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k-Factors Containing and Avoiding Specified Sets of Edges

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Abstract

Let r, k be integers with $k \geq 1$ and $r \geq 2k$, and let G_0 be an r-regular r-edge-connected graph with $k|V(G_0)|$ even. Let A, B be subsets of $E(G_0)$ with $A \cap B = \emptyset$ such that $|A|$ and $|B|$ satisfy one of the following three conditions: (I) $k/2 < |A| \leq k$ and $|A| + |B| \leq k$; (II) $1 \leq |A| \leq k/2$ and $|A| + |B| \leq \lceil r/2 \rceil$; or (III) $|A| = 0$ and $|B| \leq r - k$. Under these assumptions, we show that G_0 has a k-factor F with $E(F) \supseteq A$ and $E(F) \cap B = \emptyset$, unless $(G_0; A, B)$ belongs to an exceptional family of triples.

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1. Introduction

In this paper, we consider finite undirected graphs without loops (but possibly with multiple edges). Let G be a graph. We let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. For $e \in E(G)$, we let $V(e)$ denote the set of endvertices of e. For disjoint subsets X and Y of $V(G)$, we let $E_G(X, Y)$ denote the set of edges of G joining X and Y. For $x \in V(G)$, we let $\deg_G(x)$ denote the degree of x in G. A subset A of $E(G)$ is often identified with the spanning subgraph of G having A as its edge set; for example, if X and Y are disjoint subsets of $V(G)$, then $E_A(X, Y)$ denotes the set of those edges in A which join X and Y, and if $x \in V(G)$, then $\deg_A(x)$ denotes the number of those edges in A which are incident with x. For a subset X of $V(G)$, we let $\langle X \rangle_G$ denote the subgraph induced by X in G. Let f be an integer-valued function defined on $V(G)$. A spanning subgraph F of G such that $\deg_F(x) = f(x)$ for all $x \in V(G)$ is called an f-factor of G. Let $k \geq 1$ be an integer. If $f(x) = k$ for all $x \in V(G)$, an f-factor is called a k-factor.

Let r, k be integers with $r > k \geq 1$. Let G_0 be an r-regular r-edge-connected graph, and let A, B be subsets of $E(G_0)$ with $A \cap B = \emptyset$. In this paper, we are concerned with the existence of a k-factor F with $E(F) \supseteq A$ and $E(F) \cap B = \emptyset$. Note that if F is a k-factor with $E(F) \supseteq A$ and $E(F) \cap B = \emptyset$, then $G - E(F)$ is an $(r-k)$-factor with $E(G - E(F)) \supseteq B$ and $E(G - E(F)) \cap A = \emptyset$. Thus we confine ourselves to the case where $r \geq 2k$. Before stating our result, we make one more definition.

Let r, k be integers with $k \geq 1$ and $r \geq 2k$, and set $\theta = k/r$. Let G_0 be an r-regular r-edge-connected graph with $k|V(G_0)|$ even, and let A, B be subsets of $E(G_0)$ with $A \cap B = \emptyset$. Let $\varepsilon \in \{-1, 0, 1\}$, and let a, b, c, d be nonnegative integers such that $a + b \leq r - k$, $c + d < r/2$ and $(c-b)\theta + (d-a)(1-\theta) \leq -1$. Suppose that there exists a partition $V(G_0) = S \cup T$ with $|S| - |T| = \varepsilon$ such that $|E(\langle S \rangle_{G_0})| = a + c$, $|E(\langle T \rangle_{G_0})| = b + d$, $|E(\langle S \rangle_{G_0}) \cap A| = a$ and $|E(\langle T \rangle_{G_0}) \cap B| = b$. In this case, we say that $(G_0; A, B)$ is of type $(\varepsilon; a, b, c, d)$. A straightforward calculation (for example, apply Claim 4 in Section 2 with $c' = 2c$ and $d' = 2d$) shows that if $(G_0; A, B)$ is of type $(\varepsilon; a, b, c, d)$, then G_0 does not have a k-factor F with $E(F) \supseteq A$ and $E(F) \cap B = \emptyset$. We say that $(G_0; A, B)$ is of exceptional type if there exist an integer $\varepsilon \in \{-1, 0, 1\}$ and nonnegative integers a, b, c, d with $a + b \leq r - k$, $c + d < r/2$ and $(c-b)\theta + (d-a)(1-\theta) \leq -1$ such that G_0 is of type $(\varepsilon; a, b, c, d)$. Note that if $(G_0; A, B)$ is of exceptional type, then G_0 is nearly bipartite with balanced bipartition.

The following theorem was proved by R. E. L. Aldred, D. A. Holton and J. Sheehan:

Theorem A. [1] *Let r be an integer with $r \geq 4$, and let G_0 be an r-regular r-edge-connected graph. Let A, B be subsets of $E(G_0)$ with $A \cap B = \emptyset$, and suppose that $|A|$ and $|B|$ satisfy one of the following three conditions:*

(I) $|A| = 2$ *and* $|B| = 0$;

(II) $|A| = 1$ *and* $0 \leq |B| \leq \lceil r/2 \rceil - 1$; *or*

(III) $|A| = 0$ *and* $|B| \leq r - 2$.

Suppose further that $\deg_{A \cup B}(x) \leq 2$ for all $x \in V(G)$. Then G_0 has a 2-factor F with $E(F) \supseteq A$ and $E(F) \cap B = \emptyset$, unless $(G_0; A, B)$ is of exceptional type with $k = 2$.

The purpose of this paper is to show that the following generalization of Theorem A holds without the assumption that $\deg_{A \cup B}(x) \leq k$ for all $x \in V(G_0)$ (for the case where $k = 1$, the reader is also referred to J. Plesník [2]):

Theorem 1.1. *Let r, k be integers with $k \geq 2$ and $r \geq 2k$, and let G_0 be an r-regular r-edge-connected graph with $k|V(G_0)|$ even. Let A, B be subsets of $E(G_0)$ with $A \cap B = \emptyset$, and suppose that $|A|$ and $|B|$ satisfy one of the following three conditions.*

(I) $k/2 < |A| \leq k$ *and* $|A| + |B| \leq k$;

(II) $1 \leq |A| \leq k/2$ *and* $|A| + |B| \leq \lceil r/2 \rceil$; *or*

(III) $|A| = 0$ *and* $|B| \leq r - k$.

Then G_0 *has a* k *-factor* F *with* $E(F) \supseteq A$ *and* $E(F) \cap B = \emptyset$ *, unless* $(G_0; A, B)$ *is of exceptional type.*

In Section 3, we discuss the sharpness of conditions (I), (II), (III) in Theorem 1.1.

2. Proof of Theorem 1.1

The following criterion for the existence of an f -factor is essential for our proof:

Theorem B. [3] *Let* G *be a graph, and let* f *be an integer-valued function on* $V(G)$ *such that* $0 \leq f(x) \leq \deg_G(x)$ *for all* $x \in V(G)$ *. Then* G *has an* f *-factor if and only if*

$$\delta_G(S, T; f) := \Sigma_{x \in S} f(x) + \Sigma_{y \in T}(\deg_{G-S}(y) - f(y)) - h_G(S, T; f) \geq 0$$

for all disjoint subsets S *and* T *of* $V(G)$ *, where* $h_G(S, T; f)$ *denotes the number of components* C *of* $G - S - T$ *such that* $|E_G(T, V(C))| + \Sigma_{z \in V(C)} f(z) \equiv 1 \pmod 2$ *.*

We also make use of the following lemma:

Lemma 2.1. [3] *Under the notation of Theorem B,* $\delta_G(S, T; f) \equiv \Sigma_{x \in V(G)} f(x) \pmod 2$ *for all disjoint subsets* S *and* T *of* $V(G)$ *.*

Now let r , k , G_0 , A , B be as in Theorem 1.1. Let $G = G_0 - A - B$. For $x \in V(G)$, let $f(x) = k - \deg_A(x)$. Note that G has an f -factor if and only if G_0 has a k -factor F with $E(F) \supseteq A$ and $E(F) \cap B = \emptyset$.

Suppose that G_0 dose not have a k -factor F with $E(F) \supseteq A$ and $E(F) \cap B = \emptyset$. Then by Theorem B and Lemma 2.1, there exist subsets S and T of $V(G_0)$ with $S \cap T = \emptyset$ such that

$$\delta_G(S, T; f) \leq -2. \tag{1}$$

Set $U = V(G_0) - S - T$ and $q = h_G(S, T; f)$. Let p denote the number of components of $G - S - T$. Thus $q \leq p$.

Let

$$\begin{aligned}
a_1 &= |\{e \in A \,|\, |V(e) \cap S| = |V(e) \cap U| = 1\}|, \\
a_2 &= |\{e \in A \,|\, |V(e) \cap S| = 2\}|, \\
b_1 &= |\{e \in B \,|\, |V(e) \cap T| = |V(e) \cap U| = 1\}|, \\
b_2 &= |\{e \in B \,|\, |V(e) \cap T| = 2\}|, \\
c_1 &= |\{e \in E(G_0) - A \,|\, |V(e) \cap S| = |V(e) \cap U| = 1\}|, \\
c_2 &= |\{e \in E(G_0) - A \,|\, |V(e) \cap S| = 2\}|, \\
d_1 &= |\{e \in E(G_0) - B \,|\, |V(e) \cap T| = |V(e) \cap U| = 1\}|, \\
d_2 &= |\{e \in E(G_0) - B \,|\, |V(e) \cap T| = 2\}|.
\end{aligned}$$

Also let $a = a_1 + a_2$, $b = b_1 + b_2$, $c' = c_1 + b_1 + 2c_2$, $d' = d_1 + a_1 + 2d_2$. Note that $k \le \lceil r/2 \rceil \le r - k$ by the assumption that $r \ge 2k$. Thus one of the following holds:

$$k/2 < a \le k \text{ and } a + b \le k; \tag{2}$$

$$1 \le a \le k/2 \text{ and } a + b \le \lceil r/2 \rceil; \text{ or} \tag{3}$$

$$a = 0 \text{ and } b \le r - k \tag{4}$$

(if (III) holds, then (4) holds; if (II) holds, then (3) or (4) holds according as $a \ge 1$ or $a = 0$; if (I) holds, then (2), (3) or (4) holds according as $a > k/2$, $1 \le a \le k/2$ or $a = 0$). Note that in any case, we have

$$a \le k \text{ and } a + b \le r - k. \tag{5}$$

Claim 1. $c' + d' \ge pr$.

Proof. Since G_0 is r-edge-connected, $a_1 + c_1 + b_1 + d_1 \ge pr$, which implies the desired inequality. □

Claim 2.

(i) $r|S| = a_1 + 2a_2 + c_1 + 2c_2 + |E_{G_0}(S,T)|$.

(ii) $r|T| = b_1 + 2b_2 + d_1 + 2d_2 + |E_{G_0}(S,T)|$.

Proof. For convenience, let $H = \langle S \rangle_{G_0}$. Since G_0 is r-regular,

$$\begin{aligned} r|S| &= \Sigma_{x \in S} \deg_{G_0}(x) \\ &= \Sigma_{x \in S} \deg_H(x) + \Sigma_{x \in S} |E_{G_0}(\{x\},U)| + \Sigma_{x \in S} |E_{G_0}(\{x\},T)| \\ &= 2(a_2 + c_2) + (a_1 + c_1) + |E_{G_0}(S,T)|. \end{aligned}$$

Thus (i) is proved, and (ii) can be verified in a similar way. □

Claim 3. $(2a + c') - (2b + d') = r(|S| - |T|)$.

Proof. It follows from Claim 2 that $r(|S| - |T|) = (a_1 + 2a_2 + c_1 + 2c_2) - (b_1 + 2b_2 + d_1 + 2d_2) = (2a + c') - (2b + d')$. □

Let $\theta = k/r$. Note that

$$k(1 - \theta) = (r - k)\theta. \tag{6}$$

Claim 4. $\delta_G(S,T;f) = c'\theta + d'(1 - \theta) - 2a(1 - \theta) - 2b\theta - q$.

Proof. Let

$$a'_1 = |\{e \in A \,|\, |V(e) \cap T| = |V(e) \cap U| = 1\}|,$$
$$a'_2 = |\{e \in A \,|\, |V(e) \cap T| = 2\}|.$$

Then

$$\Sigma_{y \in T} \deg_{G-S}(y) = (\Sigma_{y \in T} \deg_{G_0 - S}(y)) - (a'_1 + 2a'_2 + b_1 + 2b_2)$$

and

$$\Sigma_{y \in T} f(y) = \Sigma_{y \in T}(k - \deg_A(y)) = k|T| - (a'_1 + 2a'_2 + |E_A(S,T)|).$$

Hence

$$\Sigma_{y \in T}(\deg_{G-S}(y) - f(y)) = (\Sigma_{y \in T} \deg_{G_0 - S}(y)) - k|T| - b_1 - 2b_2 + |E_A(S,T)|.$$

Also

$$\Sigma_{y \in S} f(x) = \Sigma_{x \in S}(k - \deg_A(x)) = k|S| - a_1 - 2a_2 - |E_A(S,T)|.$$

Consequently

$$\begin{aligned}\delta_G(S,T;f) &= (k|S| - a_1 - 2a_2) + ((\Sigma_{y \in T} \deg_{G_0 - S}(y)) - k|T| - b_1 - 2b_2) - q \\ &= (k|S| - a_1 - 2a_2) + ((r-k)|T| - b_1 - 2b_2) - |E_{G_0}(S,T)| - q.\end{aligned}$$

On the other hand,

$$k|S| - a_1 - 2a_2 - \theta|E_{G_0}(S,T)| = (c_1 + 2c_2)\theta - (a_1 + 2a_2)(1 - \theta)$$

by Claim 2(i), and

$$(r-k)|T| - b_1 - 2b_2 - (1-\theta)|E_{G_0}(S,T)| = (d_1 + 2d_2)(1-\theta) - (b_1 + 2b_2)\theta$$

by Claim 2(ii). Therefore

$$\begin{aligned}\delta_G(S,T;f) &= (c_1 + 2c_2 - b_1 - 2b_2)\theta + (d_1 + 2d_2 - a_1 - 2a_2)(1-\theta) - q \\ &= (c' - 2b)\theta + (d' - 2a)(1-\theta) - q,\end{aligned}$$

as desired. □

Claim 5. $2a(1-\theta) + 2b\theta \leq (2(r-k)+1)\theta$.

Proof. Note that $1 - \theta \geq \theta$ by the assumption that $r \geq 2k$. Hence if (2) holds, $2a(1-\theta) + 2b\theta \leq 2k(1-\theta) = 2(r-k)\theta$ by (6); if (3) holds, $2a(1-\theta) + 2b\theta \leq k(1-\theta) + (r-k+1)\theta = (2(r-k)+1)\theta$ by (6); if (4) holds, $2a(1-\theta) + 2b\theta \leq 2(r-k)\theta$. □

Claim 6. $c' + d' < 2r$.

Proof. Suppose that $c' + d' \geq 2r$. Set $p' = \max\{2, p\}$. Then by Claim 1, $c' + d' \geq p'r$. Since $1 - \theta \geq \theta$, this implies

$$c'\theta + d'(1-\theta) \geq p'r\theta = 2r\theta + (p'-2)k \geq 2r\theta + p' - 2.$$

Note that $p' \geq p \geq q$. Consequently it follows from Claims 4 and 5 that

$$\begin{aligned}\delta_G(S,T;f) &\geq 2r\theta + p' - 2 - (2(r-k)+1)\theta - p' \\ &> 2r\theta + p' - 2 - 2r\theta - p' = -2,\end{aligned}$$

which contradicts (1). □

Claim 7. *$p \leq 1$ and $q \leq 1$.*

Proof. By Claims 1 and 6, $p \leq 1$, and hence $q \leq 1$. □

Claim 8. *$c' < 2(r-k)$ and $d' < 2k$.*

Proof. Suppose that $c' \geq 2(r-k)$ or $d' \geq 2k$. Then $c'\theta \geq 2(r-k)\theta$, or $d'(1-\theta) \geq 2k(1-\theta) = 2(r-k)\theta$ by (6). Thus $c'\theta + d'(1-\theta) \geq 2(r-k)\theta$. Hence by Claims 4, 5 and 7,

$$\begin{aligned}\delta_G(S,T;f) &\geq 2(r-k)\theta - (2(r-k)+1)\theta - 1 \\ &= -\theta - 1 \geq -3/2,\end{aligned}$$

which contradicts (1). □

Claim 9. *$||S| - |T|| \leq 1$.*

Proof. Since $a \leq k$ and $b \leq r-k$ by (5), we get $2a + c' < 2r$ and $2b + d' < 2r$ by Claim 8. Hence by Claim 3,

$$r||S| - |T|| \leq \max\{2a + c', 2b + d'\} < 2r,$$

which implies $||S| - |T|| \leq 1$. □

Claim 10. *$(c' - 2b)\theta + (d' - 2a)(1-\theta) \leq -1$.*

Proof. This follows from Claims 4 and 7 and (1). □

Claim 11. *If (2) holds, then $|S| - |T| = 0$ or 1.*

Proof. Note that (2) implies $a > b$. Hence by Claims 3 and 8, $r(|S| - |T|) > -d' > -2k \geq -r$. Consequently $|S| - |T| \geq 0$ which, in view of Claim 9, implies $|S| - |T| = 0$ or 1. □

Claim 12. *If (3) holds, then $|S| - |T| = -1$ or 0.*

Proof. Suppose that $|S|-|T|\geq 1$. Then by Claim 3, $c'-2b\geq d'-2a+r$. Hence

$$\begin{aligned}(c'-2b)\theta+(d'-2a)(1-\theta)&\geq(d'-2a+r)\theta+(d'-2a)(1-\theta)\\&=d'-2a+k\geq d'\geq 0,\end{aligned}$$

which contradicts Claim 10. Thus $|S|-|T|\leq 0$, and hence $|S|-|T|=-1$ or 0 by Claim 9. □

Claim 13. *If* (4) *holds, then* $|S|-|T|=-1$.

Proof. Suppose that $|S|-|T|\geq 0$. Then by Claim 3, $c'-2b\geq d'$. Hence

$$(c'-2b)\theta+(d'-2a)(1-\theta)\geq d'\theta+d'(1-\theta)=d'\geq 0,$$

which contradicts Claim 10. Thus $|S|-|T|\leq -1$, and hence $|S|-|T|=-1$ by Claim 9. □

Claim 14. $c'+d'<r$.

Proof. Suppose that $c'+d'\geq r$. Having Claim 9 in mind, we divide the proof into the following three cases.

Case 1. $|S|-|T|=-1$.

By Claim 3, $c'-d'=2b-2a-r$, and hence $c'\geq b-a$ and $d'\geq r+a-b$. By (5), $a+b\leq r-k$. Consequently

$$\begin{aligned}(c'-2b)\theta+(d'-2a)(1-\theta)&\geq-(a+b)\theta+(r-a-b)(1-\theta)\\&\geq-(r-k)\theta+k(1-\theta)=0\end{aligned}$$

by (6), which contradicts Claim 10.

Case 2. $|S|-|T|=0$.

By Claim 3, $c'\geq r/2-a+b$ and $d'\geq r/2+a-b$. By Claim 13, (2) or (3) holds, and hence $a+b\leq(r+1)/2$. Consequently

$$\begin{aligned}(c'-2b)\theta+(d'-2a)(1-\theta)&\geq(r/2-a-b)\theta-(r/2-a-b)(1-\theta)\\&=r/2-a-b\geq-1/2,\end{aligned}$$

which contradicts Claim 10.

Case 3. $|S|-|T|=1$.

By Claim 3, $c'\geq r-a+b$ and $d'\geq a-b$. By Claims 12 and 13, (2) holds. Consequently

$$\begin{aligned}(c'-2b)\theta+(d'-2a)(1-\theta)&\geq(r-a-b)\theta-(a+b)(1-\theta)\\&\geq(r-k)\theta-k(1-\theta)=0\end{aligned}$$

by (6), which contradicts Claim 10. □

Claim 15. *$p = 0$ and $q = 0$.*

Proof. This follows from Claims 1 and 14. □

We are now in a position to complete the proof of Theorem 1.1. Set $\varepsilon = |S| - |T|$. Then $\varepsilon \in \{-1, 0, 1\}$ by Claim 9. By Claim 15, $S \cup T = V(G)$, and $a_1 = b_1 = c_1 = d_1 = 0$. Hence $a = a_2$, $b = b_2$, $c' = 2c_2$ and $d' = 2d_2$. By (5), $a_2 + b_2 \leq r - k$. By Claim 14, $c_2 + d_2 < r/2$. By Claims 4 and 15 and (1), $(c_2 - b_2)\theta + (d_2 - a_2)(1 - \theta) \leq -1$. Therefore G_0 is of type $(\varepsilon; a_2, b_2, c_2, d_2)$. This completes the proof of Theorem 1.1.

From Claim 3 and Claims 11 through 13, we obtain the following corollary.

Corollary 2.2. *Let G_0, A, B be as in Theorem 1.1. Suppose that G_0 does not have a k-factor F with $E(F) \supseteq A$ and $E(F) \cap B = \emptyset$ and, under the notation of the proof of Theorem 1.1, let $c = c_2$ and $d = d_2$. Then $(G_0; A, B)$ is of type $(\varepsilon; a, b, c, d)$, and one of the following holds:*

(i) *r and k are even, $k/2 < a \leq k$, $a + b \leq k$, $\varepsilon = 1$, and $2(a + c) - 2(b + d) = r$;*

(ii) *$k/2 < a \leq k$, $b \geq 1$, $a + b \leq k$, $\varepsilon = 0$ and $a + c = b + d$;*

(iii) *$1 \leq a \leq k/2$, $b \geq 1$, $a + b \leq \lceil r/2 \rceil$, $\varepsilon = 0$ and $a + c = b + d$;*

(iv) *r and k are even, $1 \leq a \leq k/2$, $b > (r - k)/2$, $a + b \leq r/2$, $\varepsilon = -1$ and $2(a + c) - 2(b + d) = -r$; or*

(v) *r and k are even, $a = 0$, $(r - k)/2 < b \leq r - k$, $\varepsilon = -1$ and $2(a + c) - 2(b + d) = -r$.*

Proof. Note that if $\varepsilon = \pm 1$, then $|V(G_0)| = |S| + |T|$ is odd, and hence r and k are even by the assumption of Theorem 1.1. First assume $\varepsilon = 1$. Then by Claims 12 and 13, (2) holds. Thus it follows from Claim 3 that (i) holds. Next assume $\varepsilon = 0$. Then by Claim 13, (2) or (3) holds. Since $a + c = b + d$ by Claim 3, $(c - b)\theta + (d - a)(1 - \theta) = c - b$. Since $(c - b)\theta + (d - a)(1 - \theta) \leq -1$, this implies $b \geq c + 1 \geq 1$. Thus (ii) or (iii) holds. Finally assume $\varepsilon = -1$. Then by Claim 11, (3) or (4) holds. Since $2(a + c) - 2(b + d) = -r$ by Claim 3, $(c - b)\theta + (d - a)(1 - \theta) = (r/2 - (b - c))(1 - \theta) - (b - c)\theta$. Since $(c - b)\theta + (d - a)(1 - \theta) \leq -1$, we see from (6) that $b - c > (r - k)/2$, and hence $b > (r - k)/2$. Consequently (iv) or (v) holds. □

3. Sharpness of conditions

In this section, we show that conditions (I), (II), (III) in Theorem 1.1 are the weakest possible. Let r, k be integers with $k \geq 1$ and $r \geq 2k$. In the case where $k = 1$ and $r = 2$, it is easy to see that (II) does not occur, and (I) and (III) are sharp. Thus in what follows, we exclude the case where $k=1$ and $r=2$. Let a, b be nonnegative integers which satisfy one of the following three conditions:

$$k/2 < a \le k+1 \text{ and } a+b = k+1; \tag{7}$$

$$1 \le a \le k/2 \text{ and } a+b = \lceil r/2 \rceil + 1; \text{ or} \tag{8}$$

$$a = 0 \text{ and } b = r-k+1. \tag{9}$$

We define nonnegative integers c' and d' as follows: if (7) holds, let $c' = r-a+b$ and $d' = a-b$; if (8) holds, let $c' = \lceil r/2 \rceil - a + b$ and $d' = \lceil r/2 \rceil + a - b$; if (9) holds, let $c' = r-k+1$ and $d' = k-1$. Let S, T, U be disjoint sets with sufficiently large cardinality. We choose S and T so that $|S|-|T| = 1$, 0 or -1 according as (7), (8) or (9) holds. Then $(2a+c')-(2b+d') = r(|S|-|T|)$, When k or r is odd, we choose U so that $|S|+|T|+|U|$ is even. We construct an r-regular r-edge-connected graph G_0 with $V(G_0) = S \cup T \cup U$ as follows. First let A be a set of a independent edges joining vertices in S, and let B be a set of b independent edges joining vertices in T. Next let C be a set of c' independent edges joining S and U, and let D be a set of d' independent edges joining T and U. We choose C and D so that $A \cup B \cup C \cup D$ is independent. Then we can add edges joining S and T and edges joining vertices in U so that the resulting graph G_0 is r-regular r-edge-connected, and so that $\langle U \rangle_{G_0}$ contains at least $3r/2-k$ vertex-disjoint triangles. Now let G, f, θ be as in the proof of Theorem 1.1. Then applying Claim 4, we obtain $\delta_G(S,T;f) \le c'\theta + d'(1-\theta) - 2a(1-\theta) - 2b\theta = -1$ (see (6)). Hence by Theorem B, G_0 does not have a k-factor with $E(F) \supseteq A$ and $E(F) \cap B = \emptyset$. Further from the fact that G_0 contains $3r/2-k$ disjoint triangles, it follows that for any partition $V(G_0) = S' \cup T'$ of $V(G_0)$, we have $|E(\langle S' \rangle_{G_0})| + |E(\langle T' \rangle_{G_0})| \ge 3r/2-k$. Consequently G_0 is not of exceptional type. This shows that we cannot weaken conditions (I), (II), (III) in Theorem 1.1, even if we add the assumption that $A \cup B$ is independent.

References

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