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p-NORM FRACTIONAL DOMINATION IN GRAPHS

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Abstract

If $p > 0$, the p-norm fractional domination number of a graph G is the minimum value of $\left(\sum_{v \in V(G)} f(v)^p\right)^{1/p}$ as f ranges over fractional dominating functions on G. (When $p = 1$, this is the usual fractional domination number.) When the minimum is achieved, we call f a minimum p-norm fractional dominating function on G. We show that for (finite) $p > 1$ there is a unique minimum p-norm fractional dominating function, that it depends continuously on p, and that it is constant on each cell of any equitable partition of the graph. We also investigate what happens as $p \downarrow 1$ and as $p \uparrow \infty$.

Keywords: Fractional dominating function, fractional domination number, minimum fractional dominating function, p-norm.

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1. Introduction

$G = (V, E)$ will be a finite simple graph throughout, and notation will be standard. For instance, for $v \in V$, the open neighbourhood of v is $N_G(v) = \{u \in V : uv \in E\}$, and the notation is shortened to $N(v)$ when G is the only graph under discussion. Likewise, the closed neighbourhood of v is $N[v] = \{v\} \cup N(v)$.

If $g : V \to [0,1]$ satisfies $\sum_{u \in N[v]} g(u) \geq 1$ for every $v \in V$, then g is a *fractional dominating function* (FDF) on G; let $\mathsf{FDF}(G)$ denote the set of such functions. Clearly $\mathsf{FDF}(G)$ is a closed and convex subset of $[0,1]^V$, the set of functions from V into $[0,1]$. Because it is closed and bounded, the continuous function $g \to \sum_{v \in V} g(v)$ achieves a minimum on $\mathsf{FDF}(G)$. This minimum is the *fractional domination number* of G, denoted $\gamma_f(G)$, and any $g \in \mathsf{FDF}(G)$ at which it is achieved is a *minimum fractional dominating function* (MFDF). Let $\mathsf{MFDF}(G)$ denote the set of such functions.

The role of the function $g \to \sum_{v \in V} g(v)$ can be taken by any other continuous real-valued function ρ on $\mathsf{FDF}(G)$, and we could call the minimum of ρ on $\mathsf{FDF}(G)$ the "ρ-fractional domination number" of G, denoted $\gamma_f(G, \rho)$. Of course, if we want $\gamma_f(\cdot, \rho)$ to be a graph parameter, defined for every graph, then ρ will have to be, like $\rho(x_1, \ldots, x_n) = \sum_{i=1}^n x_i$, defined generally, without particular reference to $\mathsf{FDF}(G)$ and easily adjustable for different values of $n = |V|$. We will be most interested in functions ρ of the form $\rho(x_1, \ldots, x_n) = \psi(\sum_{i=1}^n \phi(x_i))$, where ψ and ϕ are ordinary real valued functions of one variable.

Whether ρ is of this form or not, we recall that ρ is *convex* if and only if for $\bar{x}, \bar{y} \in [0,1]^n$ and $t \in [0,1]$,

$$\rho(t\bar{x} + (1-t)\bar{y}) \leq t\rho(\bar{x}) + (1-t)\rho(\bar{y}) \tag{1}$$

If equality in 1 implies that either $t = 0$ or $t = 1$ or $\bar{x} = \bar{y}$, then ρ is said to be *strictly convex.* The same definitions apply when ρ is real-valued on any convex domain in $\mathbb{R}^n$.

Let us now introduce three standard results, for which we omit the proofs. (The first follows from induction on k, the second is a straightforward application of definitions, and the third comes from elementary calculus.)

Lemma 1.1. *Suppose that ρ is real-valued on a convex domain $D \subseteq \mathbb{R}^n$. Then ρ is convex if and only if, for all $k \geq 2$, all $\bar{x}^{(1)}, \ldots, \bar{x}^{(k)} \in D$, and all $\lambda_1, \ldots, \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$,*

$$\rho\left(\sum_{i=1}^k \lambda_i \bar{x}^{(i)}\right) \leq \sum_{i=1}^k \lambda_i \rho(\bar{x}^{(i)})$$

Furthermore, ρ is strictly convex if and only if ρ is convex and equality in the above inequality implies that $\bar{x}^{(1)} = \cdots = \bar{x}^{(k)}$.

Lemma 1.2. *If $D \subseteq \mathbb{R}^n$ is convex, ϕ is a (strictly) convex function on $\mathbb{R}$, and σ is defined on D by $\sigma(x_1, \ldots, x_n) = \sum_{i=1}^n \phi(x_i)$, then σ is (strictly) convex.*

Lemma 1.3. *If ϕ is continuous on $[a,b]$ and $\phi'' \geq 0$ on (a,b), then ϕ is convex on $[a,b]$; and if $\phi'' > 0$ on (a,b) then ϕ is strictly convex on $[a,b]$.*

Corollary 1.4. *If $p > 1$, $\phi(x) = x^p$ defines a strictly convex function on $[0, \infty)$.*

For $1 \leq p < \infty$, the ℓ^p norm (or p-norm) on $\mathbb{R}^n$ is defined by:

$$\|(x_1, \dots, x_n)\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

It is well known that $\|\cdot\|_p$ is a *norm*, meaning that for vectors $u, v \in \mathbb{R}^n$ and a scalar $a \in \mathbb{R}$:

1. $\|u\|_p \geq 0$ with equality only if $u = \bar{0}$;
2. $\|au\|_p = |a| \cdot \|u\|_p$;
3. $\|u + v\|_p \leq \|u\|_p + \|v\|_p$.

(It easily follows from properties 2 and 3 that $\|\cdot\|_p$ is a convex function on $\mathbb{R}^n$.) The ℓ^∞ norm (∞-norm) is defined by:

$$\|(x_1, \dots, x_n)\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

The following is well-known, and easily shown using a bit of calculus.

Lemma 1.5. *If $v \in \mathbb{R}^n$, $\|v\|_p$ is a continuous non-increasing function of p on $[1, \infty)$ and $\|v\|_p \to \|v\|_\infty$ as $p \to \infty$.*

The next lemma is also elementary, but vital enough to our purposes that we supply its proof.

Lemma 1.6. *Suppose that F is a non-empty, closed, bounded, and convex subset of $\mathbb{R}^n$, ϕ is a continuous strictly convex function with domain a closed real interval I such that $I^n \cap F \neq \emptyset$, and ψ is a continuous strictly increasing function on an interval J containing the range of σ, defined by $\sigma(x_1, \dots, x_n) = \sum_{i=1}^n \phi(x_i)$, $x_1, \dots, x_n \in I$. Then $\rho = \psi \circ \sigma$ achieves a minimum value on F at a unique point.*

Proof. ρ is defined and continuous on I^n, and its values on F will be its values on $I^n \cap F$, which is closed and bounded (as well as convex). Therefore ρ achieves a minimum on F. Because ψ is strictly increasing, the minimum of ρ on F will be achieved where σ achieves its minimum. By Lemma 1.2, σ is strictly convex on $I^n \cap F$. If σ achieves its minimum m at $\bar{x}, \bar{y} \in I^n \cap F$ then, because $I^n \cap F$ is convex, $m \leq \sigma(\frac{1}{2}(\bar{x} + \bar{y})) \leq \frac{1}{2}\sigma(\bar{x}) + \frac{1}{2}\sigma(\bar{y}) = m$. Thus, $\sigma(\frac{\bar{x}+\bar{y}}{2}) = \frac{1}{2}\sigma(\bar{x}) + \frac{1}{2}\sigma(\bar{y})$, and therefore $\bar{x} = \bar{y}$. □

Corollary 1.7. *For any graph G and $p \in (1, \infty)$, $\|\cdot\|_p$ achieves its minimum on $\mathsf{FDF}(G)$ at a unique fractional dominating function.*

Let us call the minimum of $\|\cdot\|_p$ on $\mathsf{FDF}(G)$ the p-norm fractional dominating number of G and denote it by $\gamma_f(G, p)$ (rather than the more cumbersome $\gamma_f(G, \|\cdot\|_p)$), and if $1 < p < \infty$ let the unique $g \in \mathsf{FDF}(G)$ for which $\|g\|_p$ is a minimum be denoted by g_p, or $g_p(G)$ if the dependence on G need be noted. As observed in the proof of Lemma 1.6 in more general circumstances, minimizing $\|g\|_p$ amounts to minimizing $\|g\|_p{}^p = \sum_{v \in V} g(v)^p$.

As usual, $\delta(G)$ denotes the minimum degree of vertices in G.

Lemma 1.8. $\gamma_f(G, \infty) = \frac{1}{\delta(G)+1}$.

Proof. The constant function $\frac{1}{\delta(G)+1}$ is a fractional dominating function on G, so $\gamma_f(G, \infty) \leq \frac{1}{\delta(G)+1}$. On the other hand, if $g \in \mathsf{FDF}(G)$ then for any $v \in V$ with degree $\delta(G)$,

$$\begin{aligned} 1 &\leq \sum_{u \in N[v]} g(u) \\ &\leq \|g\|_\infty |N[v]| \\ &= \|g\|_\infty (\delta(G) + 1) \end{aligned}$$

which proves that $\frac{1}{\delta(G)+1} \leq \gamma_f(G, \infty)$. □

It is well-known that the conclusion of Corollary 1.7 fails when $p = 1$ (note that $\gamma_f(G, 1) = \gamma_f(G)$; see [2] for results about $\mathsf{MFDF}(G)$), and now it is easy to see that it fails when $p = \infty$ as well. P_5 is a graph for which the conclusion of Corollary 1.7 fails for both $p = 1$ and $p = \infty$. It is shown in [2] that the MFDFs on G are of the form depicted in Figure 1 part (a), where $0 \leq s, t \leq 1$ and $s + t \geq 1$; it is an easy exercise to see that the FDFs g on P_5 for which $\|g\|_\infty = \gamma_f(P_5, \infty) = \frac{1}{2}$ all have the form depicted in part (b) of the figure, where $0 \leq x \leq \frac{1}{2}$.

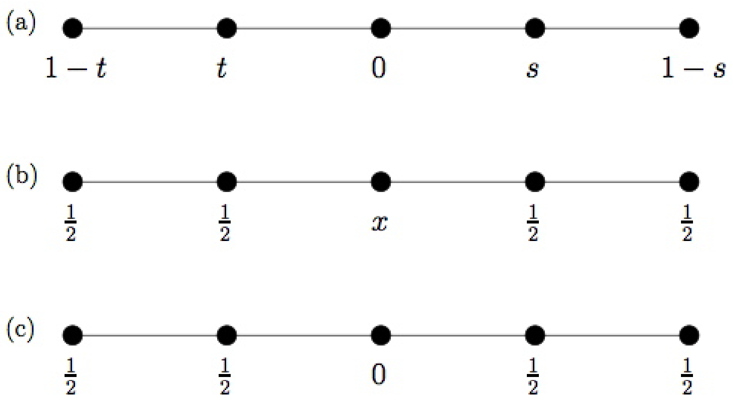

Figure 1: Families of minimum fractional dominating functions on P_5.

The function depicted in part (c) of the figure is the unique function g_p which instantiates $\gamma_f(P_5, p)$ for p strictly between 1 and ∞; this can be demonstrated using some of the results in the subsequent section together with some straightforward calculus. Note that this is also the unique FDF in the intersection of the minimizing sets for $p = 1$ and $p = \infty$; we trust that the reader is on the edge of his or her seat wondering whether this is an example of some more general phenomenon.

We have already defined $\mathsf{MFDF}(G) = \{g \in \mathsf{FDF}(G) : \|g\|_1 = \gamma_f(G, 1)\}$; let us also define the notation $\mathsf{MFDF}(G, \infty) = \{g \in \mathsf{FDF}(G) : \|g\|_\infty = \gamma_f(G, \infty)\}$.

2. Results about $\gamma_f(G, p)$ and g_p

Theorem 2.1. *$\gamma_f(G, p)$ is a continuous non-increasing function of p, $1 \le p \le \infty$.*

Proof. Suppose that $1 \le p < q \le \infty$. If $p = 1$ then let g_p be any member of $\mathsf{MFDF}(G)$, and observe that, by Lemma 1.5 and the definition of $\gamma_f(G, \cdot)$,

$$\gamma_f(G, q) \le \|g_p\|_q \le \|g_p\|_p = \gamma_f(G, p)$$

so $\gamma_f(G, \cdot)$ is a non-increasing function.

Next we show that $\gamma_f(G, \cdot)$ is continuous from the left. Suppose that $1 < p \le \infty$, and (p_k) is a sequence increasing to p. If $p = \infty$, let g_p be any member of $\mathsf{MFDF}(G, \infty)$. Since (p_k) is increasing to p, $\gamma_f(G, p_k)$ is non-increasing to a limit $L \ge \gamma_f(G, p)$. On the other hand, by Lemma 1.5 and the definition of $\gamma_f(G, p_k)$, for each k we have

$$\gamma_f(G, p_k) \le \|g_p\|_{p_k} \longrightarrow \|g_p\|_p = \gamma_f(G, p)$$

as $k \to \infty$. Since $\gamma_f(G, p_k) \to L$ as $k \to \infty$, it follows that $L \le \gamma_f(G, p)$, so $L = \gamma_f(G, p)$.

Showing that $\gamma_f(G, \cdot)$ is continuous from the right is a little trickier. Suppose that $1 \le p < \infty$, and that (p_k) is a sequence tending to p from above. Then $\gamma_f(G, p_k)$ is non-decreasing to a limit $L \le \gamma_f(G, p)$. Since $\mathsf{FDF}(G)$ is compact, some sub-sequence of $\{g_{p_k}\}$ converges (with respect to the usual topology, i.e. co-ordinatewise) to some $g \in \mathsf{FDF}(G)$. Rename so that it is $\{g_{p_k}\}$ which converges to g; we still have that $\gamma_f(G, p_k) \to L$ as $k \to \infty$. Using the "triangle inequality" (which was the third property of a norm given above), we have:

$$\begin{aligned}\gamma_f(G, p) &\le \|g\|_p \le \|g - g_{p_k}\|_p + \|g_{p_k}\|_p \\ &= \|g - g_{p_k}\|_p + (\|g_{p_k}\|_p - \|g_{p_k}\|_{p_k}) + \|g_{p_k}\|_{p_k} \\ &= \|g - g_{p_k}\|_p + (\|g_{p_k}\|_p - \|g_{p_k}\|_{p_k}) + \gamma_f(G, p_k)\end{aligned}$$

Since $g_{p_k}(v) \to g(v)$ as $k \to \infty$ for all $v \in V$, $\|g - g_{p_k}\|_p \to 0$ as $k \to \infty$. Also, as $k \to \infty$, $\gamma_f(G, p_k) \to L$. If we can show that $\|g_{p_k}\|_p - \|g_{p_k}\|_{p_k} \to 0$ as $k \to \infty$, it will follow that $\gamma_f(G, p) \le L$ and we will be done. But clearly, for $0 < q < \infty$ and

$h \in \mathbb{R}^n$, $\|h\|_q$ is a continuous function of q and the co-ordinates of h. Therefore, since $g_{p_k} \to g$ co-ordinatewise and $p_k \to p$ as $k \to \infty$, both $\|g_{p_k}\|_{p_k}$ and $\|g_{p_k}\|_p$ tend to $\|g\|_p$ as $k \to \infty$, so $\|g_{p_k}\|_p - \|g_{p_k}\|_{p_k} \to 0$ as $k \to \infty$. □

Theorem 2.2. *g_p is a continuous vector-valued function of p, $1 < p < \infty$.*

Proof. Suppose (p_k) is a sequence in $(1, \infty)$ tending to $p \in (1, \infty)$. We want to show that $g_{p_k} \to g_p$ as $k \to \infty$, co-ordinatewise. It suffices to show that every convergent subsequence of (g_{p_k}) converges to g_p. So suppose that (g_{p_k}) itself converges to $h \in \mathsf{FDF}(G)$, co-ordinatewise; if we show that $h = g_p$, then we are done.

Since $p_k \to p$ and $g_{p_k} \to h$, by previous arguments we have that $\gamma_f(G, p_k) = \|g_{p_k}\|_{p_k} \to \|h\|_p$ as $k \to \infty$. On the other hand, by Theorem 2.1, $\gamma_f(G, p_k) \to \gamma_f(G, p)$ as $k \to \infty$. Therefore $\|h\|_p = \gamma_f(G, p)$, so by Corollary 1.7, $h = g_p$. □

Theorems 2.1 and 2.2 are not really about graphs as such; they are applications to the case $F = \mathsf{FDF}(G)$ of some fundamental results concerning compact convex sets $F \subseteq \mathbb{R}^n$ and the norms $\|\cdot\|_p$, $1 \le p \le \infty$. We supplied the proofs because the proofs were easy, and we thought that those in our audience not so familiar with analytic arguments of this sort might find them interesting.

The next theorem is also an application of more general results, but the proofs in this case are deeper and we give references instead: [1], Theorem 3.1, and [4], Theorem 2.

Theorem 2.3. *For each G, as $p \downarrow 1$ $g_p = g_p(G)$ tend to a limit $g_1 \in \mathsf{MFDF}(G)$ and as $p \uparrow \infty$ g_p tends to a limit $g_\infty \in \mathsf{MFDF}(G, \infty)$.*

That the limits g_1, g_∞ are in $\mathsf{MFDF}(G), \mathsf{MFDF}(G, \infty)$ respectively follows by a simple argument from Theorem 2.1.

The references cited above also reveal to us something about the structure of g_1 and g_∞ for a given graph.

Theorem 2.4 ([4], Theorem 2(ii)). *The function $g_1 \in \mathsf{MFDF}(G)$ is the unique function which minimizes $\sum_{v \in V} g(v) \ln g(v)$ over all $g \in \mathsf{MFDF}(G)$.*

Of course, the sum being minimized in this result will always be negative; note that we treat $0 \cdot \ln 0 = 0$. This formula is somewhat suggestive of the entropy function on a discrete probability distribution in information theory, leading us to understand g_1 as the "most random" function in $\mathsf{MFDF}(G)$.

For g_1 we gave a characterization; for g_∞, we have an algorithm.

Input: A graph $G = (V, E)$.
Output: A function $g_\infty \in \mathsf{MFDF}(G, \infty)$ such that $g_p \to g_\infty$ as $p \to \infty$.
Define $g : V \to [0, 1]$ by $g(v) = \frac{1}{\delta(G)+1}$ for all $v \in V$;
$S := \{v \in V : \sum_{u \in N_G[v]} g(u) = 1\}$;
$Z := \{v \in V : g(v) = 0\}$;
$S' := (N[S] \cup Z) - S$;
$T := V - (S \cup S')$;
while $T \neq \emptyset$ **do**
  Define $h : (V - S) \to (0, 1]$ by:
    $h(v) = \big((\sum_{u \in N[v]} g(u)) - 1\big)/(d_T(v) + 1)$ if $v \in T$;
    $h(v) = \big((\sum_{u \in N[v]} g(u)) - 1\big)/|N_G[v] \cap T|$ if $v \in S'$;
    `// Treat this as` ∞ `if` $N[v]$ `and` T `are disjoint`
  $\epsilon := \min_{v \in V - S} h(v)$;
  Define $g' : V \to [0, 1]$ by:
    $g'(v) = \max\{g(v) - \epsilon, 0\}$ for $v \in T$;
    $g'(v) = g(v)$ for $v \notin T$;
  $g := g'$; refresh S, Z, S', T ;
end
return $g_\infty := g$

Algorithm 1: The Polya algorithm as applied to finding g_∞

Theorem 2.5 ([1], Section 3). *Algorithm 1 applied to a graph G returns g_∞.*

Note that we could have also specified the algorithm purely in terms of $\mathsf{MFDF}(G, \infty)$, as it more or less is in the original text.

We conclude this section with a theorem that takes into account G's properties as a graph. An *equitable partition* of G is a partition $U_1, \ldots, U_k$ of $V = V(G)$ for which there exist integers $d_{i,j}, 1 \leq i, j \leq k$, such that each vertex of U_i has exactly $d_{i.j}$ neighbours (in G) in U_j. Observe that this implies that the subgraph of G induced by U_i is $d_{i,i}$-regular, and if $1 \leq i < j \leq k$, then we have $d_{i,j}|U_i| = d_{j,i}|U_j|$ by counting the edges between U_i and U_j two ways.

The partition of $V(G)$ into automorphism classes, the orbits of the vertices under the action of the automorphism group $\mathrm{Aut}(G)$, is an example of an equitable partition of G. It is shown in [6] that every graph has a coarsest equitable partition; the following theorem is most useful when applied to that coarsest equitable partition.

Theorem 2.6. *If $U_1, \ldots, U_k$ is an equitable partition of G then for each $p \in (1, \infty)$, g_p is constant on each $U_i, i = 1, \ldots, k$.*

Proof. Fix $p \in (1, \infty)$ and let $g = g_p$. Let $n_i = |U_i|, i = 1, \ldots, k$. Define h on V by $h(v) = \frac{1}{n_i} \sum_{u \in U_i} g(u)$ for all $v \in U_i, i = 1, \ldots, k$; that is, h is constantly equal on U_i to the average value of g over U_i. Now, for each $i \in \{1, \ldots, k\}$ we have (by Lemma 1.1, with $\rho(x) = x^p$):

$$\begin{aligned}\sum_{u \in U_i} h(u)^p &= n_i \left(\frac{1}{n_i} \sum_{u \in U_i} g(u) \right)^p \\ &\leq n_i \cdot \frac{1}{n_i} \sum_{u \in U_i} g(u)^p \\ &= \sum_{u \in U_i} g(u)^p\end{aligned}$$

And hence

$$\begin{aligned}\sum_{u \in V} h(u)^p &= \sum_{i=1}^{k} \sum_{u \in U_i} h(u)^p \\ &\leq \sum_{i=1}^{k} \sum_{u \in U_i} g(u)^p \\ &= \sum_{u \in V} g(u)^p\end{aligned}$$

Thus, if $h \in \mathsf{FDF}(G)$ then since $g = g_p$ we would have that $g = h$, and hence that g is constant on each U_i. Our proof that $h \in \mathsf{FDF}(G)$ is essentially present in [5] and [3]. In what follows, we will make extensive use of adjacency matrices and vectors of vertex weights; we shall tacitly assume an ordering $v_1, \ldots, v_n$ of the vertices of G and employ the convention that all matrices and vectors are indexed with respect to this ordering.

A doubly-stochastic matrix S is a *fractional automorphism* of a graph G with adjacency matrix A if $SA = AS$. It is shown in [3] that if f is a fractional dominating function of G and S a fractional automorphism, then $S\bar{f}$ is also a fractional dominating function of G (where $\bar{f}$ denotes the characteristic vector of f), and further that $\|S\bar{f}\|_1 = \|\bar{f}\|_1$; specifically, if $f \in \mathsf{MFDF}(G)$ then $Sf \in \mathsf{MFDF}(G)$.

Define the matrix S by

$$S_{i,j} = \begin{cases} \frac{1}{n_m} & \text{if } \{v_i, v_j\} \subseteq U_m\} \\ 0 & \text{otherwise} \end{cases}$$

Let A be the adjacency matrix of G; to show that $SA = AS$, it suffices to show that either of these products is symmetric. Consider the element $(AS)_{i,j} = \sum_k A_{i,k} S_{k,j}$ and its image under transposition. Let us say that $v_i \in U_a$ and $v_j \in U_b$; by the construction of the two matrices, it is clear that $(AS)_{i,j} = \frac{d_{a,b}}{n_b}$, and similarly $(AS)_{j,i} = \frac{d_{b,a}}{n_a}$. If $a = b$ then these two quantities are equal, since $G[U_a]$ is regular. If $a \neq b$, then we observe the two quantities to be equal from $d_{a,b}n_a = d_{b,a}n_b$; this equation results from counting the edges of the bipartite graph $G[U_a, U_b]$ two different ways. Therefore S is a fractional automorphism of G.

To conclude the proof, we note that if $\bar{g}, \bar{h}$ are the characteristic vectors of g, h respectively, then $S\bar{g} = \bar{h}$, and hence $h \in \mathsf{MFDF}(G)$. □

Corollary 2.7. *If G is regular with degree d, then g_p is the constant function $\frac{1}{d+1}$ for all $1 \le p \le \infty$.*

Proof. It suffices to see the result for $1 < p < \infty$. Since $U_1 = V$ is itself an equitable partition of G (with $d_{1,1} = d$), and since $\frac{1}{d+1}$ is the smallest constant function in $\mathsf{FDF}(G)$, we must have $g_p \equiv \frac{1}{d+1}$. □

Corollary 2.8. *Each of $\mathsf{MFDF}(G)$ and $\mathsf{MFDF}(G,\infty)$ contains a function constant on each cell of the coarsest equitable partition of G.*

3. Problems and examples

In order to motivate further discussion of these parameters and their optimal solutions, let us now present some examples for $\gamma_f(G,p)$ and g_p for $1 \le p \le \infty$ and some small graphs, specifically the paths P_n for $n = 2, \ldots, 6$.

P_2 is a regular graph, and hence is covered by Corollary 2.7: $g_p = \frac{1}{2}$ for all p, with $\gamma_f(P_2, p) = 2^{(1/p)-1}$.

P_3 : $\mathsf{MFDF}(P_3)$ consists of a single point, the function which assigns 1 to the vertex of maximum degree and 0 to the others; likewise, $\mathsf{MFDF}(P_3,\infty)$ is also a singleton containing the constant function assigning $\frac{1}{2}$ to all vertices. For g_p when $1 < p < \infty$, Theorem 2.6 together with a little calculus and the definition of $\mathsf{FDF}(P_3)$ yields that g_p assigns $x_p = (2^{\frac{1}{p-1}} + 1)^{-1}$ to the two pendant vertices and $1 - x_p$ to the central vertex.

P_4 : The constant function which assigns $\frac{1}{2}$ to all vertices is the sole member of $\mathsf{MFDF}(P_4,\infty)$ and also an element of $\mathsf{MFDF}(P_4)$ which is constant on each automorphism class; it is also, unsurprisingly, g_p for all p between 1 and ∞ as well.

P_5 was dispatched in section 1.

P_6 : For convenience let $V(P_6) = \{v_1, \ldots, v_6\}$ with $E(P_6) = \{v_i v_{i+1} : i = 1, \ldots, 5\}$. [2] showed that $\mathsf{MFDF}(P_6)$ consists of the single function g_1 which assigns 1 to v_2, v_5 and 0 to all other vertices. The functions in $\mathsf{MFDF}(G,\infty)$ which are constant on each automorphism class all have the form:

$$g(v) = \begin{cases} \frac{1}{2} & v \in \{v_1, v_2, v_5, v_6\} \\ t & v \in \{v_3, v_4\}, t \in [\frac{1}{4}, \frac{1}{2}] \end{cases}$$

Again, we draw upon Theorem 2.6 and elementary calculus to discover that

$$g_p(v_1) = g_p(v_6) = 1 - y_p$$
$$g_p(v_2) = g_p(v_5) = y_p$$
$$g_p(v_3) = g_p(v_4) = \frac{1 - y_p}{2}$$

for $1 < p < \infty$, where $y_p = (1 + 2^{-p})^{\frac{1}{p-1}}/[1 + (1 + 2^{-p})^{\frac{1}{p-1}}]$. Taking the limit as $p \uparrow \infty$ gives a value of $t = \frac{1}{4}$ for the correct $g_\infty \in \mathsf{MFDF}(G, \infty)$.

In the preceding set of examples, note that g_p was either constant for all p or else steadily varying (as in P_3 and P_6) as a vector-valued function of p. Here is an example in which g_p is constant on an interval, but not constant over $[1, \infty]$. Let $G = K_{2,3}$ with bipartition A, B (where $|A| = 2$); by Theorem 2.6 we may assume that any g_p only takes on two values a, b on the vertices of A, B respectively. The unique function in $\mathsf{MFDF}(G)$ has $a = \frac{2}{5}, b = \frac{1}{5}$; the unique function in $\mathsf{MFDF}(G, \infty)$ has $a = b = \frac{1}{3}$. Without going into the gory (but elementary) details, we find that $g_p = g_1$ when $1 \le p \le \log_2 6$; when $p > \log_2 6$ we have a g_p with $a = 3^{\frac{1}{p-1}}/[1 + 2 \cdot 3^{\frac{1}{p-1}}]$ and $b = 1 - 2a$. We wonder if there exists a graph G such that the function $p \to g_p(G)$ is constant on multiple intervals in $[1, \infty]$ while varying in between.

Note that in all of the examples presented above, the set $\{g_p : 1 \le p \le \infty\}$ has formed a line segment in the boundary of $\mathsf{FDF}(G)$. This seems intuitively to be true in general, based on the fact that $\mathsf{FDF}(G)$ is a polytope together with the geometry of the situation, but we have no proof at present. If this is true, then it would simplify the computation of $\gamma_f(G, p)$ for general p considerably.

Regarding the behaviour of $p \to \gamma_f(G, p)$, there is a refinement of Lemma 1.5 which leads to an improvement of Theorem 2.1, which in turn leads to something of interest. The refinement of Lemma 1.5: if $v \in \mathbb{R}^n$ has at least two non-zero components, then $\|v\|_p$ is a strictly decreasing function of p, $0 < p < \infty$. The improvement of Theorem 2.1: if G has at least two vertices, then $\gamma_f(G, p)$ is a strictly decreasing function of p, $1 \le p \le \infty$.

To see this, we return to the proof of Theorem 2.1. Suppose that $1 \le p < q \le \infty$. As before, $\gamma_f(G, q) \le \|g_p\|_q \le \|g_p\|_p = \gamma_f(G, p)$, applying Lemma 1.5 to g_p. Now, if $\gamma_f(G, q) = \gamma_f(G, p)$ then $\|g_p\|_q = \|g_p\|_p$, so (by the sharpened Lemma 1.5) g_p can have at most one non-zero component. But since $g_p \in \mathsf{FDF}(G)$, this implies that $g_p(v) = 1$ for some $v \in V$ with $N[v] = V$, and $g_p(u) = 0$ for all $u \in V - \{v\}$. Then $H \cong K_{1,n-1}, n = |V|$ is a spanning subgraph of G, so $\mathsf{FDF}(H) \subseteq \mathsf{FDF}(G)$, and hence $\gamma_f(G, q) \le \gamma_f(H, q)$. However, an elementary calculation shows that if $n \ge 2$:

$$\gamma_f(H, q) = \left[\frac{n - 1}{\left(1 + (n-1)^{\frac{1}{q-1}}\right)^{q-1}} \right]^{\frac{1}{q}}$$
$$< 1 = \|g_p\|_p = \gamma_f(G, p)$$

Thus it is not possible that $\gamma_f(G,q) = \gamma_f(G,p)$ if G has at least two vertices; so $\gamma_f(G,p)$ is a strictly decreasing function of p if $G \neq K_1$.

Now, $\gamma_f(G,1) = \gamma_f(G) \geq 1$ and $\gamma_f(G,\infty) = \frac{1}{\delta(G)+1} \leq 1$, so if $G \neq K_1$ the strict monotonicity and continuity of $\gamma_f(G,\cdot)$ implies that there is a unique $p_1 = p_1(G) \in [1,\infty]$ such that $\gamma_f(G,p_1) = 1$. Let us call $p_1(G)$ the *unitary domination index* of G. This index appears to measure some sort of connectedness or centrality in the graph: $p_1(G) = 1$ if and only if G has a vertex of degree $|V(G)| - 1$, and $p_1(G) = \infty$ if and only if G has an isolated vertex.

Lemma 3.1. *Let G be a d-regular graph on $n > 1$ vertices. Then $p_1(G) = \log_{d+1} n$.*

Proof. Recall that Corollary 2.7 showed that $g_p(G) \equiv \frac{1}{d+1}$ for all p. The result then follows from solving $n \cdot (\frac{1}{d+1})^p = 1$ for p. □

Conjecture 3.2. *Let G be any graph on at least two vertices and $\bar{G}$ its complement. Then $p_1(G)^{-1} + p_1(\bar{G})^{-1} \geq 1$, with equality precisely when G or $\bar{G}$ contains an isolated vertex.*

Finally, we note that what we looked at here for fractional domination and $1 \leq p \leq \infty$ can be looked at for any fractional graph parameter definable by a linear programme, and $0 < p \leq \infty$. In particular, it might be interesting to investigate the p-norm fractional (neighbourhood) packing numbers $\pi_f(G,p)$, defined to be the maximum of the values of $\|g\|_p$, where

$$g \in \mathsf{FPF}(G) = \{g \in [0,1]^V : \sum_{u \in N_G[v]} g(u) \leq 1 \text{ for each } v \in V\}$$

It is a famous fact that, by linear programming duality, $\pi_f(G,1) = \pi_f(G) = \gamma_f(G) = \gamma_f(G,1)$. It may seem that we would always have $\pi_f(G,p) \leq \gamma_f(G,p)$ in view of how $\mathsf{FDF}(G)$ and $\mathsf{FPF}(G)$ are defined, but this is far from true: any function that assigns 1 to one vertex of G and 0 to the others is in $\mathsf{FPF}(G)$, so $\pi_f(G,p) \geq 1$ for all $p \in (0,\infty]$, and $\pi_f(G,\infty) = 1$ for all G. Therefore, if G has no isolated vertices, then $\gamma_f(G,p) < \pi_f(G,p)$ for all p sufficiently large. (A characterization of "sufficiently large" here would be nice to find; clearly this is true for $p > p_1(G)$, but it might also hold for smaller p.)

It is nonetheless true that $\pi_f(G,p)$ is a non-increasing, continuous function of p. It is not true that $\pi_f(G,p)$ is necessarily achieved at a unique function in $\mathsf{FPF}(G)$ for $1 < p < \infty$, and we wonder if there are any graphs besides K_1 such that $\pi_f(G,p)$ is achieved by $\|g\|_p$ for only one $g \in \mathsf{FPF}(G)$ for some or all $p \in (1,\infty)$.

$\pi_f(G,p)$ is achieved at a unique $g \in \mathsf{FPF}(G)$ for $0 < p < 1$ (for such p, $x \to x^p$ is strictly concave on $(0,\infty)$), call if $g^{(p)}$; how does it behave as $p \uparrow 1$ and as $p \downarrow 0$? Does the relation between $\pi_f(G,p)$ and $\gamma_f(G,p)$ for $0 < p < 1$ in any way mirror their relation in reverse order when $1 < p < \infty$?

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