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On the intersection of two regular hyperovals in projective planes of even order

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Abstract

A k-arc $\mathcal{K}$ in a projective plane $\mathrm{PG}(2,q)$ of order q is a set of k distinct points such that any line in the plane meets $\mathcal{K}$ in at most two points. For considering some applications, such as optical orthogonal codes, we are interested in the number of intersection points between two regular hyperovals, which are $(q+2)$-arcs consisting of a conic and its nucleus when q is even. In this paper, the number of intersection points of two regular hyperovals can be classified according to how the corresponding conics intersect.

Keywords: conic, nucleus, regular hyperoval, optical orthogonal code.

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1. Introduction

In a projective plane $\mathrm{PG}(2,q)$ over a finite field of order q, a k-arc is a set $\mathcal{K}$ of k points no three of which are collinear. The maximum value of k for a k-arc to exist is $q+1$ or $q+2$ when q is odd or even, respectively. A $(q+1)$-arc is called an *oval*, and a $(q+2)$-arc a *hyperoval.* Let $\mathbb{F}_q$ be a finite field of order q and $f(x,y,z)$ be a homogeneous function of degree two. If f is irreducible over the algebraic closure of $\mathbb{F}_q$, the set of points $P(x,y,z)$ in the plane $\mathrm{PG}(2,q)$ such that $f(x,y,z)=0$ is called a *conic.* It is well-known that any conic is a $(q+1)$-arc in $\mathrm{PG}(2,q)$. When q is even, the $(q+1)$ unisecants to a conic in $\mathrm{PG}(2,q)$ are concurrent. The point of concurrency of the unisecants is called the *nucleus*. It is easy to see that any conic with its nucleus is a $(q+2)$-arc, that is, a hyperoval. Such a hyperoval is said to be *regular*. While there exist some other classes of hyperovals, we only consider in this paper intersections of regular hyperovals.

Let Λ be a set of nonnegative integers. A family of k-arcs is called *a family of mutually Λ-intersecting k-arcs* if any pair of k-arcs in the family has m points in common, where m is an integer from Λ.

Theorem 1.1. [4] *Let $\mathcal{A}$ be a family of mutually Λ-intersecting k-arcs. Then there exists an optical orthogonal code consisting of $|\mathcal{A}|$ codewords with length q^3+q^2+q+1, weight k, auto-correlation peak 2, and cross-correlation peak $\lambda_c = \max(2, m)$, where m is a maximum value of $\mathcal{A}$ and $|\mathcal{A}|$ is the number of k-arcs contained in $\mathcal{A}$.*

Optical orthogonal codes are used as spreading sequences for optical code division multiple access networks. The codes have good auto- and cross-correlation properties. As shown in Theorem 1.1, the number of intersection points of two distinct arcs corresponds to the cross-correlation of the distinct codewords in the obtained optical orthogonal codes. In this paper, we give the number of intersection points between two hyperovals. In addition, we are interested in finding the condition such that the number is as small as possible.

Let $\overline{C}$ be a hyperoval consisting of a conic C and its nucleus $\mathcal{N}(C)$ over $\mathrm{PG}(2,q)$, that is, $\overline{C} = C \cup \{\mathcal{N}(C)\}$. Consider the intersection of two hyperovals $\overline{C_1}$ and $\overline{C_2}$. The intersection of two hyperovals can be regarded as the union of four sets, that is,

$$\overline{C_1} \cap \overline{C_2} = (C_1 \cap C_2) \cup (C_1 \cap \{\mathcal{N}(C_2)\}) \cup (C_2 \cap \{\mathcal{N}(C_1)\}) \cup (\{\mathcal{N}(C_1)\} \cap \{\mathcal{N}(C_2)\}).$$

We now count the number of intersection points of two hyperovals as specified by the number of intersection points of corresponding two conics C_1 and C_2. Let $\delta = |\overline{C_1} \cap \overline{C_2}| - |C_1 \cap C_2|$. Then δ is also expressed as

$$\delta = |C_1 \cap \{\mathcal{N}(C_2)\}| + |C_2 \cap \{\mathcal{N}(C_1)\}| + |\{\mathcal{N}(C_1)\} \cap \{\mathcal{N}(C_2)\}|. \tag{1}$$

Lemma 1.2. *For any two conics over $\mathrm{PG}(2,q)$, $\delta \leq 2$.*

Proof. When $\mathcal{N}(C_1) = \mathcal{N}(C_2)$, it is clear that each nucleus is not on another conic. Thus δ is equal to 1. When $\mathcal{N}(C_2) \in C_1$ or $\mathcal{N}(C_1) \in C_2$, the nuclei are not the same point. Therefore, there is no case in which all of $|C_1 \cap \{\mathcal{N}(C_2)\}|$, $|C_2 \cap \{\mathcal{N}(C_1)\}|$ and $|\{\mathcal{N}(C_1)\} \cap \{\mathcal{N}(C_2)\}|$ in (1) are equal to 1. Thus δ is at most 2. □

In the rest of this paper, we show the value of δ can be smaller than 2 in some cases.

2. Intersection points of two hyperovals

The value of δ in (1) can be classified according to how two conics can intersect. Let M be a multiset of intersection multiplicities over $\overline{\mathbb{F}}_q$ between two conics over $\mathbb{F}_q$, where $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$. For example, $M = \{1,1,2\}$ means that two conics intersect once at two points and twice at a point. Note that their intersection points may be contained in $\mathrm{PG}(2,q^i)$ but not in $\mathrm{PG}(2,q^j)$ for $j < i$. Because the sum of intersection multiplicities between two conics is 4, we can obtain the next Lemma immediately.

Lemma 2.1. *M is one of the following sets:*

$$\{1,1,1,1\}$$
$$\{1,1,2\}$$
$$\{2,2\}$$
$$\{1,3\}$$
$$\{4\}$$

In order to prove our results on the value of δ for each of the above classifications, we need to show some lemmata. Denote the intersection multiplicity at Q of C_1 and C_2 by $I(Q, C_1 \cap C_2)$.

Lemma 2.2. *If two conics C_1 and C_2 intersect once at a point, then their nuclei are distinct points.*

Proof. Let Q be an intersection point of C_1 and C_2 with $I(Q, C_1 \cap C_2) = 1$. Then a tangent line l_1 to C_1 at Q is not coincident with a tangent line l_2 to C_2 at Q. Since each nucleus lies on l_1 and l_2, respectively, they are distinct. □

Lemma 2.3. *If two distinct conics C_1 and C_2 intersect at least twice at a point, then each nucleus is not on another conic, that is, $\mathcal{N}(C_1) \notin C_2$ and $\mathcal{N}(C_2) \notin C_1$.*

Proof. Let Q be an intersection point of C_1 and C_2 with $I(Q, C_1 \cap C_2) \geq 2$. Then the two conics have a common tangent line l at Q. Both of the nuclei $\mathcal{N}(C_1)$ and $\mathcal{N}(C_2)$ lie on l, which implies the fact that $\mathcal{N}(C_1) \notin C_2$ and $\mathcal{N}(C_2) \notin C_1$. □

From Lemmata 2.2 and 2.3, we can obtain the values of δ for all cases other than $M = \{1,1,1,1\}$.

Theorem 2.4. *If two conics C_1 and C_2 intersect once at two points and twice at a point, that is, $M = \{1,1,2\}$, then $\delta = 0$.*

Proof. By Lemma 2.2, the nuclei of C_1 and C_2 are distinct points. We have $|\{\mathcal{N}(C_1)\} \cap \{\mathcal{N}(C_2)\}| = 0$. In addition, by Lemma 2.3, we have $|C_1 \cap \{\mathcal{N}(C_2)\}| = 0$ and $|C_2 \cap \{\mathcal{N}(C_1)\}| = 0$. Thus from (1), we can show $\delta = 0$. □

Theorem 2.5. *If two conics C_1 and C_2 intersect twice at two points, that is, $M = \{2,2\}$, then $\delta = 1$.*

Proof. Let Q_i be intersection points of C_1 and C_2 with $I(Q_i, C_1 \cap C_2) = 2$ for $i = 1,2$. By Lemma 2.3, we have $|C_1 \cap \{\mathcal{N}(C_2)\}| = 0$ and $|C_2 \cap \{\mathcal{N}(C_1)\}| = 0$. In addition, the two conics have a common tangent line l_i at Q_i for each $i = 1,2$. Since the nuclei $\mathcal{N}(C_1)$ and $\mathcal{N}(C_2)$ are the intersection points of l_1 and l_2, it must be $\mathcal{N}(C_1) = \mathcal{N}(C_2)$. Thus we can obtain $\delta = 1$. □

Theorem 2.6. *If two conics C_1 and C_2 intersect once at a point and three times at a point, that is, $M = \{1,3\}$, then $\delta = 0$.*

Proof. In analogy with Theorem 2.4, we can obtain $\delta = 0$. □

Theorem 2.7. *If two conics C_1 and C_2 intersect four times at a point, that is, $M = \{4\}$, then $\delta \leq 1$.*

Proof. From Lemma 2.3, each nucleus is not on another conic. It may occur that $\mathcal{N}(C_1) = \mathcal{N}(C_2)$. So we have $\delta \leq 1$. □

3. The case when $M = \{1,1,1,1\}$

In order to show the value of δ for the case that all intersection points have multiplicity 1, we need to classify more precisely the order of the projective plane on where intersection points lie. Let $\pi_i = \mathrm{PG}(2, q^i)$ and $\pi_i \subset \pi_{i+1}$. For any intersection point P between two conics, we denote $P \in P(i)$ if P lies on π_i but not on π_{i-1}. Let $M(i)$ be the multiset of intersection multiplicities of given two conics at all points in $P(i)$, that is, $M(i) = \{I(P, C_1 \cap C_2); P \in P(i)\}$. Let $\mathcal{M}$ be the family of $M(i)$, that is, $\mathcal{M} = \{M(1); M(2); \cdots; M(n)\}$, where n is the maximum number of i such that $M(i) \neq \emptyset$. The brackets of $M(i)$ are omitted for convenience. For example, $\mathcal{M} = \{1,1;1,1\}$ means that two conics intersect once at two points in π_1, once at two points in π_2, and they have no intersection points in π_i for $i > 2$.

Lemma 3.1. *If any two conics have an intersection points in $P(i)$, then they intersect at a multiple of i points in $P(i)$.*

Proof. Let $P = (a,b,c)$ be an intersection point in $P(i)$. Then the points $P^q = (a^q, b^q, c^q)$, $P^{q^2} = (a^{q^2}, b^{q^2}, c^{q^2})$, $\ldots$, $P^{q^{i-1}} = (a^{q^{i-1}}, b^{q^{i-1}}, c^{q^{i-1}})$ are also intersection points in $P(i)$. □

Using the notations of $\mathcal{M}$ and from Lemma 3.1, the sets of M in Lemma 2.1 except for $M = \{1,1,1,1\}$ are also expressed as follows.

Lemma 3.2. *$\mathcal{M}$ is one of the following sets:*

$$\{1,1,2\}$$
$$\{2;1,1\}$$
$$\{2,2\}$$
$$\{\emptyset;2,2\}$$
$$\{1,3\}$$
$$\{4\}$$

When $M=\{1,1,1,1\}$, we obtain the next Lemma.

Lemma 3.3. *$\mathcal{M}$ is one of the following sets:*

$$\{1,1,1,1\}$$
$$\{1,1;1,1\}$$
$$\{1;\emptyset;1,1,1\}$$
$$\{\emptyset;1,1,1,1\}$$

In order to show Lemma 3.3, the following results and Lemma 3.6 are required.

Result 3.4. [2] *Let $D(t)=t+t^2+\cdots t^{2^{h-1}}$. The number of solutions of the quadratic equation*

$$ax^2+bx+c=0 \tag{1}$$

over $\mathbb{F}_q$ with $q=2^h$ is determined as follows. If $b=0$, then the equation (1) has one solution. For $b\neq 0$, the number of solutions is zero if $D(\frac{ac}{b^2})=1$, and two if $D(\frac{ac}{b^2})=0$.

Result 3.5. [2] *Let $\tau_0=\{t\in\mathbb{F}_q : D(t)=0\}$ and $\tau_1=\{t\in\mathbb{F}_q : D(t)=1\}$, for $q=2^h$. Then*

(i) $0\in\tau_0$,

(ii) h *is even* $\Rightarrow 1\in\tau_0$,

(iii) h *is odd* $\Rightarrow 1\in\tau_1$, *and*

(iv) $s\in\tau_i, t\in\tau_j \Rightarrow s+t\in\tau_0$ *if* $i=j$, $s+t\in\tau_1$ *if* $i\neq j$.

Lemma 3.6. *If any two conics have no intersection points in π_1, then they intersect at a point in $\pi_2\setminus\pi_1$.*

Proof. Suppose that two conics C_1 and C_2 are expressed as the following equations, respectively,

$$x^2+xy+y^2+z^2=0,$$
$$ax^2+bxy+cxz+dy^2+eyz+fz^2=0.$$

If an intersection point of C_1 and C_2 lies on $x = 0$, then the point is given by $(0, 1, 1)$ and contained in π_1 which is contrary to our assumption. Assuming that $(1, s, t)$ is an intersection point of C_1 and C_2 and not contained in π_1, we have the quadratic equation for s

$$s^2 + s + t^2 + 1 = 0. \tag{2}$$

Then (2) has no solutions in $\mathbb{F}_q$ which leads to

$$D_q(\delta_1) = 1, \quad \text{where } \delta_1 = t^2 + 1 \tag{3}$$

from Result 3.4. Similarly, the quadratic equation for s

$$ds^2 + (b + et)s + ft^2 + ct + a = 0,$$

has no solutions in $\mathbb{F}_q$ which also leads to

$$D_q(\delta_2) = 1, \quad \text{where } \delta_2 = \frac{d(ft^2 + ct + a)}{(b + et)^2}. \tag{4}$$

From (3) and (4), it is satisfied that

$$D_{q^2}(\delta_1) = D_q(\delta_1) + D_q(\delta_1)^2 = 0$$

and

$$D_{q^2}(\delta_2) = D_q(\delta_2) + D_q(\delta_2)^2 = 0$$

which means s is an element in $\mathbb{F}_{q^2}$ and not in $\mathbb{F}_q$. Thus the $(1, s, t)$ necessarily lies on $\pi_2 \setminus \pi_1$. □

Proof of Lemma 3.3. There are four intersection points Q_i with $I(Q_i, C_1 \cap C_2) = 1$ for $i = 1, \ldots, 4$. If four intersection points exist in π_1, that is, $M(1) = \{1, 1, 1, 1\}$, then there exist no intersection points in π_i for $i \geq 2$. When $M(1) = \{1, 1, 1\}$, there exists one intersection point $P = (a, b, c)$ in π_i for $i \geq 2$. Then the point $P^q = (a^q, b^q, c^q)$ is also intersection points in π_i for $i \geq 2$ from Lemma 3.1. So there are at least five intersection points between two conics, which is a contradiction. Hence there is no case that $M(1) = \{1, 1, 1\}$. When $M(1) = \{1, 1\}$, there exist two intersection points in π_i for $i \geq 2$. From Lemma 3.1, the two intersection points must lie on π_2 which implies that $\mathcal{M} = \{1, 1; 1, 1\}$. When $M(1) = \{1\}$, we can see that there exist three intersection points in π_3 which means that $\mathcal{M} = \{1; \emptyset; 1, 1, 1\}$. When $M(1) = \{\emptyset\}$, from Lemma 3.6, all intersection points must lie on π_2. □

Let $q = 2^h$. We show that for all cases in Lemma 3.3 the values of δ may be less than or equal to 1 depending on whether h is even or odd.

Theorem 3.7. *Let $\mathcal{M} = \{1, 1, 1, 1\}$. If h is odd, then $\delta \leq 1$.*

Proof. The necessary and sufficient condition that $\delta = 2$ is that each nucleus is contained in another conic, that is,

$$\mathcal{N}(C_1) \in C_2 \text{ and } \mathcal{N}(C_2) \in C_1.$$

Without loss of generality, let $P_1 = (0,0,1)$ and $P_2 = (0,1,\alpha)$ be two intersection points of C_1 and C_2 for $\alpha \in \mathbb{F}_q$. Then any conic passing through P_1 and P_2 can be expressed as

$$ax^2 + bxy + cxz + \alpha ey^2 + eyz = 0$$

for $a, b, c, e \in \mathbb{F}_q$. Suppose that C_1 is given by

$$x^2 + \alpha y^2 + yz = 0. \tag{5}$$

Then the nucleus of C_1 is $\mathcal{N}(C_1) = (1,0,0)$. Since C_2 passes through P_1, P_2 and $\mathcal{N}(C_1)$, C_2 is defined by

$$bxy + cxz + \alpha y^2 + yz = 0, \tag{6}$$

and the nucleus of C_2 is $\mathcal{N}(C_2) = (1, c, b)$. From $\mathcal{N}(C_2) \in C_1$, we can obtain

$$1 + \alpha c^2 + bc = 0. \tag{7}$$

Any intersection points of C_1 and C_2 satisfy the equation $x(x + by + cz) = 0$ from (5) and (6). Since any other intersection points of C_1 and C_2 except for P_1 and P_2 are not on $x = 0$, we can put $x = 1$. Scalar multiplying equation (5) by c and substituting $cz = 1 + by$ give us

$$(\alpha c + b)y^2 + y + c = 0,$$

and moreover from (7)

$$y^2 + cy + c^2 = 0.$$

Thus the necessary and sufficient condition that C_1 and C_2 intersect in π_1 at any other two points except for P_1 and P_2 is the above quadratic equation for y has two solutions. Since $c \neq 0$ from (7), we have the condition $D(\frac{1 \cdot c^2}{c^2}) = D(1) = 0$. By Result 3.5, this is true precisely when h is even. $\square$

Theorem 3.8. *Let* $\mathcal{M} = \{0; 1, 1, 1, 1\}$. *If* h *is odd, then* $\delta \leq 1$.

Theorem 3.9. *Let* $\mathcal{M} = \{1, 1; 1, 1\}$. *If* h *is even, then* $\delta \leq 1$.

Proof of Theorem 3.8 and Theorem 3.9. Let ω be a root of an irreducible polynomial $x^2 + \beta x + \alpha$ over $\mathbb{F}_q$. Then we have the condition that

$$\delta_1 = \frac{\alpha}{\beta^2} \in \tau_1. \tag{8}$$

Without loss of generality, let $P = (1, \omega, 0)$ be a point of π_2 but not on π_1. Assume that two distinct conics C_1, C_2 have two points P and $P^q = (1, \omega^q, 0)$ in common and their nuclei are contained in each other simultaneously. Any conic C_1 passing through P is defined by the equation

$$\alpha x^2 + \beta xy + y^2 + z^2 = 0, \tag{9}$$

and the nucleus $\mathcal{N}(C_1)$ of C_1 is the point $(0,0,1)$. Since C_2 passes through the points P and $\mathcal{N}(C_1)$, it is defined by the equation

$$\alpha x^2 + \beta xy + cxz + y^2 + eyz = 0, \tag{10}$$

and its nucleus $\mathcal{N}(C_2)$ is expressed by (e,c,β). From our assumption, the point (e,c,β) is on C_1, which leads the quadratic equation

$$\alpha e^2 + \beta ce + c^2 + \beta^2 = 0. \tag{11}$$

From (9) and (10), $(cx+ey+z)z$ vanishes at each intersection point of C_1 and C_2. Since P and P^q lie on the line $z=0$, the other intersection points must be on $cx+ey+z=0$ with $z \neq 0$. Scalar multiplying equation (9) by c^2 and substituting $cx = ey+z$ with $z=1$ give us

$$(\alpha e^2 + \beta ce + c^2)y^2 + \beta cy + \alpha + c^2 = 0.$$

Moreover from (11),

$$\beta^2 y^2 + \beta cy + \alpha + c^2 = 0. \tag{12}$$

First we show Theorem 3.8. The necessary and sufficient condition so that C_1 and C_2 have no intersection points in π_1 is the equation (12) has no solutions for y in $\mathbb{F}_q$. This implies that $c \neq 0$ and that the invariant of (12) satisfies the condition

$$\frac{\beta^2(\alpha + c^2)}{\beta^2 c^2} = \frac{\alpha}{c^2} + 1 \in \tau_1. \tag{13}$$

Since equation (11) in e has some solutions in $\mathbb{F}_q$, the condition

$$\delta_2 = \frac{\alpha(c^2+\beta^2)}{\beta^2 c^2} = \frac{\alpha}{\beta^2} + \frac{\alpha}{c^2} \in \tau_0 \tag{14}$$

holds. From Result 3.5, and from the conditions (8) and (14), it can be shown that

$$\frac{\alpha}{c^2} = \delta_1 + \delta_2 \in \tau_1. \tag{15}$$

When h is odd for $q = 2^h$, 1 is an element of τ_1 which contradicts the conditions (13) and (15). Hence there is no case that $\mathcal{N}(C_1) \in C_2$ and $\mathcal{N}(C_2) \in C_1$ when h is odd.

Second we show Theorem 3.9. The assumption of Theorem 3.9 implies that C_1 and C_2 have two intersection points in $\mathrm{PG}(2,q)$ which means

$$\frac{\alpha}{c^2} + 1 \in \tau_0. \tag{16}$$

When h is even for $q = 2^h$, 1 is an element of τ_0 which contradicts the conditions (15) and (16). Hence there is no case that $\mathcal{N}(C_1) \in C_2$ and $\mathcal{N}(C_2) \in C_1$ when h is even. $\square$

Theorem 3.10. *Let* $\mathcal{M} = \{1; \emptyset; 1, 1, 1\}$ *. If* h *is odd,* $\delta \leq 1$ *.*

Proof. Let P be a point of π_3 but not on π_2 and not on any line of π_1 . There are exactly q^2+q+1 conics through P , P^q and P^{q^2} (see for example [1]). We denote by $\mathcal{C}_2$ the set of such conics. Let α be a primitive root of $\mathbb{F}_{q^3}$. Introduce new coordinates so that P , P^q and P^{q^2} are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, respectively. Then the coordinate of any point of π_1 can be represented as $(\alpha^i, \alpha^{qi}, \alpha^{q^2 i})$, for $i = 0, \ldots, q^2+q$. Moreover $\mathcal{C}_2$ and the set of points of a conic C_i in $\mathcal{C}_2$ are expressed as

$$\mathcal{C}_2 = \{C_i : i = 0, \ldots, q^2+q\},$$

$$C_i = \{(x,y,z) : \alpha^i yz + \alpha^{qi} xz + \alpha^{q^2 i} xy = 0\}.$$

Consider two hyperoval $\overline{C}_0$ and $\overline{C_i}$ for C_0 , $C_i \in \mathcal{C}_2$. The necessary and sufficient condition that $\delta = 2$ is each nucleus is contained in another conic, that is,

$$\mathcal{N}(C_i) \in C_0 \text{ and } \mathcal{N}(C_0) \in C_i, \tag{17}$$

where $\mathcal{N}(C_0) = (1,1,1)$ and $\mathcal{N}(C_i) = (\alpha^i, \alpha^{qi}, \alpha^{q^2 i})$. Assuming that (17), we can obtain the following conditions

$$\mathcal{N}(C_0) \in C_i \Leftrightarrow \alpha^i + \alpha^{qi} + \alpha^{q^2 i} = \mathrm{Tr}(\alpha^i) = 0 \tag{18}$$

and

$$\mathcal{N}(C_i) \in C_0 \Leftrightarrow \alpha^{(q+1)i} + \alpha^{(q^2+q)i} + \alpha^{(q^2+1)i} = \mathrm{Tr}(\alpha^{(q+1)i}) = 0. \tag{19}$$

Thus, for any α^i satisfying (18) and (19), the next equation has three distinct solutions over $\mathbb{F}_{q^3}$

$$(x-\alpha^i)(x-\alpha^{qi})(x-\alpha^{q^2 i}) = x^3 - \alpha^{(q^2+q+1)i} = 0.$$

We have $x^{3(q-1)} = (\alpha^i)^{q^3-1} = 1$ over $\mathbb{F}_{q^3}$, which means that q^3-1 must be divisible by $3(q-1)$. So q^2+q+1 must be divisible by three. For $q = 2^h$, if $h = 2m$ then

$$q^2+q+1 = 2^{4m} + 2^{2m} + 1 = 4^{2m} + 4^m + 1 \equiv 0 \pmod 3.$$

When $h = 2m+1$,

$$q^2+q+1 = 2^{4m+2} + 2^{2m+1} + 1 = 4^{2m+1} + 4^m \cdot 2 + 1 \equiv 1 \pmod 3.$$

The necessary and sufficient condition for $\delta = 2$ is that h is even. □

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