if every vertex in $V-S$ has at least two neighbors in S and every vertex of S has a neighbor in S. The *double domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality of a double dominating set of G. A double dominating set of G with minimum cardinality is called a $\gamma_{\times 2}(G)$-set. Double domination was introduced by Harary and Haynes [2]. For a comprehensive survey of domination in graphs and its variations, see [3, 4].

In this note we give a lower bound on the double domination number of a tree T in terms of the order n, the number of leaves ℓ, and the number of support vertices, namely $\gamma_{\times 2}(T) \geq (2n+\ell-s+2)/3$, and we characterize the extremal trees.

2. Main results

We begin by giving a straightforward property of double dominating sets.

Observation 2.1. *Every DDS of a graph contains all its leaves and support vertices.*

We show that if T is a tree of order n with ℓ leaves and s support vertices, then $\gamma_{\times 2}(T)$ is bounded below by $(2n+\ell-s+2)/3$. For the purpose of characterizing the trees attaining this bound, we introduce a family $\mathcal{F}$ of trees $T=T_k$ that can be obtained as follows. Let $T_1=P_2=uv$ and $A(T_1)=\{u,v\}$. If $k\geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the following operations.

- **Operation** $\mathcal{O}_1$**:** Attach a vertex z by joining it to any support of T_i. Let $A(T_{i+1})=A(T_i)\cup\{z\}$.
- **Operation** $\mathcal{O}_2$**:** Attach a path $P_3=abc$ by joining c to any vertex d of $A(T_i)$ with the condition that if d is a leaf of T_i then its support vertex is not strong in T_i. Let $A(T_{i+1})=A(T_i)\cup\{a,b\}$.

Lemma 2.2. *If* $T\in\mathcal{F}$*, then* $A(T)$ *is a* $\gamma_{\times 2}(T)$*-set of size* $(2n+\ell-s+2)/3$.

Proof. We use the terminology of the construction for the tree $T=T_k$, and the set $A(T)$. To show that $A(T)$ is a $\gamma_{\times 2}(T)$-set of cardinality $(2n+\ell-s+2)/3$, we use induction on the number of operations k performed to construct T. The property is true for $T_1=P_2$. Suppose the property is true for all trees of $\mathcal{F}$ constructed with $k-1\geq 0$ operations. Let $T=T_k$ with $k\geq 2$, $T'=T_{k-1}$, and assume that T' has order n', with ℓ' leaves, and s' support vertices.

If T is a star $K_{1,p}$ with $p\geq 2$, then $A(T)=V(T)$ is a $\gamma_{\times 2}(T)$-set of cardinality $(2n+\ell-s+2)/3=n$. Thus we assume that T is not a star.

If T was obtained from T' by Operation $\mathcal{O}_1$, then $\gamma_{\times 2}(T)=\gamma_{\times 2}(T')+1$, $n=n'+1$, $\ell=\ell'+1$, and $s'=s$. By induction on T', $A(T')$ is a $\gamma_{\times 2}(T')$-set of cardinality $(2n'+\ell'-s'+2)/3$ and so $A(T)=A(T')\cup\{z\}$ is a $\gamma_{\times 2}(T)$-set of cardinality $(2n+\ell-s+2)/3$.

Assume now that T was obtained from T' using Operation $\mathcal{O}_2$. Then we have $n=n'+3$, and either $\ell=\ell'$, $s'=s$ or $\ell=\ell'+1$ and $s=s'+1$, respectively. Since $A(T)=A(T')\cup\{a,b\}$ is a DDS of T, $\gamma_{\times 2}(T)\leqslant|A(T)|=\gamma_{\times 2}(T')+2$. Now by Observation 2.1, every $\gamma_{\times 2}(T)$-set D contains a and b, and we assume that $c\notin D$ (else c would be replaced by d or a neighbor of d). Thus, $D-\{a,b\}$ is a DDS of T' and $\gamma_{\times 2}(T')\leqslant\gamma_{\times 2}(T)-2$. It follows that $\gamma_{\times 2}(T\ )=\gamma_{\times 2}(T')+2$ and $A(T)$ is a $\gamma_{\times 2}(T\ )$-set of size $(2n+\ell-s+2)/3$. □

We now are ready to establish our main result.

Theorem 2.3. *If* T *is a nontrivial tree of order* n*, with* ℓ *leaves and* s *support vertices, then* $\gamma_{\times 2}(T)\geq(2n+\ell-s+2)/3$ *with equality if and only if* $T\in\mathcal{F}$.

Proof. If $T \in \mathcal{F}$, then by Lemma 2.2, $\gamma_{\times 2}(T\ ) = (2n + \ell - s + 2)/3$. To prove that if T is a tree of order $n \geq 2$, then $\gamma_{\times 2}(T) \geq (2n + \ell - s + 2)/3$ with equality only if $T \in \mathcal{F}$, we proceed by induction on the order n. If $\text{diam}(T) = 1$, then $\gamma_{\times 2}(P_2) = 2 = (2n+\ell-s+2)/3$ and $T \in \mathcal{F}$. If $\text{diam}(T) = 2$, then T is star $K_{1,p}(p \geq 2)$ with $\gamma_{\times 2}(T) = n = (2n + \ell - s + 2)/3$, and $T \in \mathcal{F}$ since it is obtained from T_1 by using Operation $\mathcal{O}_1$. If $\text{diam}(T) = 3$, then T is a double star with $\gamma_{\times 2}(T) = n > (2n + \ell - s + 2)/3$. This establishes the base cases.

Assume that every tree T' of order $2 \leqslant n' < n$ and with ℓ' leaves and s' support vertices satisfies $\gamma_{\times 2}(T') \geq (2n' + \ell' - s' + 2)/3$ with equality only if $T' \in \mathcal{F}$. Let T be a tree of order n and diameter at least four having ℓ leaves, s support vertices and let S a $\gamma_{\times 2}(T)$-set.

If T contains any strong support vertex, say u, then let T' be the tree obtained from T by removing a leaf adjacent to u. Then $\gamma_{\times 2}(T') = \gamma_{\times 2}(T) - 1, n' = n - 1$, $\ell' = \ell - 1$, and $s' = s$. Applying the inductive hypothesis to T', we obtain $\gamma_{\times 2}(T) \geq (2n + \ell - s + 2)/3$. Further if $\gamma_{\times 2}(T) = (2n + \ell - s + 2)/3$, then $\gamma_{\times 2}(T') = (2n' + \ell' - s' + 2)/3$, and $T' \in \mathcal{F}$. Thus, $T \in \mathcal{F}$ and is obtained from T' by using Operation $\mathcal{O}_1$. Thus we can assume that every support vertex of T is adjacent to exactly one leaf.

We now root T at a vertex r of maximum eccentricity $\text{diam}\,(T) \geqslant 4$. Let v be a support vertex at maximum distance from r, u the parent of v, and w the parent of u in the rooted tree. Let v' be the unique leaf adjacent to v. Note that $\deg_T(w) \geq 2$ since T has diameter at least four. Let T_x denote the subtree induced by a vertex x and its descendants in the rooted tree T. We distinguish between three cases.

Case 1. u is a support vertex. Let $T' = T - \{v, v'\}$. Then $n' = n - 2$, $\ell' = \ell - 1$, and $s' = s - 1$. By Observation 2.1, S contains v, u and their adjacent leaves. So $S - \{v, v'\}$ is a DDS of T' and $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 2$. Applying the inductive hypothesis to T', $\gamma_{\times 2}(T) \geq \gamma_{\times 2}(T') + 2 \geq (2n' + \ell' - s' + 2)/3 + 2$ implying that $\gamma_{\times 2}(T) > (2n + \ell - s + 2)/3$.

Case 2. u is not a support vertex and has at least a child $b \neq v$ that is a support vertex. If $p = \deg_T(u) - 1$ is the number of children of u, then T_u is a subdivided star of order $2p + 1$. Let $T' = T - T_u$. Then $n' = n - 2p - 1$, $\ell' \geq \ell - p$, and $s' \leq s - p + 1$. By Observation 2.1, $V(T_u) - \{u\} \subset S$ and without loss of generality $u \notin S$ (else we replace u by w or a neighbor of w). Thus $S - V(T_u)$ is a DDS of T' and so $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 2p$. By induction on T', we obtain $\gamma_{\times 2}(T) \geq \gamma_{\times 2}(T') + 2p \geq (2n' + \ell' - s' + 2)/3 + 2p$ implying that $\gamma_{\times 2}(T) \geq (2n + \ell - s + 2 + 2p - 3)/3 > (2n + \ell - s + 2)/3$ since $p \geq 2$.

Case 3. $\deg_T(u) = 2$. Let $T' = T - T_u$. Clearly, $n' = n - 3$. If $n' = 2$, then $T = P_5$ and $P_5 \in \mathcal{F}$ since it is obtained from T_1 by using Operation $\mathcal{O}_2$. Thus let $n' \geq 3$. By Observation 2.1, $v', v \in S, u \notin S$, and $w \in S$. So $S - \{v, v'\}$ is a DDS of T' and $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 2$. By induction on T', we have $\gamma_{\times 2}(T) \geq \gamma_{\times 2}(T') + 2 \geqslant (2n' + \ell' - s' + 2)/3 + 2$.

Now if $\deg_T(w) \geq 3$, then $\ell = \ell' - 1$ and $s' = s - 1$. It follows that $\gamma_{\times 2}(T) \geq (2n + \ell - s + 2)/3$.

Further if $\gamma_{\times 2}(T) = (2n + \ell - s + 2)/3$, then $\gamma_{\times 2}(T') = (2n' + \ell' - s' + 2)/3$, and so $S' = S - \{v', v\}$ is a $\gamma_{\times 2}(T')$-set where $w \in S'$. By induction on T', we have $T' \in \mathcal{F}$, implying that $T \in \mathcal{F}$ and is obtained from T' by using Operation $\mathcal{O}_2$.

Assume now that $\deg_T(w) = 2$. Then $n' \geq 4$, for otherwise $T = P_6$ and $\gamma_{\times 2}(P_6) = 5 > (2n+\ell-s+2)/3$. Thus $\ell' = \ell$ and $s' \leq s$. It follows that $\gamma_{\times 2}(T) \geq (2n+\ell-s+2)/3$.

Further if $\gamma_{\times 2}(T) = (2n+\ell-s+2)/3$, then we have equality throughout this inequality chain. In particular $\gamma_{\times 2}(T') = (2n' + \ell' - s' + 2)/3$, $\ell' = \ell$ and $s' = s$, that is, w is a leaf in T' and its support vertex is not strong. By induction on T', we have $T' \in \mathcal{F}$. Thus, $T \in \mathcal{F}$ and is obtained from T' by using Operation $\mathcal{O}_2$. □

Note that in [1], Blidia et al. showed that every tree T of order $n \geq 3$ with ℓ leaves and s support vertices, satisfies $\gamma_{\times 2}(T) \leq (2n + \ell + s)/3$. So Theorem 2.3 gives in some sense a good framing on the double domination number in trees.

Acknowledgement

I thank the referee for pointing out a mistake in the previous version.

References

[1] M. Blidia, M. Chellali, T.W. Haynes, and M.A. Henning, Independent and double domination in trees. *Utilitas Math.,* **70**(2006), 159-173.

[2] F. Harary and T.W. Haynes, Double domination in graphs. *Ars Combin.* **55** (2000), 201–213.

[3] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[4] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.