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Order and Radius of 3-Connected Graphs

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Abstract

We show that if G is a 3-connected graph with radius r, then $|V(G)| \geq 4r - 4$.

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1. Introduction

By a graph, we mean a finite, undirected, simple graph without loops or multiple edges. Let G be a graph. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $v, w \in V(G)$, let $d_G(v, w)$ denote the usual distance between v and w. Set

$$r(G) := \min_{v \in V(G)} \max_{w \in V(G)} d_G(v, w).$$

The number $r(G)$ is called the radius of G. A vertex $z \in V(G)$ is called a central vertex of G if $\max_{w \in V(G)} d_G(z, w) = r(G)$.

In [3], Harant and Walter proved that there is a constant $C > 0$ such that every 3-connected graph G with radius r satisfies $|V(G)| + C \log |V(G)| > 4r$. Subsequently it was proved by Harant in [2] and by Inoue in [4] that if G is a 3-connected graph with radius r, then $|V(G)| \geq 4r - 15$. The purpose of this paper is to prove the following theorem.

Theorem. Let $r \geq 1$ be an integer, and let G be a 3-connected graph with radius r. Then $|V(G)| \geq 4r - 4$.

As is remarked in [3], the following graph shows that the bound $4r-4$ in the Theorem is sharp for $r \geq 3$. Let $r \geq 3$, and defined a graph G of order $4r-4$ by $V(G) = \{x_i, y_i | 1 \leq i \leq 2r-2\}, E(G) = \{x_{i-1}x_i, y_{i-1}y_i, x_iy_i | 1 \leq i \leq 2r-2\}$
(we take $x_0 = x_{2r-2}$ and $y_0 = y_{2r-2}$). Then G is 3-connected and $r(G) = r$.

In passing, we mention that in [1], Egawa and Inoue proved that if k is an integer with $k \geq 2$ and G is a $(2k-1)$-connected graph with radius r, then $|V(G)| \geq 2kr-(2k+9)$.

The organization of the paper is as follows. Section 2 contains prelimininary lemmas. We prove two main propositions, Propositions 1 and 2, in Sections 3 through 6. We complete the proof of the Theorem in Section 7.

2. Preliminary Results

Throughout the rest of the paper, we let G, r be as in the Theorem. If $r \leq 2$, we clearly have $|V(G)| \geq 4 \geq 4r-4$. Thus we may assume $r \geq 3$. For a vertex $v \in V(G)$ and a nonnegative integer i, let $N_i(v) := \{w | w \in V(G), d_G(v,w) = i\}$. We write $N(v)$ for $N_1(v)$. Fix a central vertex z, and let $X_i := N_i(z)$ for $0 \leq i \leq r$. Note that for each i with $1 \leq i \leq r-1$, we have $N(w) \subset X_{i-1} \cup X_i \cup X_{i+1}$ for every $w \in X_i$.

Lemma 1. *Let* $1 \leq i \leq r-1$. *Then* $|\{y \in X_i | N(y) \cap X_{i+1} \neq \phi\}| \geq 3$.

Proof. Since $G - \{y \in X_i | N(y) \cap X_{i+1} \neq \phi\}$ is disconnected, the desired conclusion follows from the assumption that G is 3-connected.

The following two lemmas immediately follow from Lemma 1.

Lemma 2. $|X_i| \geq 3$ *for each* $1 \leq i \leq r-1$.

Lemma 3. *Let* $1 \leq i \leq r-1$, *and suppose that* $|X_i| = 3$. *Then* $N(y) \cap X_{i+1} \neq \phi$ *for every* $y \in X_i$.

Lemma 4. *Let* $1 \leq i \leq r-1$, *and set* $A = \{y \in X_i | N(y) \cap X_{i+1} \neq \phi\}$. *Suppose that* $|A| = 3$ *and* $|X_{i+1}| \geq 3$. *Then there exist three independent edges joining* A *and* X_{i+1}.

Proof. Suppose not. Then by the Matching Theorem of Hall, there exists $Y \subseteq A$ such that $|(\cup_{y \in Y} N(y)) \cap X_{i+1}| \leq |Y|-1$. Set $Z = ((\cup_{y \in Y} N(y)) \cap X_{i+1}) \cup (A-Y)$. Then $G-Z$ is disconnected and $|Z| \leq 2$, which contradicts the assumption that G is 3-connected.

Lemma 5. $|V(G)| \geq |X_r| + 3r - 2$.

Proof. By Lemma 2, $|V(G)| = \sum_{i=0}^{r} |X_i| \geq 1 + 3(r-1) + |X_r| = |X_r| + 3r - 2$.

Lemma 6. *Set* $A = \{y \in X_{r-1} | N(y) \cap X_r \neq \phi\}$, *and suppose that we have either* $|X_r| = 1$, *or* $|A| = 3$ *and* $|X_r| = 2$. *Then the following hold.*

(i) $d_G(y, y') \leq 3$ *for every* $y, y' \in A \cup X_r$.

(ii) *If* B *is a subset of* A *with* $|B| = 2$ *, then for each* $y \in X_r$ *, there exists* $x \in B$ *such that* $d_G(x, y) = 1$ *and, for each* $y \in A$ *, there exists* $x \in B$ *such that* $d_G(x, y) \leq 2$ *.*

(iii) *There exists* $w \in A$ *such that* $d_G(w, y) = 1$ *for every* $y \in X_r$ *and* $d_G(w, y) \leq 2$ *for every* $y \in A$ *.*

Proof. If $N(v) \supseteq A$ for each $v \in X_r$, the desired conclusions follow immediately. Thus we may assume that there exists $v \in X_r$ such that $N(v) \not\supseteq A$. Since

$$\bigcup_{v \in X_r} N(v) \supseteq A \tag{2.1}$$

by the definition of A , this implies $|X_r| \geq 2$, and hence $|A| = 3$ and $|X_r| = 2$ by the assumption of the lemma. Write $A = \{y_1, y_2, y_3\}$, $X_r = \{z_1, z_2\}$. Note that $|N(z_i)| \geq 3$ for each $1 \leq i \leq 2$ by the assumption that G is 3-connected. Consequently, in view of (2.1), we may assume $z_1y_1, z_1y_2, z_2y_1, z_2y_3 \in E(G)$. Thus (iii) holds with $w = y_1$. Again since $|N(z_i)| \geq 3$ for each $1 \leq i \leq 2$, we also have $z_1z_2 \in E(G)$. Hence (i) holds. To prove (ii), let B be a subset of A with $|B| = 2$. If $y_1 \in B$, the desired conclusions hold with $x = y_1$ throughout. Thus we may assume $B = \{y_2, y_3\}$. Now for $y \in X_r$, the desired conclusion holds with $x = y_2$ or $x = y_3$ according as $y = z_1$ or $y = z_2$ and, for $y \in A$, the desired conclusion holds with $x = y_2$ or $x = y_3$ according as $y \in \{y_1, y_2\}$ or $y = y_3$.

3. Statement of Proposition 1 and Initial Reduction

We continue with the notation of the preceding section. The bulk of the proof of the Theorem is devoted to the verification of Propositions 1 and 2 (we prove Proposition 1 in Sections 3 through 5, and prove Proposition 2 in Section 6). Both of these two propositions roughly say that the average of the $|X_i|$ is only slightly less than 4 , if it is less than 4 .

Proposition 1. *Let* a, b *be integers with* $a + 2 \leq b$ *, and suppose that* $|X_a| = |X_b| = 3$ *and* $|X_i| > 3$ *for each* $a + 2 \leq i \leq b - 1$ *.*

(1) *Suppose that* $r \geq 7$ *,* $a \geq 3$ *and* $b \leq r - 2$ *. Then* $\sum_{i=a}^{b-1} |X_i| \geq 4(b - a)$ *.*

(2) *Suppose that* $r \geq 6$ *,* $a \geq 3$ *,* $b \leq r - 1$ *and* $|X_r| = 2$ *. Then* $\sum_{i=a}^{b-1} |X_i| \geq 4(b - a)$ *.*

(3) *Suppose that* $r \geq 5$ *,* $a \geq 2$ *,* $b \leq r - 1$ *and* $|X_r| = 1$ *. Then* $\sum_{i=a}^{b-1} |X_i| \geq 4(b - a)$ *.*

Proof. We prove (1), (2) and (3) simultaneously. By way of contradiction, suppose that $\sum_{i=a}^{b-1} |X_i| < 4(b - a)$. Then one of the following two situations must occur:

(A) $|X_i| = 4$ for each $a+1 \leq i \leq b-1$;or

(B) $|X_{a+1}| = 3$, and $|X_i| = 4$ or 5 for each $a+2 \leq i \leq b-1$, and the number of X_i with $|X_i| = 5$ is at most one.

We now introduce a graph structure $\mathscr{G}$ on X_{a+1} by joining v and w if and only if $d_G(v,w) \leq 2$ and $v \neq w$. Let α denote the independence number of $\mathscr{G}$.

Claim 3.1. $\alpha \leq 2$.

Proof. Suppose that $\alpha \geq 3$. Then there exist $v_1, v_2, v_3 \in X_{a+1}$ such that $d_G(v_i, v_j) \geq 3$ for all $1 \leq i < j \leq 3$. This implies that $(\{v_i\} \cup N(v_i)) \cap (\{v_j\} \cup N(v_j)) = \phi$ for all $1 \leq i < j \leq 3$. On the other hand, whether (A) holds or (B) holds, $|X_{a+1}| + |X_{a+2}| \leq 8$. Since $|N(v_i)| \geq 3$ for each i by the assumption that G is 3-connected, we now obtain $12 \leq \sum_{1 \leq i \leq 3} |\{v_i\} \cup N(v_i)| = |\cup_{1 \leq i \leq 3} (\{v_i\} \cup N(v_i))| \leq |X_a \cup X_{a+1} \cup X_{a+2}| \leq 3+8 = 11$, a contradiction.

In the rest of this section, we consider the case where $\mathscr{G}$ is connected.

Case 1. $\mathscr{G}$ is connected.

Since $|X_{a+1}| \leq 4$ by (A) and (B), we have $r(\mathscr{G}) \leq 2$. Let w_0 be a central vertex of $\mathscr{G}$. Note that if $r(\mathscr{G}) = 2$, then $|X_{a+1}| = 4$ and $\mathscr{G}$ is either a path of length 3 or a cycle of length 4 . Thus in the case where $r(\mathscr{G}) = 2$, we write $X_{a+1} = \{w_0, w_1, w_2, w_3\}$ so that $\{w_0w_1, w_0w_3, w_1w_2\} \subseteq E(\mathscr{G}) \subseteq \{w_0w_1, w_0w_3, w_1w_2, w_2w_3\}$.

Claim 3.2. *One of the following holds:*

(1) *there exist* $x \in X_{a+1}$ *and* c *with* $a+1 \leq c \leq b$ *such that* $d_G(x,y) \leq c-a+1$ *for every* $y \in X_c$ *and* $d_G(x,y') \leq r-a+1$ *for every* $y' \in \cup_{a \leq i \leq c-1} X_i$ *;or*

(2) $r(\mathscr{G}) = 2$ *, and there exists* c *with* $a+1 \leq c \leq b-1$ *such that* $d_G(w_0,y) \leq c-a+2$ *and* $d_G(w_1,y) \leq c-a+2$ *for every* $y \in X_c$ *, and* $d_G(w_0,y') \leq r-a+1$ *and* $d_G(w_1,y') \leq r-a+1$ *for every* $y' \in \cup_{a \leq i \leq c-1} X_i$ *;or*

(3) $r(\mathscr{G}) = 2$ *, and there exist* $x \in N(w_0) \cap N(w_1) \cap (X_a \cup X_{a+2})$ *and* c *with* $a+1 \leq c \leq b-1$ *such that either* $d_G(x,y) \leq c-a+1$ *for every* $y \in N_{c-(a+1)}(w_2) \cap X_c$ *, or* $d_G(x,y') \leq c-a+1$ *for every* $y' \in N_{c-(a+1)}(w_3) \cap X_c$ *.*

Proof. Suppose that none of (1), (2), (3) holds. Let w_0 be as in the paragraph preceding the statement of Claim 3.2. Suppose that $r(\mathscr{G}) = 1$. Then $d_G(w_0,w) \leq 2$ for every $w \in X_{a+1}$. Since $N(y) \cap X_{a+1} \neq \phi$ for every $y \in X_a$ by Lemma 3, this implies that $d_G(w_0,y) \leq 3 < r-a+1$ for every $y \in X_a$. Hence (1) holds with $x = w_0$ and $c = a+1$, which contradicts the assumption that none of (1), (2), (3) holds. Thus $r(\mathscr{G}) = 2$. This implies that (A) holds. Let w_1 , w_2 , w_3 be also as in the paragraph preceding the statement of Claim 3.2. If w_0w_1 or $w_2w_3 \in E(G)$, then $d_G(w_0,w) \leq$

3 and $d_G(w_1, w) \le 3$ for every $w \in X_{a+1}$, and hence it follows from Lemma 3 that $d_G(w_0, y) \le 4$ and $d_G(w_1, y) \le 4$ for every $y \in X_a$, which implies that (2) holds with $c = a+1$, a contradiction. Thus w_0w_1, $w_2w_3 \notin E(G)$. Since $d_{\mathscr{G}}(w_0, w_1) = 1$, this implies $N(w_0) \cap N(w_1) \cap (X_a \cup X_{a+2}) \ne \phi$. If w_0w_3 or $w_1w_2 \in E(G)$, then (3) holds with $c = a+1$. Thus w_0w_3, $w_1w_2 \notin E(G)$. Consequently $E(G[X_{a+1}]) = \phi$ (here $G[X_{a+1}]$ denotes the subgraph of G induced by X_{a+1}).

For each $j = 0, 1, 2, 3$, set $N_j = N(w_j) - (\cup_{w \in X_{a+1} - \{w_j\}} N(w))$. Note that

$$d_G(w_0, y) \le 4 \text{ for every } y \in X_{a+1} \tag{3.1}$$

and, in view of Lemma 3, we also have

$$d_G(w_0, y) \le 3 \text{ for every } y \in (X_a \cup X_{a+2}) - N_2, \tag{3.2}$$

and

$$d_G(w_0, y) \le 5 \text{ for every } y \in X_a. \tag{3.3}$$

If $N_2 = \phi$, then by (3.1) and (3.2), (1) holds with $x = w_0$ and $c = a+2$. Thus $N_2 \ne \phi$. Simiarly $N_3 \ne \phi$. On the other hand, since $N(w_0) \cap N(w_2) = \phi$ and $N(w_1) \cap N(w_3) = \phi$, we have $7 \ge |X_a \cup X_{a+2}| \ge |N_3| + |N(w_0)| + |N(w_2)|$ and $7 \ge |X_a \cup X_{a+2}| \ge |N_2| + |N(w_1)| + |N(w_3)|$. Since $|N(w_j)| \ge 3$ for each j by the assumption that G is 3-connected, this forces $|X_{a+2}| = 4$, $|N_2| = |N_3| = 1$, and $|N(w_j)| = 3$ for each j. Note that this implies $b \ge a+3$, and hence $r - a + 1 \ge 5$. If $N(w_2) \cap N(w_3) \ne \phi$, then $\mathscr{G}$ is a cycle of length 4, and hence we can argue as above to get $N_0 \ne \phi$ and $N_1 \ne \phi$, which implies $7 = |X_a \cup X_{a+2}| \ge |N_1| + |N_3| + |N(w_0)| + |N(w_2)| \ge 8$, a contradiction. Thus $N(w_2) \cap N(w_3) = \phi$. Write $N_2 = \{x_2\}$, $N_3 = \{x_3\}$, $N(w_2) - \{x_2\} = \{x_1, z_2\}$, $N(w_3) - \{x_3\} = \{x_0, z_3\}$ and $(X_a \cup X_{a+2}) - (N(w_2) \cup N(w_3)) = \{z_1\}$. Then $N(w_0) = \{x_0, z_1, z_3\}$ and $N(w_1) = \{x_1, z_1, z_2\}$.

If $x_2 \in X_a$, then $d_G(w_0, y) \le 3$ for every $y \in X_{a+2}$ by (3.2), and hence it follows from (3.1) and (3.3) that (1) holds with $x = w_0$ and $c = a+2$. Thus $x_2 \in X_{a+2}$. Similarly $x_3 \in X_{a+2}$. Note that this implies that

$$d_G(w, y) \le 5 \text{ for every } w \in X_{a+1} \text{ and every } y \in X_a. \tag{3.4}$$

Now since $N(w_2) \cap X_a \ne \phi$ and $N(w_3) \cap X_a \ne \phi$ by the definition of X_{a+1}, we have $\{x_1, z_2\} \cap X_a \ne \phi$ and $\{x_0, z_3\} \cap X_a \ne \phi$. At the cost of relabeling, we may assume $z_2, z_3 \in X_a$. Since $|X_a| = 3$, we have $|\{x_1, z_2\} \cap X_a| = 1$ or $|\{x_0, z_3\} \cap X_a| = 1$. We may assume $|\{x_1, z_2\} \cap X_a| = 1$.

Suppose that $z_1 \in X_{a+2}$. Then $X_a = \{x_0, z_2, z_3\}$ and $X_{a+2} = \{x_1, x_2, x_3, z_1\}$. If $z_1x_2 \in E(G)$ or $N(z_1) \cap N(x_2) \ne \phi$, then $d_G(w_0, y) \le 3$ for every $y \in X_a \cup X_{a+2}$ by (3.2), and hence it follows from (3.1) that (1) holds with $x = w_0$ and $c = a+2$. Thus

$z_1x_2 \notin E(G)$ and $N(z_1) \cap N(x_2) = \phi$. Similarly $z_1x_3 \notin E(G)$ and $N(z_1) \cap N(x_3) = \phi$. If $x_3x_2 \in E(G)$ or $x_3x_1 \in E(G)$ or $N(x_3) \cap N(x_2) \neq \phi$, then $d_G(w_2, y) \leq 3$ for every $y \in X_{a+2}$ and $d_G(w_2, y) \leq 4$ for every $y \in X_{a+1}$, and hence it follows from (3.4) that (1) holds with $x = w_2$ and $c = a + 2$. Thus x_3x_2, $x_3x_1 \notin E(G)$ and $N(x_3) \cap N(x_2) = \phi$. Since $|N(x_3)| \geq 3$, we now get $|N(x_3) \cap X_{a+3}| \geq 2$. If $x_2x_1 \in E(G)$, then $d_G(z_1, x_1) \leq 2$ and $d_G(z_1, x_2) \leq 3$, and hence (3) holds with $c = a + 2$. Thus $x_2x_1 \notin E(G)$. Since $|N(x_2)| \geq 3$, we get $|N(x_2) \cap X_{a+3}| \geq 2$. Since we already know $N(x_3) \cap N(x_2) = \phi$ and since $|X_{a+3}| \leq 4$, this implies $X_{a+3} = (N(x_3) \cap X_{a+3}) \cup (N(x_2) \cap X_{a+3})$. Since $|N(z_1)| \geq 3$, we now obtain $z_1x_1 \in E(G)$. But then $d_G(z_1, x_1) = 1$ and $d_G(z_1, x_2) \leq 3$, and hence (3) holds with $c = a + 2$, a contradiction. Consequently $z_1 \in X_a$, and hence $X_a = \{z_1, z_2, z_3\}$ and $X_{a+2} = \{x_0, x_1, x_2, x_3\}$.

We prove the following subclaim.

Subclaim. Let $a + 1 \leq i \leq b - 1$. Then the following hold.

(i) We have $|X_i| = 4$ and $E(G[X_i]) = \phi$, and we can write $X_i = \{x_{ij} | 1 \leq j \leq 4\}$ so that $x_{ij} \in N_{i-(a+1)}(w_j)$ for each $1 \leq j \leq 4$.

(ii) We have $|N(x_{i2}) \cap X_{i+1}| = |N(x_{i3}) \cap X_{i+1}| = 2$, $(N(x_{i2}) \cap X_{i+1}) \cap (N(x_{i3}) \cap X_{i+1}) = \phi$, $|N(x_{i1}) \cap X_{i+1}| = |N(x_{i0}) \cap X_{i+1}| = 1$, $N(x_{i1}) \cap X_{i+1} \subseteq N(x_{i2}) \cap X_{i+1}$ and $N(x_{i0}) \cap X_{i+1} \subseteq N(x_{i3}) \cap X_{i+1}$.

(iii) If $a + 2 \leq i \leq b - 1$, then $N_{i-(a+1)}(w_2) \cap X_i = \{x_{i2}, x_{i1}\}$ and $N_{i-(a+1)}(w_3) \cap X_i = \{x_{i3}, x_{i0}\}$.

Proof. We proceed by induction on i. If we simply let $x_{a+1,j} = w_j$ for each $1 \leq j \leq 4$, then (i) and (ii) hold for $i = a + 1$. Thus let $i \geq a + 2$, and assume that the subclaim is proved for $i - 1$. Let $y \in \cup_{a+1 \leq j \leq i} X_j$. Then by (3.1), $d_G(w_0, y) \leq 4 + (i - a - 1) \leq 4 + (r - 2 - a - 1) = r - a + 1$. Likewise $d_G(w_1, y) \leq r - a + 1$. Hence in view of (3.4), we have

$$d_G(w_0, y) \leq r - a + 1 \text{ and } d_G(w_1, y) \leq r - a + 1 \text{ for every } y \in \cup_{a \leq j \leq i} X_j. \quad (3.5)$$

Similarly

$$d_G(w_2, y) \leq r - a + 1 \text{ and } d_G(w_3, y) \leq r - a + 1 \text{ for every } y \in \cup_{a \leq j \leq i-2} X_j. \quad (3.6)$$

Now write $N(x_{i-1,1}) \cap X_i = \{x_{i1}\}$, $(N(x_{i-1,2}) \cap X_i) - \{x_{i1}\} = \{x_{i2}\}$, $N(x_{i-1,0}) \cap X_i = \{x_{i0}\}$ and $(N(x_{i-1,3}) \cap X_i) - \{x_{i0}\} = \{x_{i3}\}$. Then $X_i = \{x_{ij} | 1 \leq j \leq 4\}$, and (iii) holds.

Suppose that $x_{i0}x_{i1} \in E(G)$ or $x_{i2}x_{i3} \in E(G)$. Then $d_G(w_0, x_{i2}) \leq d_G(w_0, x_{i0}) + d_G(x_{i0}, x_{i1}) + d_G(x_{i1}, x_{i2}) \leq (i - a - 1) + 1 + 2$ or $d_G(w_0, x_{i2}) \leq d_G(w_0, x_{i0}) + d_G(x_{i0}, x_{i3}) +$

$d_G(x_{i3}, x_{i2}) \leq (i-a-1)+2+1$ according as $x_{i0}x_{i1} \in E(G)$ or $x_{i2}x_{i3} \in E(G)$. Hence in either case, $d_G(w_0, x_{i2}) \leq i-a+2$. Similarly $d_G(w_1, x_{i3}) \leq i-a+2$. Since we clearly have $d_G(w_0, y) \leq i-a+1$ for every $y \in X_i - \{x_{i2}\}$ and $d_G(w_1, y) \leq i-a+1$ for every $y \in X_i - \{x_{i3}\}$, it follows from (3.5) that (2) holds with $c = i$. Thus $x_{i0}x_{i1}$, $x_{i2}x_{i3} \notin E(G)$. Suppose now that $x_{i0}x_{i2} \in E(G)$ or $x_{i1}x_{i2} \in E(G)$. Then $d_G(z_1, x_{i2}) \leq i-a+1$. Since $d_G(z_1, x_{i1}) \leq i-a$, this implies that (3) holds with $c = i$. Thus $x_{i0}x_{i2}$, $x_{i1}x_{i2} \notin E(G)$. Similarly $x_{i0}x_{i3}$, $x_{i1}x_{i3} \notin E(G)$. Consequently $E(G[X_i]) = \phi$, which proves (i).

Since $|N(x)| \geq 3$ for each $x \in X_i$, we have $|N(x_{i2}) \cap X_{i+1}| \geq 2$, $|N(x_{i3}) \cap X_{i+1}| \geq 2$, $|N(x_{i1}) \cap X_{i+1}| \geq 1$ and $|N(x_{i0}) \cap X_{i+1}| \geq 1$. If $N(x_{i2}) \cap X_{i+1} \subseteq N(x_{i3}) \cap X_{i+1}$, then by (3.5), (1) holds with $x = w_0$ and $c = i+1$. Thus $N(x_{i2}) \cap X_{i+1} \not\subseteq N(x_{i3}) \cap X_{i+1}$, and hence $|(N(x_{i2}) \cup N(x_{i3})) \cap X_{i+1}| \geq 3$. Suppose that $N(x_{i2}) \cap N(x_{i3}) \cap X_{i+1} \neq \phi$. Then $d_G(w_2, y) \leq i-a+2$ and $d_G(w_3, y) \leq i-a+2$ for every $y \in (N(x_{i2}) \cup N(x_{i3})) \cap X_{i+1}$. We also get $d_G(w_2, x_{i3}) \leq i-a+1 < r-a+1$ and $d_G(w_2, x_{i-1,3}) \leq i-a+2 < r-a+1$ and, similarly, $d_G(w_3, x_{i2}) < r-a+1$ and $d_G(w_3, x_{i-1,2}) < r-a+1$. Since $|X_{i+1} - (N(x_{i2}) \cup N(x_{i3}))| \leq 1$, we have $X_{i+1} - (N(x_{i2}) \cup N(x_{i3})) \subseteq N(x_{i1})$ or $X_{i+1} - (N(x_{i2}) \cup N(x_{i3})) \subseteq N(x_{i0})$. We may assume $X_{i+1} - (N(x_{i2}) \cup N(x_{i3})) \subseteq N(x_{i1})$. Then $d_G(w_2, y) \leq i-a+2$ for every $y \in X_{i+1}$. Since we clearly have $d_G(w_2, y) \leq 4+(i-a-1) \leq r-a+1$ for every $y \in (X_i \cup X_{i-1}) - \{x_{i3}, x_{i-1,3}\}$, it now follows from (3.6) that (1) holds with $x = w_2$ and $c = i+1$. Thus $(N(x_{i2}) \cap X_{i+1}) \cap (N(x_{i3}) \cap X_{i+1}) = \phi$. Since $|X_{i+1}| \leq 4$, this implies $|N(x_{i2}) \cap X_{i+1}| = |N(x_{i3}) \cap X_{i+1}| = 2$. If $(N(x_{i1}) \cap X_{i+1}) \cap (N(x_{i3}) \cap X_{i+1}) \neq \phi$, then by (3.5), (1) holds with $x = w_1$ and $c = i+1$. Thus $(N(x_{i1}) \cap X_{i+1}) \cap (N(x_{i3}) \cap X_{i+1}) = \phi$, and hence $N(x_{i1}) \cap X_{i+1} \subseteq N(x_{i2}) \cap X_{i+1}$. If $N(x_{i1}) \cap X_{i+1} = N(x_{i2}) \cap X_{i+1}$, then by (3.5), (1) holds with $x = w_0$ and $c = i+1$. Thus $|N(x_{i1}) \cap X_{i+1}| = 1$. Similarly we obtain $N(x_{i0}) \cap X_{i+1} \subseteq N(x_{i3}) \cap X_{i+1}$ and $|N(x_{i0}) \cap X_{i+1}| = 1$. This proves (ii), and completes the proof of the subclaim.

We can now easily complete the proof of Claim 3.2. Let $i = b-1$ in the Subclaim. Then from (ii), we obtain $|X_b| \geq |N(x_{b-1,2}) \cap X_b| + |N(x_{b-1,3}) \cap X_b| = 4$, which contradicts the assumption that $|X_b| = 3$. This completes the proof of Claim 3.2.

Having Claim 3.2 in mind, we divide the rest of the proof for Case 1 into three cases. In each case, we derive a contradiction by showing that there exists $u \in V(G)$ such that $d_G(u, v) < r$ for every $v \in V(G)$.

Case 1.1. Claim 3.2(1) holds.

Let x, c be as in Claim 3.2(1), and let u be a vertex in X_3 which is on a shortest $z - x$ path. We show that $d_G(u, v) < r$ for every $v \in V(G)$. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq a-1$, $d_G(u, v) \leq d_G(u, z) + d_G(z, v) = 3 + i \leq 3 + (a-1) = a+2 \leq r-1 < r$. If $a \leq i \leq c-1$, then $d_G(x, v) \leq r-a+1$ by Claim 3.2(1), and hence $d_G(u, v) \leq d_G(u, x) + d_G(x, v) \leq \{(a+1)-3\} + (r-a+1) = r-1 < r$. Thus we may assume $c \leq i \leq r$. Let y be a vertex in X_c which is on a shortest $z - v$ path. Then $d_G(x, y) \leq c-a+1$ by Claim 3.2(1), and hence $d_G(u, v) \leq d_G(u, x) + d_G(x, y) + d_G(y, v) \leq \{(a+1)-3\} + (c-a+1) + (i-c) = i-1 \leq r-1 < r$.

Case 1.2. Claim 3.2(2) holds.

Let c be as in Claim 3.2(2).

Subcase 1.2.1. $a \geq 3$ and $b \leq r-2$.

Let u be a vertex in X_4 which is on a shortest $z-w_0$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq a-1$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = 4+i \leq 4+(a-1) = a+3 \leq r-1 < r$. If $a \leq i \leq c-1$, then $d_G(w_0,v) \leq r-a+1$ by Claim 3.2(2), and hence $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,v) \leq \{(a+1)-4\}+(r-a+1) = r-2 < r$. If $c \leq i \leq r$, then letting y be a vertex in X_c which is on a shortest $z-v$ path, and we get $d_G(w_0,y) \leq c-a+2$ by Claim 3.2(2), and hence $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,y)+d_G(y,v) \leq \{(a+1)-4\}+(c-a+2)+(i-c) = i-1 \leq r-1 < r$.

Subcase 1.2.2. $a=2$ or $b=r-1$.

By the assumptions of the proposition, we have either $|X_r| = 1$, or $|X_{r-1}| = 3$ and $|X_r| = 2$. By Lemma 6(iii) (and Lemma 3), there exists $v_{r-1} \in X_{r-1}$ such that $d_G(v_{r-1},v') = 1$ for every $v' \in X_r$. Let w' be a vertex in X_{a+1} which is on a shortest $z-v_{r-1}$ path. By symmetry, we may assume $w' \in \{w_0,w_3\}$. Then $d_G(w_0,w') \leq 2$ (see the paragraph preceding Claim 3.2). Let u be a vertex in X_3 which is on a shortest $z-w_0$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq a-1$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = 3+i \leq 3+(a-1) = a+2 \leq r-1 < r$. If $a \leq i \leq c-1$, then $d_G(w_0,v) \leq r-a+1$ by Claim 3.2(2), and hence $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,v) \leq \{(a+1)-3\}+(r-a+1) = r-1 < r$. If $c \leq i \leq r-1$, then letting y be a vertex in X_c which is on a shortest $z-v$ path, we get $d_G(w_0,y) \leq c-a+2$ by Claim 3.2(2), and hence $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,y)+d_G(y,v) \leq \{(a+1)-3\}+(c-a+2)+(i-c) = i \leq r-1 < r$. If $i=r$, then letting v_{r-1}, w' be as above, we obtain $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,w')+d_G(w',v_{r-1})+d_G(v_{r-1},v) \leq \{(a+1)-3\}+2+\{(r-1)-(a+1)\}+1 = r-1 < r$.

Case 1.3. Claim 3.2(3) holds.

Let x, c be as in Claim 3.2(3). We may assume that $d_G(x,y) \leq c-a+1$ for every $y \in N_{c-(a+1)}(w_2) \cap X_c$.

Subcase 1.3.1. $a \geq 3$ and $b \leq r-2$.

Let u be a vertex in X_4 which is on a shortest $z-w_0$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq a-1$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = 4+i \leq 4+(a-1) = a+3 \leq r-1 < r$. If $i=a$, then taking $w \in N(v) \cap X_{a+1}$ (see Lemma 3), we get $d_G(w_0,w) \leq 4$ (see the paragraph preceding Claim 3.2), and hence $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,w)+d_G(w,v) \leq \{(a+1)-4\}+4+1 = a+2 \leq r-2 < r$. If $a+1 \leq i \leq r-1$, then letting w' be a vertex in X_{a+1} which is on a shortest $z-v$ path, we get $d_G(w_0,w') \leq 4$, and hence $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,w')+d_G(w',v) \leq \{(a+1)-4\}+4+\{i-(a+1)\} = i \leq r-1 < r$. Thus we may assume $i=r$. Let P be a shortest $z-v$ path, and let w'' be the vertex in X_{a+1} which is on P. If $w'' \in \{w_0,w_1,w_3\}$, then $d_G(w_0,w'') \leq 2$, and hence $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,w'')+d_G(w'',v) \leq \{(a+1)-4\}+2+\{r-(a+1)\} = r-2 < r$. Thus we may assume $w'' = w_2$. Let y be the vertex in X_c which is on P. Then $d_G(w_2,y) = c-(a+1)$, and hence $d_G(x,y) \leq c-a+1$ by the assumption made at the beginning of Case 1.3. Therefore $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,x)+d_G(x,y)+d_G(y,v) \leq \{(a+1)-4\}+1+(c-a+1)+(r-c) = r-1 < r$.

Subcase 1.3.2. $a = 2$ or $b = r-1$.

We have $|X_r| = 1$, or $|X_{r-1}| = 3$ and $|X_r| = 2$. By Lemma 6(iii), there exists $v_{r-1} \in X_{r-1}$ such that $d_G(v_{r-1}, v') = 1$ for every $v' \in X_r$. Let Q be a shortest $z - v_{r-1}$ path, and let w' be the vertex in X_{a+1} which is on Q.

Subcase 1.3.2.1. $w' \in \{w_0, w_1, w_3\}$.

Let u be a vertex in X_3 which is on a shortest $z - w_0$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq a-1$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = 3+i \leq 3+(a-1) = a+2 \leq r-1 < r$. Thus we may assume $a \leq i \leq r$. First we consider the case where $i = a$. If $i = a \leq r-4$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = 3+i \leq 3+(r-4) = r-1 < r$. Thus we may assume $i = a = r-3$. Then $c = r-2$, and hence $d_G(x,w_2) \leq 2$ by the assumption made at the beginning of Case 1.3. This implies that $d_G(w_0,w) \leq 3$ for every $w \in X_{a+1}$. Now having Lemma 3 in mind, take $w \in N(v) \cap X_{a+1}$. Then $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,w)+d_G(w,v) \leq \{(a+1)-3\}+3+1 = a+2 = r-1 < r$. Next we consider the case where $a+1 \leq i \leq c$. Let w'' be a vertex in X_{a+1} which is on a shortest $z-v$ path. Then $d_G(w_0,w'') \leq 4$. Hence $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,w'')+d_G(w'',v) \leq \{(a+1)-3\}+4+\{i-(a+1)\} = i+1 \leq c+1 \leq r-1 < r$. Now we consider the case where $c+1 \leq i \leq r-1$. Let P be a shortest $z-v$ path, and let w''' be the vertex in X_{a+1} which is on P. If $w''' \in \{w_0,w_1,w_3\}$, then $d_G(w_0,w''') \leq 2$, and hence $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,w''')+d_G(w''',v) \leq \{(a+1)-3\}+2+\{i-(a+1)\} = i-1 \leq r-2 < r$. Thus we may assume $w''' = w_2$. Let y be the vertex in X_c which is on P. Then $d_G(w_2,y) = c-(a+1)$, and hence $d_G(x,y) \leq c-a+1$. Therefore $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,x)+d_G(x,y)+d_G(y,v) \leq \{(a+1)-3\}+1+(c-a+1)+(i-c) = i \leq r-1 < r$. Finally we consider the case where $i = r$. Let v_{r-1}, w' be as defined at the beginning of Subcase 1.3.2. Then $d_G(w_0,w') \leq 2$ by the assumption of Subcase 1.3.2.1. Hence $d_G(u,v) \leq d_G(u,w_0)+d_G(w_0,w')+d_G(w',v_{r-1})+d_G(v_{r-1},v) \leq \{(a+1)-3\}+2+\{(r-1)-(a+1)\}+1 = r-1 < r$.

Subcase 1.3.2.2.1. $w' = w_2$ and $x \in X_a$.

Note that $d_G(x,w) \leq 3$ for every $w \in X_{a+1}$ (see the paragraph preceding Claim 3.2). Let u be a vertex in X_2 which is on a shortest $z-x$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq a$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = 2+i \leq 2+a \leq r-1 < r$. If $a+1 \leq i \leq r-1$, then letting w be a vertex in X_{a+1} which is on a shortest $z-v$ path, we obtain $d_G(u,v) \leq d_G(u,x)+d_G(x,w)+d_G(w,v) \leq (a-2)+3+\{i-(a+1)\} = i \leq r-1 < r$. Thus we may assume $i = r$. Let y be the vertex in X_c which is on Q, where Q is as defined at the beginning of Subcase 1.3.2. Then $d_G(w_2,y) = c-(a+1)$ by the assumption of Subcase 1.3.2.2.1, and hence $d_G(x,y) \leq c-a+1$. Therefore $d_G(u,v) \leq d_G(u,x)+d_G(x,y)+d_G(y,v_{r-1})+d_G(v_{r-1},v) \leq (a-2)+(c-a+1)+\{(r-1)-c\}+1 = r-1 < r$.

Subcase 1.3.2.2.2.1. $w' = w_2$ and $x \in X_{a+2}$ and $a \leq r-4$.

Note that $d_G(x,w) \leq 3$ for every $w \in X_{a+1}$. Let u be a vertex in X_4 which is on a shortest $z-x$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le a-1$, $d_G(u,v) \le d_G(u,z)+d_G(z,v) = 4+i \le a+3 \le r-1 < r$. If $i = a$, then taking $w \in N(v) \cap X_{a+1}$, we obtain $d_G(u,v) \le d_G(u,x)+d_G(x,w)+d_G(w,v) \le \{(a+2)-4\}+3+1 = a+2 \le r-2 < r$. If $a+1 \le i \le r-1$, then letting w be a vertex in X_{a+1} which is on a shortest $z-v$ path, we obtain $d_G(u,v) \le d_G(u,x)+d_G(x,w)+d_G(w,v) \le \{(a+2)-4\}+3+\{i-(a+1)\} = i \le r-1 < r$. If $i = r$, then letting y be the vertex in X_c which is on Q, we get $d_G(x,y) \le c-a+1$, and hence $d_G(u,v) \le d_G(u,x)+d_G(x,y)+d_G(y,v_{r-1})+d_G(v_{r-1},v) \le \{(a+2)-4\}+(c-a+1)+\{(r-1)-c\}+1 = r-1 < r$.

Subcase 1.3.2.2.2.2. $w' = w_2$ and $x \in X_{a+2}$ and $a = r-3$.

We have $b = r-1$ and $c = r-2$, and hence $d_G(x,w_2) \le 2$. This implies that $d_G(w_0,w) \le 3$ for every $w \in X_{r-2}$. Since $b = r-1$, we have $|X_{r-1}| = 3$ and $|X_r| \le 2$. Hence by Lemma 6(i) (and Lemma 3), $d_G(x,y) \le 3$ for every $y \in X_{r-1} \cup X_r$. Consequently $d_G(w_0,y') \le 4$ for every $y' \in X_{r-2} \cup X_{r-1} \cup X_r$. Now let u be a vertex in X_3 which is on a shortest $z-w_0$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le r-4$, $d_G(u,v) \le d_G(u,z)+d_G(z,v) = 3+i \le r-1 < r$. If $i = r-3$, then taking $w \in N(v) \cap X_{r-2}$ (recall that $r-3 = a$ by the assumption of Subcase 1.3.2.2.2.2), we obtain $d_G(u,v) \le d_G(u,w_0)+d_G(w_0,w)+d_G(w,v) \le \{(r-2)-3\}+3+1 = r-1 < r$. If $r-2 \le i \le r$, $d_G(u,v) \le d_G(u,w_0)+d_G(w_0,v) \le \{(r-2)-3\}+4 = r-1 < r$.

This concludes the discussion for the case where $\mathscr{G}$ is connected.

4. The Case Where $\mathscr{G}$ Is Disconnected

In this section and the next section, we consider the case where $\mathscr{G}$ is disconnected. The main results of this section are Claims 4.11 through 4.13, which correspond to Claim 3.2 in Case 1.

Case 2. $\mathscr{G}$ is disconnected.

By Claim 3.1, $\mathscr{G}$ consists of two components. Let S_{a+1} and T_{a+1} be the vertex sets of the components of $\mathscr{G}$. For j with $a \le j \le b$, set $S_j = (\cup_{v \in S_{a+1}} N_{|a+1-j|}(v)) \cap X_j$, $T_j = (\cup_{w \in T_{a+1}} N_{|a+1-j|}(w)) \cap X_j$.

Since $S_{a+1} \cup T_{a+1} = X_{a+1}$, it immediately follows from the definition of S_j and T_j that $S_j \cup T_j = X_j$ for each $a+1 \le j \le b$. Applying Lemma 3 with $i = a$, we also see that $S_a \cup T_a = X_a$. Thus $S_j \cup T_j = X_j$ for each $a \le j \le b$. Also since $d_G(v,w) \ge 3$ for every $v \in S_{a+1}$ and every $w \in T_{a+1}$, $S_j \cap T_j = \phi$ for each $a \le j \le a+2$. Note that $S_a \ne \phi$ and $T_a \ne \phi$ by the definition of X_j. Thus we may assume $|S_a| = 1$ and $|T_a| = 2$.

Claim 4.1. *Let $a+1 \le i \le b$, and suppose that for each h with $a+1 \le h \le i-1, d_G(v,w) \ge 3$ for every $v \in S_h$ and every $w \in T_h$. Then the following hold.*

(1) (a) $|S_i| \ge 1$. (b)*If* $i \ge a+2, |S_i| \ge 2$.

(2) $|T_i| \ge 2$.

Proof. From the assumption of Claim 4.1, it follows that $G-(S_i \cup T_a)$ is disconnected, and hence (1)(a) follows from the assumption that G is 3-connected. Similarly, $G-(S_a \cup T_i)$ is disconnected and, in the case where $i \geq a+2, G-(S_a \cup S_i)$ is also disconnected, and hence (1)(b) and (2) also follow from the assumption that G is 3-connected.

We define an integer c as follows. Set

$Q := \{i | a+1 \leq i \leq b-1$, there exists $w_1 \in S_i$ and there exists $w_2 \in T_i$ such that $d_G(w_1, w_2) \leq 2\}$.

We have $Q \neq \phi$ because if $Q = \phi$, then $|X_b| = |S_b| + |T_b| \geq 2+2 = 4$ by Claim 4.1, which contradicts the assumption that $|X_b| = 3$. Now set

$$c = \min Q .$$

Then $a+2 \leq c \leq \max Q \leq b-1$ by the definition of S_{a+1} and T_{a+1}. The following remarks immediately follow from the definition of c.

Remark. For each $a \leq i \leq c$, we have $X_i - S_i = T_i$.

Remark. Let $a+1 \leq i \leq c-1$. Then $N(v) \subset S_{i-1} \cup S_i \cup S_{i+1}$ for every $v \in S_i$, and $N(w) \subset T_{i-1} \cup T_i \cup T_{i+1}$ for every $w \in T_i$.

The following two claims immediately follow from Claim 4.1.

Claim 4.2.

(1) *If* (A) *holds, then* $|S_{a+1}| = 1$ *or* 2, *and* $|S_i| = 2$ *for each* $a+2 \leq i \leq c$.

(2) *If* (B) *holds, then* $|S_{a+1}| = 1, |S_i| = 2$ *or* 3 *for each* $a+2 \leq i \leq c$, *and the number of those indices* i *with* $a+2 \leq i \leq c$ *for which* $|S_i| = 3$ *is at most one.*

Claim 4.3.

(1) *If* (A) *holds, then* $|T_{a+1}| = 2$ *or* 3, *and* $|T_i| = 2$ *for each* $a+2 \leq i \leq c$.

(2) *If* (B) *holds, then* $|T_i| = 2$ *or* 3 *for each* $a+1 \leq i \leq c$, *and the number of those indices* i *with* $a+1 \leq i \leq c$ *for which* $|T_i| = 3$ *is at most one.*

Claim 4.4. $|S_{i-1} \cup S_i \cup S_{i+1}| \leq 7$ *for each* $a+1 \leq i \leq c-1$.

Proof. Since Claim 4.2 implies that $|S_i| \leq 3$ for each $a \leq i \leq c$, and that the number of indices i with $a \leq i \leq c$ such that $|S_i| = 3$ is at most one, the desired inequality follows immediately.

Claim 4.5. $|T_{i-1} \cup T_i \cup T_{i+1}| \leq 7$ *for each* $a+1 \leq i \leq c-1$.

Proof. Since Claim 4.3 implies that $|T_i| \leq 3$ for each $a \leq i \leq c$, and that the number of indices i with $a \leq i \leq c$ such that $|T_i| = 3$ is at most one, the desired inequality follows immediately.

Claim 4.6. *Let $a+1 \le i \le c-1$. Then the following hold.*

(1) $d_G(v,v') \le 2$ *for every* $v, v' \in S_i$.

(2) $d_G(w,w') \le 2$ *for every* $w, w' \in T_i$.

Proof. Suppose that there exist $v, v' \in S_i$ such that $d_G(v,v') \ge 3$. Then $(\{v\} \cup N(v)) \cap (\{v'\} \cup N(v')) = \phi$. Since $|N(v)| \ge 3$ and $|N(v')| \ge 3$ by the assumption that G is 3-connected, it follows from Claim 4.4 that $8 \le |\{v,v'\} \cup N(v) \cup N(v')| \le |S_{i-1} \cup S_i \cup S_{i+1}| \le 7$, a contradiction. Thus (1) is proved. We can prove (2) in exactly the same way by using Claim 4.5 in place of Claim 4.4.

Claim 4.7. *Let $a \le i < j \le c$. Then the following hold.*

(1) $d_G(w,v) \le j-i+2$ *for every* $w \in S_j$ *and every* $v \in S_i$.

(2) $d_G(w,v) \le j-i+2$ *for every* $w \in T_j$ *and every* $v \in T_i$.

Proof. Let $w \in T_j$ and $v \in T_i$. If $a+1 \le i \le c-1$, then letting v' be a vertex in T_i which is on a shortest $z-w$ path, we get $d_G(v',v) \le 2$ by Claim 4.6(2), and hence $d_G(w,v) \le d_G(w,v') + d_G(v',v) \le j-i+2$. Thus we may assume $i = a$. Take $v'' \in N(v) \cap T_{a+1}$ (see Lemma 3). Let v''' be a vertex in T_{a+1} which is on a shortest $z-w$ path. Then $d_G(v''',v'') \le 2$ by Claim 4.6(2), and hence $d_G(w,v) \le d_G(w,v''') + d_G(v''',v'') + d_G(v'',v) \le \{j-(a+1)\} + 2 + 1 = j-a+2$. This proves (2). Note that we did not make use of the assumption that $|S_a| = 1$ and $|T_a| = 2$ in the proof of (2). Thus (1) similarly follows from Claim 4.6(1).

Letting $i = c-1$ and $j = c$ in Claim 4.7, we obtain the following claim.

Claim 4.8.

(1) $d_G(u,v) \le 3$ *for every* $u \in S_{c-1}$ *and every* $v \in S_c$.

(2) $d_G(u,v) \le 3$ *for every* $u \in T_{c-1}$ *and every* $v \in T_c$.

Claim 4.9.

(1) $d_G(w,w') \le 4$ *for every* $w, w' \in S_c$.

(2) $d_G(w,w') \le 4$ *for every* $w, w' \in T_c$.

Proof. Let $w, w' \in S_c$. Let u be a vertex in S_{c-1} which is on a shortest $z-w$ path (i.e., $u \in N(w) \cap S_{c-1}$). Then by Claim 4.8(1), $d_G(w,w') \le 1 + d_G(u,w') \le 4$. Thus (1) is proved, and (2) similarly follows from Claim 4.8(2).

Throughout the rest of the discussion for Case 2, we fix two vertices w_1, w_2 with $w_1 \in S_c$ and $w_2 \in T_c$ such that $d_G(w_1,w_2) \le 2$. If possible, we choose w_1 and w_2 so that $d_G(w_1,w_2) = 1$. We prove one more auxiliary result.

Claim 4.10. *Let $a \le i \le c-1$, and let $v \in X_i$.*

(1) *If $v \in S_i$, then $d_G(w_1, v) \le c-i+2$ and $d_G(w_2, v) \le c-i+4$.*

(2) *If $v \in T_i$, then $d_G(w_1, v) \le c-i+4$ and $d_G(w_2, v) \le c-i+2$.*

(3) *If $i = a$ and $v \in S_a$, then $d_G(w_1, v) = c-a$ and $d_G(w_2, v) \le c-a+2$.*

Proof. Since $d_G(w_1, w_2) \le 2$, (1) and (2) follow from Claim 4.7. To prove (3), let $v \in S_a$. Then since $|S_a| = 1$, it follows from the definition of S_j that v is on a shortest $z - w_1$ path. Thus (3) immediately follows from the assumption that $d_G(w_1, w_2) \le 2$.

We now prove the three main claims of this section.

Claim 4.11. *Suppose that $a+3 \le c \le r-3$ and $d_G(w_1, w_2) = 2$. Then one of the following holds:*

(1) *$d_G(w_1, y) \le 4$ for every $y \in X_c$; or*

(2) *$d_G(w_2, y) \le 4$ for every $y \in X_c$; or*

(3) *there exists $x \in X_{c-1}$ such that $d_G(x, y) \le 7$ for every $y \in X_{c-1} \cup X_c$ and $d_G(x, y') \le 4$ for every $y' \in X_{c+1}$.*

Proof. Suppose that none of (1), (2), (3) holds. First note that the assumption that $d_G(w_1, w_2) = 2$, together with the choice of w_1 and w_2 mentioned in the paragraph preceding Claim 4.10, implies that there is no edge between S_c and T_c. By Claims 4.2 and 4.3, $|S_{c-1}| = |S_c| = 2$ or $|T_{c-1}| = |T_c| = 2$. By symmetry, we may assume $|S_{c-1}| = |S_c| = 2$ (we do not make use of the assumption that $|S_a| = 1$ and $|T_a| = 2$ in the proof of Claim 4.11). Write $S_c = \{w_1, w_3\}$, $S_{c-1} = \{u_1, u_3\}$. We may assume $u_1w_1 \in E(G)$. If $w_1w_3 \in E(G)$ or $N(w_1) \cap N(w_3) \ne \phi$, then $d_G(w_1, w_3) \le 2$, and hence it follows from Claim 4.9(2) that (2) holds. Thus $w_1w_3 \notin E(G)$ and $N(w_1) \cap N(w_3) = \phi$. Since $N(w_3) \cap S_{c-1} \ne \phi$ by the definition of S_c, this forces $N(w_1) \cap S_{c-1} = \{u_1\}$ and $N(w_3) \cap S_{c-1} = \{u_3\}$. Since $|N(w_1)| \ge 3$ and $|N(w_3)| \ge 3$, we now obtain $|N(w_1) \cap X_{c+1}| \ge 2$ and $|N(w_3) \cap X_{c+1}| \ge 2$. Note that by Claim 4.6, 4.8 and 4.9, we have $d_G(u, v) \le 4$ for every $u, v \in S_{c-1} \cup S_c$, and $d_G(u, v) \le 4$ for every $u, v \in T_{c-1} \cup T_c$. Since $d_G(u_1, w_2) \le 3$, this implies that

$$d_G(u_1, y) \le 7 \text{ for every } y \in X_{c-1} \cup X_c. \tag{4.1}$$

Since (3) does not hold with $x = u_1$, it follows from (4.1) and Claim 4.8(1) that there exists $y_0 \in X_{c+1}$ such that $y_0 \notin N(w_1) \cup N(w_3)$. This means that (B) holds, and $X_{c+1} = (N(w_1) \cap X_{c+1}) \cup (N(w_3) \cap X_{c+1}) \cup \{y_0\}$, and hence $|T_c| = |T_{c-1}| = 2$. Write $T_c = \{w_2, w_4\}$. Since (3) does not hold with $x = u_1$, we have $y_0 \notin N(w_2)$. Hence $y_0 \in N(w_4)$. Now arguing as above with the roles of S_c and T_c replaced by each other, we obtain $|N(w_4) \cap X_{c+1}| \ge 2$. If $(N(w_4) \cap X_{c+1}) \cap (N(w_1) \cap X_{c+1}) \ne \phi$, then $d_G(u_1, w_4) \le 3$, and hence it follows from (4.1) and Claim 4.8(1) that (3) holds with $x = u_1$. Thus $(N(w_4) \cap X_{c+1}) \cap (N(w_1) \cap X_{c+1}) = \phi$. If $(N(w_4) \cap X_{c+1}) \cap (N(w_3) \cap X_{c+1}) \neq \phi$, then $d_G(u_3, w_4) \leq 3$, and hence $d_G(u_3, y) \leq 7$ for every $y \in X_{c-1} \cup X_c$, which together with Claim 4.8(1) implies that (3) holds with $x = u_3$. Thus $(N(w_4) \cap X_{c+1}) \cap (N(w_3) \cap X_{c+1}) = \phi$. Consequently $N(w_4) \cap X_{c+1} = \{y_0\}$. But this contradicts the earlier assertion that $|N(w_4) \cap X_{c+1}| \geq 2$.

Claim 4.12. *Suppose that $a + 2 = c \leq r - 3$. Then one of the following holds:*

(1) $d_G(w_1, w) \leq 3, d_G(w_2, w) \leq 4$ *for every* $w \in S_{a+2}$ *and* $d_G(w_1, w') \leq 4, d_G(w_2, w') \leq 3$ *for every* $w' \in T_{a+2}$*; or*

(2) $d_G(w_2, w) \leq 3$ *for every* $w \in S_{a+2}$*; or*

(3) *there exists* $x \in X_{a+1}$ *such that* $d_G(x, y) \leq 5$ *for every* $y \in X_{a+2}$ *and* $d_G(x, y') \leq 4$ *for every* $y' \in X_{a+3}$.

Proof. Suppose that none of (1), (2), (3) holds. By Claim 4.2, $|S_{a+1}| = 1$ or 2. We first consider the case where $|S_{a+1}| = 1$. In this case,

$$d_G(w, w') \leq 2 \text{ for every } w, w' \in S_{a+2}. \tag{4.2}$$

If $d_G(w_1, w_2) = 1$, then it follows from (4.2) that (2) holds. Thus $d_G(w_1, w_2) = 2$. By our choice of w_1 and w_2, this implies that there is no edge between S_{a+2} and T_{a+2}. If $d_G(w_2, w) \leq 2$ for every $w \in T_{a+2}$, then it follows from (4.2) that (1) holds. Thus there exists $w_4 \in T_{a+2}$ such that $d_G(w_2, w_4) \geq 3$. Then $w_2 w_4 \notin E(G)$ and $N(w_2) \cap N(w_4) = \phi$. Note that

$$d_G(u, y') \leq 4 \text{ for every } u \in T_{a+1} \text{ and every } y' \in (N(w_2) \cup N(w_4)) \cap X_{a+3} \tag{4.3}$$

by Claim 4.8(2). Take $u_2 \in N(w_2) \cap T_{a+1}$. Then

$$d_G(u_2, y) \leq 5 \text{ for every } y \in X_{a+2}. \tag{4.4}$$

by (4.2) and Claim 4.8(2). Since (3) does not hold with $x = u_2$, it follows from (4.3) and (4.4) that $X_{a+3} - (N(w_2) \cup N(w_4)) \neq \phi$. Since $|N(w_2)| \geq 3$ and $|N(w_4)| \geq 3$, we now obtain $|X_{a+3}| + |T_{a+2}| + |T_{a+1}| \geq |X_{a+3} - (N(w_2) \cup N(w_4))| + |\{w_2\} \cup N(w_2)| + |\{w_4\} \cup N(w_4)| \geq 1 + 4 + 4 = 9$. Hence $|X_{a+3}| \geq 5$ or $|T_{a+2}| \geq 3$ or $|T_{a+1}| \geq 3$. By Claims 4.2 and 4.3, this implies that $|X_{a+3}| + |T_{a+2}| + |T_{a+1}| = 9$, which forces $|X_{a+3} - (N(w_2) \cup N(w_4))| = 1$. In view of Claims 4.2 and 4.3, we also get $|S_{a+2}| = 2$. Write $X_{a+3} - (N(w_2) \cup N(w_4)) = \{y_0\}$ and $S_{a+2} = \{w_1, w_3\}$. If $w_1 w_3 \in E(G)$, then (2) holds. Thus $w_1 w_3 \notin E(G)$. If $y_0 \in N(w_1)$, then $d_G(u_2, y_0) \leq d_G(u_2, w_1) + 1 \leq 4$, and hence it follows from (4.3) and (4.4) that (3) holds with $x = u_2$. Thus $y_0 \notin N(w_1)$, which implies

$y_0 \in N(w_3)$. If $N(w_3) \cap (N(w_2) \cap X_{a+3}) \neq \phi$, then $d_G(u_2, y_0) \leq d_G(u_2, w_3) + 1 \leq 4$, and hence it follows from (4.3) and (4.4) that (3) holds with $x = u_2$. Thus $N(w_3) \cap (N(w_2) \cap X_{a+3}) = \phi$. Suppose that $N(w_3) \cap (N(w_4) \cap X_{a+3}) \neq \phi$. Take $u_4 \in N(w_4) \cap T_{a+1}$. Then $d_G(u_4, w_3) \leq 3$ and $d_G(u_4, y_0) \leq 4$. Since we have $d_G(u_4, y) \leq 5$ for every $y \in T_{a+2} \cup \{w_1\}$ by Claim 4.8(2), this together with (4.3) implies that (3) holds with $x = u_4$. Thus $N(w_3) \cap (N(w_4) \cap X_{a+3}) = \phi$. Consequently $|N(w_3)| = |S_{a+1} \cup \{y_0\}| = 2$, which contradicts the assumption that G is 3-connected. This concludes the discussion for the case where $|S_{a+1}| = 1$.

We now consider the case where $|S_{a+1}| = 2$. In this case, (A) holds by Claim 4.2, and hence $|S_{a+1}| = |S_{a+2}| = |T_{a+1}| = |T_{a+2}| = 2$ by Claim 4.2(1) and Claim 4.3(1). Recall that $w_1 \in S_{a+2}$ and $w_2 \in T_{a+2}$. Write $S_{a+2} = \{w_1, w_3\}, S_{a+1} = \{u_1, u_3\}, T_{a+2} = \{w_2, w_4\}, T_{a+1} = \{u_2, u_4\}$. We may assume $u_1w_1, u_2w_2 \in E(G)$. We show that $d_G(w_1, w_2) \geq 2$. Suppose that $d_G(w_1, w_2) = 1$. If $w_1w_3 \in E(G)$ or $N(w_1) \cap N(w_3) \neq \phi$, then $d_G(w_1, w_3) \leq 2$, and hence (2) holds. Thus $w_1w_3 \notin E(G)$ and $N(w_1) \cap N(w_3) = \phi$. This implies $N(w_1) \cap S_{a+1} = \{u_1\}$ and $N(w_3) \cap S_{a+1} = \{u_3\}$. Since $|N(u_1)| \geq 3$, $|N(u_3)| \geq 3$ and $|S_a| \leq 1$, this forces $u_1u_3 \in E(G)$. Hence

$$d_G(u, y) \leq 2 \text{ for every } u \in S_{a+1} \text{ and every } y \in S_{a+2}, \tag{4.5}$$

$$d_G(u, w_2) \leq 3 \text{ for every } u \in S_{a+1}, \text{ and} \tag{4.6}$$

$$d_G(w_1, w_3) \leq 3. \tag{4.7}$$

If $w_1w_4 \in E(G)$ or $w_3w_4 \in E(G)$, then $d_G(u_1, w_4) \leq 3$ by (4.5), and hence it follows from (4.5) and (4.6) that $d_G(u_1, y) \leq 3$ for every $y \in X_{a+2}$, which implies that (3) holds with $x = u_1$. Thus $w_1w_4, w_3w_4 \notin E(G)$. If $w_2w_3 \in E(G)$, then (2) holds. Thus $w_2w_3 \notin E(G)$. Since $|N(w_1)| \geq 3$ and $|N(w_3)| \geq 3$, we get $|N(w_1) \cap X_{a+3}| \geq 1$ and $|N(w_3) \cap X_{a+3}| \geq 2$. We consider $N(w_4)$. If $N(w_4) \cap (\{u_2, w_2\} \cup (N(w_1) \cap X_{a+3})) \neq \phi$, then $d_G(w_2, w_4) \leq 3$, and hence it follows from (4.7) that (1) holds. Thus $N(w_4) \cap (\{u_2, w_2\} \cup (N(w_1) \cap X_{a+3})) = \phi$. Since $|N(w_4)| \geq 3$ and $|X_{a+3}| \leq 4$, we now obtain $N(w_4) \cap (N(w_3) \cap X_{a+3}) \neq \phi$. Hence $d_G(u_3, w_4) \leq 3$. Consequently it follows from (4.5) and (4.6) that $d_G(u_3, y) \leq 3$ for every $y \in X_{a+2}$, which implies that (3) holds with $x = u_3$, a contradiction. Thus $d_G(w_1, w_2) = 2$, as desired. This implies that there is no edge between S_{a+2} and T_{a+2}. If $d_G(w_1, w_3) \leq 2$ and $d_G(w_2, w_4) \leq 2$, then (1) holds. Thus we have $d_G(w_1, w_3) \geq 3$ or $d_G(w_2, w_4) \geq 3$. Assume $d_G(w_1, w_3) \geq 3$. Then $w_1w_3 \notin E(G)$, $N(w_1) \cap S_{a+1} = \{u_1\}$ and $N(w_3) \cap S_{a+1} = \{u_3\}$. Since $|N(w_1)| \geq 3$ and $|N(w_3)| \geq 3$, this implies $|N(w_1) \cap X_{a+3}| \geq 2$ and $|N(w_3) \cap X_{a+3}| \geq 2$. Since $|X_{a+3}| \leq 4$, we obtain $X_{a+3} = (N(w_1) \cup N(w_3)) \cap X_{a+3}$. Note that by Claim 4.8(1), $d_G(u_1, y) \leq 3$ for every $y \in S_{a+2}$. Consequently $d_G(u_1, y') \leq 4$ for every $y' \in X_{a+3}$, and $d_G(u_1, y) \leq 3$ for every $y \in S_{a+2} \cup \{w_2\}$. Now if $N(w_4) \cap (\{u_2, w_2\} \cup X_{a+3}) \neq \phi$, then $d_G(u_1, w_4) \leq 5$, and hence (3) holds with $x = u_1$. Thus $N(w_4) \cap (\{u_2, w_2\} \cup X_{a+3}) = \phi$. Therefore $N(w_4) = \{u_4\}$, which contradicts the assumption that G is 3-connected. If

$d_G(w_2, w_4) \geq 3$, then we similarly obtain $N(w_3) = \{u_3\}$, which again contradicts the assumption that G is 3-connected, completing the proof of Claim 4.12.

Claim 4.13. *Suppose that* $c = r-2$. *Then* $d_G(w_2, y) \leq 5$ *for every* $y \in X_{r-2} \cup X_{r-1} \cup X_r$.

Proof. Suppose that the claim is false. By Claim 4.9(2), this means that

$$\text{there exists } v \in S_{r-2} \cup X_{r-1} \cup X_r \text{ such that } d_G(w_2, v) \geq 6. \tag{4.8}$$

Note that from the assumption that $c = r-2$, it follows that $b = r-1$, and hence $|X_{r-1}| = 3$ and $|X_r| \leq 2$ by the assumptions of the proposition. Suppose that $N(w_2) \cap X_{r-1} \neq \phi$. Then by Lemma 6(i) (and Lemma 3), $d_G(w_2, y) \leq 4$ for every $y \in X_{r-1} \cup X_r$, and hence it follows from (4.8) that there exists $w_3 \in S_{r-2}$ such that $d_G(w_2, w_3) \geq 6$. By Claim 4.9(1), this implies $d_G(w_1, w_2) = 2$, and hence there is no edge between S_{r-2} and T_{r-2}. Since $N(w_2) \cap X_{r-1} \neq \phi$, we also get $N(w_3) \cap X_{r-1} = \phi$ by Lemma 6(i). Furthermore since $d_G(w_1, w_3) \geq d_G(w_2, w_3) - d_G(w_2, w_1) \geq 4$, we have $w_1 w_3 \notin E(G)$ and $N(w_1) \cap N(w_3) = \phi$. Since $N(w_1) \cap S_{r-3} \neq \phi$ and $|N(w_3)| \geq 3$, we now obtain $|S_{r-2}| + |S_{r-3}| \geq |\{w_1\} \cup (N(w_1) \cap S_{r-3})| + |\{w_3\} \cup N(w_3)| \geq 2 + 4 = 6$, which contradicts Claim 4.2.

Thus $N(w_2) \cap X_{r-1} = \phi$. This implies $d_G(w_1, w_2) = 1$. In view of Claim 4.9(1), it follows from (4.8) that there exists $v \in X_{r-1} \cup X_r$ such that $d_G(w_2, v) \geq 6$. If $N(w_1) \cap X_{r-1} \neq \phi$, it follows from Lemma 6(i) that $d_G(w_2, y) \leq 5$ for every $y \in X_{r-1} \cup X_r$, a contradiction. Thus $N(w_1) \cap X_{r-1} = \phi$. Hence by Lemma 1, $|X_{r-2} - \{w_1, w_2\}| \geq 3$, which implies that (B) holds and $|X_{r-2}| = 5$. By Claim 4.2(2) and Claim 4.3(2), we have $|T_{r-3}| = 2$. Write $X_{r-2} = \{w_1, w_2, w_3, w_4, w_5\}, X_{r-1} = \{x_1, x_2, x_3\}$. By Lemma 4, we may assume that $w_3 x_1$, $w_4 x_2$, $w_5 x_3 \in E(G)$. At the cost of relabeling, we may also assume $w_3 \in S_{r-2}$ and $w_4 \in T_{r-2}$. If $N(w_2) \cap \{w_3, w_4, w_5\} \neq \phi$, it follows from Lemma 6(i) that $d_G(w_2, y) \leq 5$ for every $y \in X_{r-1} \cup X_r$. Thus $N(w_2) \cap \{w_3, w_4, w_5\} = \phi$. Since $|N(w_2)| \geq 3$ and $|T_{r-3}| = 2$, this forces $N(w_2) = \{w_1\} \cup T_{r-3}$, which implies that $d_G(w_2, w) \leq 2$ for every $w \in T_{r-2}$. If $w_5 \in T_{r-2}$, then $d_G(w_2, x) \leq 3$ for each $x \in \{x_2, x_3\}$, and hence we see that $d_G(w_2, y) \leq 5$ for every $y \in X_{r-1} \cup X_r$ by applying Lemma 6(ii) with $B = \{x_2, x_3\}$. Thus $w_5 \in S_{r-2}$, and hence $S_{r-2} = \{w_1, w_3, w_5\}$ and $T_{r-2} = \{w_2, w_4\}$. If $d_G(w_1, w_3) \leq 2$ and $d_G(w_1, w_5) \leq 2$, then $d_G(w_2, w) \leq 3$ for every $w \in \{w_3, w_4, w_5\}$, and hence $d_G(w_2, y) \leq 5$ for every $y \in X_{r-1} \cup X_r$. Thus we have $d_G(w_1, w_3) \geq 3$ or $d_G(w_1, w_5) \geq 3$. Since $|S_{r-3}| \leq 2$ by Claim 4.2(2), this implies that $|S_{r-3}| = 2$ and $|N(w_1) \cap S_{r-3}| = 1$. Write $S_{r-3} = \{u_1, u_3\}$ so that $N(w_1) \cap S_{r-3} = \{u_1\}$. If $(N(u_1) \cup N(w_1)) \cap \{w_3, w_5\} = \phi$, then $G - \{u_3, w_4\}$ is disconnected, which contradicts the assumption that G is 3-connected. Thus $(N(u_1) \cup N(w_1)) \cap \{w_3, w_5\} \neq \phi$. We may assume $w_3 \in N(u_1) \cup N(w_1)$. Then $d_G(w_2, x) \leq 4$ for every $x \in \{x_1, x_2\}$, and hence we see that $d_G(w_2, y) \leq 5$ for every $y \in X_r$ by applying Lemma 6(ii) with $B = \{x_1, x_2\}$. Thus $d_G(w_2, x_3) \geq 6$. Since $d_G(w_2, x_2) \leq 3$ and since $N(x_2) \cap X_r \neq \phi$ by Lemma 3, this implies $N(x_3) \not\supseteq X_r$, and hence $|N(x_3) \cap X_r| \leq 1$. We also obtain

$N(x_3) \cap \{w_3, w_4, x_1, x_2\} = \phi$. Consequently $|N(x_3)| = |\{w_5\} \cup (N(x_3) \cap X_r)| \le 2$. But this contradicts the assumption that G is 3-connected.

5. Proof of Proposition 1

We continue with the notation of the preceding section, and complete the proof of Proposition 1.

We divide the rest of the proof for Case 2 into seven cases. In each case, we derive a contradiction by showing that there exists $u \in V(G)$ such that $d_G(u,v) < r$ for every $v \in V(G)$.

Case 2.1. $a+3 \le c \le r-3$ and $d_G(w_1,w_2) = 1$.

Subcase 2.1.1. $a \ge 3$.

Let u be a vertex in X_6 which is on a shortest $z - w_2$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le c-4$, $d_G(u,v) \le d_G(u,z) + d_G(z,v) = 6+i \le 6+(c-4) \le 6+\{(r-3)-4\} = r-1 < r$. If $c-3 \le i \le c-1$, then since $c-3 \ge a$ by the assumption of Case 2.1, it follows from (1) and (2) of Claim 4.10 that $d_G(w_2,v) \le c-i+4 \le c-(c-3)+4 = 7$, and hence $d_G(u,v) \le d_G(u,w_2) + d_G(w_2,v) \le (c-6)+7 = c+1 \le r-2 < r$. If $c \le i \le r$, then letting w be a vertex in X_c which is on a shortest $z - v$ path, we get $d_G(w_2,w) \le 5$ by Claim 4.9, and hence $d_G(u,v) \le d_G(u,w_2) + d_G(w_2,w) + d_G(w,v) \le (c-6)+5+(i-c) \le (c-6)+5+(r-c) = r-1 < r$.

Subcase 2.1.2. $a = 2$.

We have $|X_r| = 1$ by the assumptions of the proposition. Write $X_r = \{v_r\}$. Let w' be a vertex in X_c which is on a shortest $z - v_r$ path. By symmetry, we may assume $w' \in T_c$ (in this subcase, we do not make use of the assumption that $|S_a| = 1$ and $|T_a| = 2$). Then $d_G(w_2,w') \le 4$ by Claim 4.9(2). Let u be a vertex in X_5 which is on a shortest $z - w_2$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le c-3$, $d_G(u,v) \le d_G(u,z) + d_G(z,v) = 5+i \le 5+(c-3) \le 5+\{(r-3)-3\} = r-1 < r$. If $c-2 \le i \le c-1$, then $d_G(w_2,v) \le c-i+4 \le 6$ by (1) and (2) of Claim 4.10, and hence $d_G(u,v) \le d_G(u,w_2) + d_G(w_2,v) \le (c-5)+6 = c+1 \le r-2 < r$. If $c \le i \le r-1$, then letting w be a vertex in X_c which is on a shortest $z-v$ path, we get $d_G(w_2,w) \le 5$ by Claim 4.9, and hence $d_G(u,v) \le d_G(u,w_2) + d_G(w_2,w) + d_G(w,v) \le (c-5)+5+(i-c) \le (c-5)+5+\{(r-1)-c\} = r-1 < r$. If $i = r$, then $v = v_r$, and hence $d_G(u,v) = d_G(u,v_r) \le d_G(u,w_2)+d_G(w_2,w')+d_G(w',v_r) \le (c-5)+4+(r-c) = r-1 < r$.

Case 2.2. $a+3 \le c \le r-3$ and $d_G(w_1,w_2) = 2$ and (1) or (2) of Claim 4.11 holds.

We may assume (2) of Claim 4.11 holds. Let u be a vertex in X_5 which is on a shortest $z - w_2$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le c-3$, $d_G(u,v) \le d_G(u,z) + d_G(z,v) = 5+i \le 5+(c-3) \le 5+\{(r-3)-3\} = r-1 < r$. If $c-2 \le i \le c-1$, then $d_G(w_2,v) \le c-i+4 \le 6$ by (1) and (2) of Claim 4.10, and hence $d_G(u,v) \le d_G(u,w_2) + d_G(w_2,v) \le (c-5)+6 = c+1 \le r-2 < r$. If $c \le i \le r$, then letting w be a vertex in X_c which is on a shortest $z-v$ path, we get $d_G(w_2,w) \le 4$ by

Claim 4.11(2), and hence $d_G(u,v) \le d_G(u,w_2)+d_G(w_2,w)+d_G(w,v) \le (c-5)+4+(i-c) \le (c-5)+4+(r-c) = r-1 < r$.

Case 2.3. $a+3 \le c \le r-3$ and $d_G(w_1,w_2)=2$ and Claim 4.11(3) holds.

Let x be as in Claim 4.11(3), and let u be a vertex in X_4 which is on a shortest $z-x$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le c-2$, $d_G(u,v) \le d_G(u,z)+d_G(z,v) = 4+i \le 4+(c-2) \le 4+\{(r-3)-2\} = r-1 < r$. If $c-1 \le i \le c$, then $d_G(x,v) \le 7$ by Claim 4.11(3), and hence $d_G(u,v) \le d_G(u,x)+d_G(x,v) \le \{(c-1)-4\}+7 \le \{\{(r-3)-1\}-4\}+7 = r-1 < r$. If $c+1 \le i \le r$, then letting y' be a vertex in X_{c+1} which is on a shortest $z-v$ path, we get $d_G(x,y') \le 4$ by Claim 4.11(3), and hence $d_G(u,v) \le d_G(u,x)+d_G(x,y')+d_G(y',v) \le \{(c-1)-4\}+4+\{i-(c+1)\} \le \{(c-1)-4\}+4+\{r-(c+1)\} = r-2 < r$.

Case 2.4. $a+2 = c \le r-3$ and Claim 4.12(1) holds.

Note that we have $a \le r-5$.

Subcase 2.4.1. $a \ge 3$.

Let u be a vertex in X_5 which is on a shortest $z-w_2$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le a-1$, $d_G(u,v) \le d_G(u,z)+d_G(z,v) = 5+i \le 5+(a-1) \le 5+\{(r-5)-1\} = r-1 < r$. If $a \le i \le a+1$, then $d_G(w_2,v) \le (a+2)-i+4 \le 6$ by (1) and (2) of Claim 4.10, and hence $d_G(u,v) \le d_G(u,w_2)+d_G(w_2,v) \le \{(a+2)-5\}+6 = a+3 \le (r-5)+3 = r-2 < r$. If $a+2 \le i \le r$, then letting w be a vertex in X_{a+2} which is on a shortest $z-v$ path, we have $d_G(w_2,w) \le 4$ by Claim 4.12(1), and hence $d_G(u,v) \le d_G(u,w_2)+d_G(w_2,w)+d_G(w,v) \le \{(a+2)-5\}+4+\{i-(a+2)\} \le \{(a+2)-5\}+4+\{r-(a+2)\} = r-1 < r$.

Subcase 2.4.2. $a=2$.

We have $|X_r| = 1$. Write $X_r = \{v_r\}$. Note that $c = a+2 = 4$ and $r \ge c+3 = 7$. Let w' be a vertex in X_4 which is on a shortest $z-v_r$ path. We may assume $w' \in T_4$. Then $d_G(w_2,w') \le 3$ by Claim 4.12(1). Let $u = w_2$. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le 2$, $d_G(u,v) \le d_G(u,z)+d_G(z,v) = 4+i \le 6 \le r-1 < r$. If $i=3$, then $d_G(w_2,v) \le 1+4 = 5$ by (1) and (2) of Claim 4.10, and hence $d_G(u,v) = d_G(w_2,v) \le 5 \le r-2 < r$. If $4 \le i \le r-1$, then letting w be a vertex in X_4 which is on a shortest $z-v$ path, we get $d_G(w_2,w) \le 4$ by Claim 4.12(1), and hence $d_G(u,v) \le d_G(w_2,w)+d_G(w,v) \le 4+(i-4) \le 4+\{(r-1)-4\} = r-1 < r$. If $i=r$, then $v=v_r$, and hence $d_G(u,v) = d_G(w_2,v_r) \le d_G(w_2,w')+d_G(w',v_r) \le 3+(r-4) = r-1 < r$.

Case 2.5. $a+2 = c \le r-3$ and Claim 4.12(2) holds.

We have $a \le r-5$.

Subcase 2.5.1. $a \ge 3$.

Let u be a vertex in X_5 which is on a shortest $z-w_2$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le a-1$, $d_G(u,v) \le d_G(u,z)+d_G(z,v) = 5+i \le 5+(a-1) \le 5+\{(r-5)-1\} = r-1 < r$. If $a \le i \le a+1$, then $d_G(w_2,v) \le 2+4 = 6$ by (1) and (2) of Claim 4.10, and hence $d_G(u,v) \le d_G(u,w_2)+d_G(w_2,v) \le \{(a+2)-5\}+6 \le$

$\{\{(r-5)+2\}-5\}+6=r-2<r$. If $a+2\le i\le r$, then letting w be a vertex in X_{a+2} which is on a shortest $z-v$ path, we get $d_G(w_2,w)\le 4$ by Claim 4.9(2) and Claim 4.12(2), and hence $d_G(u,v)\le d_G(u,w_2)+d_G(w_2,w)+d_G(w,v)\le \{(a+2)-5\}+4+\{i-(a+2)\}\le \{(a+2)-5\}+4+\{r-(a+2)\}=r-1<r$.

Subcase 2.5.2. $a=2$.

We have $|X_r|=1$. Write $X_r=\{v_r\}$. Note that $c=4$ and $r\ge 7$. Let w' be a vertex in X_4 which is on a shortest $z-v_r$ path.

Subcase 2.5.2.1. $w'\in S_4$.

We have $d_G(w_2,w')\le 3$ by Claim 4.12(2). Let $u=w_2$. Take $v\in V(G)$, and let $v\in X_i$. If $0\le i\le 2$, $d_G(u,v)\le d_G(u,z)+d_G(z,v)=4+i\le 6\le r-1<r$. If $i=3$, then by (1) and (2) of Claim 4.10, $d_G(u,v)\le 1+4=5\le r-2<r$. If $4\le i\le r-1$, then letting w be a vertex in X_4 which is on a shortest $z-v$ path, we get $d_G(w_2,w)\le 4$ by Claim 4.9(2) and Claim 4.12(2), and hence $d_G(u,v)\le d_G(w_2,w)+d_G(w,v)\le 4+(i-4)\le 4+\{(r-1)-4\}=r-1<r$. If $i=r$, then $v=v_r$, and hence $d_G(u,v)=d_G(w_2,v_r)\le d_G(w_2,w')+d_G(w',v_r)\le 3+(r-4)=r-1<r$.

Subcase 2.5.2.2. $w'\in T_4$.

Let u be a vertex in X_3 which is on a shortest $z-w_2$ path. Then $d_G(u,w')\le 3$ by Claim 4.8(2). Take $v\in V(G)$, and let $v\in X_i$. If $0\le i\le 3$, $d_G(u,v)\le d_G(u,z)+d_G(z,v)=3+i\le 6\le r-1<r$. If $i=r$, then $v=v_r$, and hence $d_G(u,v)=d_G(u,v_r)\le d_G(u,w')+d_G(w',v_r)\le 3+(r-4)=r-1<r$. Thus we may assume $4\le i\le r-1$. Let w be a vertex in X_4 which is on a shortest $z-v$ path. If $w\in S_4$, then $d_G(w_2,w)\le 3$ by Claim 4.12(2), and hence $d_G(u,v)\le d_G(u,w_2)+d_G(w_2,w)+d_G(w,v)\le 1+3+(i-4)\le 1+3+\{(r-1)-4\}=r-1<r$; if $w\in T_4$, then $d_G(u,w)\le 3$ by Claim 4.8(2), and hence $d_G(u,v)\le d_G(u,w)+d_G(w,v)\le 3+(i-4)\le 3+\{(r-1)-4\}=r-2<r$.

Case 2.6. $a+2=c\le r-3$ and Claim 4.12(3) holds.

We have $a\le r-5$. Let x be as in Claim 4.12(3), and let u be a vertex in X_3 which is on a shortest $z-x$ path. Take $v\in V(G)$, and let $v\in X_i$. If $0\le i\le a+1$, $d_G(u,v)\le d_G(u,z)+d_G(z,v)=3+i\le 3+(a+1)\le 3+\{(r-5)+1\}=r-1<r$. If $i=a+2$, then $d_G(x,v)\le 5$ by Claim 4.12(3), and hence $d_G(u,v)\le d_G(u,x)+d_G(x,v)\le \{(a+1)-3\}+5\le \{\{(r-5)+1\}-3\}+5=r-2<r$. If $a+3\le i\le r$, then letting y' be a vertex in X_{a+3} which is on a shortest $z-v$ path, we get $d_G(x,y')\le 4$ by Claim 4.12(3), and hence $d_G(u,v)\le d_G(u,x)+d_G(x,y')+d_G(y',v)\le \{(a+1)-3\}+4+\{i-(a+3)\}\le \{(a+1)-3\}+4+\{r-(a+3)\}=r-1<r$.

Case 2.7. $c=r-2$.

Since $a+2\le c$, we have $a\le r-4$. Let u be a vertex in X_{r-a} which is on a shortest $z-w_2$ path. Take $v\in V(G)$, and let $v\in X_i$. If $0\le i\le a-1$, $d_G(u,v)\le d_G(u,z)+d_G(z,v)=(r-a)+i\le (r-a)+(a-1)=r-1<r$. If $r-2\le i\le r$, then $d_G(w_2,v)\le 5$ by Claim 4.13, and hence $d_G(u,v)\le d_G(u,w_2)+d_G(w_2,v)\le \{(r-2)-(r-a)\}+5\le \{(r-2)-\{r-(r-4)\}\}+5=r-1<r$. Thus we may assume $a\le i\le r-3$. If $v\in T_i$,

then $d_G(w_2, v) \le (r-2) - i + 2 \le (r-2) - a + 2 = r - a$ by Claim 4.10(2); if $v \in S_i$ and $a+1 \le i \le r-3$, then $d_G(w_2, v) \le (r-2) - i + 4 \le (r-2) - (a+1) + 4 = r - a + 1$ by Claim 4.10(1); if $v \in S_i$ and $i = a$, then $d_G(w_2, v) \le (r-2) - a + 2 = r - a$ by Claim 4.10(3). Hence $d_G(w_2, v) \le r - a + 1$. Therefore $d_G(u, v) \le d_G(u, w_2) + d_G(w_2, v) \le \{(r-2) - (r-a)\} + (r - a + 1) = r - 1 < r$.

This concludes the discussion for the case where $\mathscr{G}$ is disconnected, and completes the proof of Proposition 1.

6. Proof of Proposition 2

We continue with the notation of Section 2. Throughout the rest of this paper, we assume that z is chosen so that $|X_r|$ is minimum, so that $|X_{r-1}|$ is maximum, subject to the condition that $|X_r|$ is minimum. Under this assumption, we prove the following proposition.

Proposition 2. *Let a, b be integers with $a+2 \le b$, and suppose that $|X_a| = |X_{a+1}| = 3$ and $|X_i| > 3$ for each $a+2 \le i \le b-1$.*

(1) *Suppose that $r \ge 6$, $3 \le a \le r-3$, $b = r-1$ and $|X_{r-1}| = |X_r| = 3$. Then $\sum_{i=a}^{b-1} |X_i| \ge 4(b-a) - 1$.*

(2) *Suppose that $r \ge 5$, $a \ge 3$, $b = r$ and $|X_r| = 2$. Then $\sum_{i=a}^{b-1} |X_i| \ge 4(b-a) - 1$.*

(3) *Suppose that $r \ge 4$, $a \ge 2$, $b = r$ and $|X_r| = 1$. Then $\sum_{i=a}^{b-1} |X_i| \ge 4(b-a) - 1$.*

Proof. We prove (1), (2) and (3) simultaneously. By way of contradiction, suppose that $\sum_{i=a}^{b-1} |X_i| < 4(b-a) - 1$. Then $|X_i| = 4$ for each $a+2 \le i \le b-1$. Note that this together with the assumptions of the proposition implies that we have $|X_{r-1}| \le 4$ and $|X_{r-1}| + |X_r| \le 6$.

Claim 6.1. *Suppose that $a = r-2$. Then there exists $x \in X_{r-2}$ such that $d_G(x, y) \le 3$ for every $y \in X_{r-2} \cup X_{r-1} \cup X_r$.*

Proof. We have $b = r$, $|X_{r-2}| = |X_{r-1}| = 3$ and $|X_r| \le 2$ by the assumptions of the proposition. Write $X_{r-2} = \{x_0, x_1, x_2\}$, $X_{r-1} = \{y_0, y_1, y_2\}$. By Lemma 4 (and Lemma 3), we may assume that $x_i y_i \in E(G)$ for each $0 \le i \le 2$. We first show that $|X_r| = 1$. Suppose that $|X_r| = 2$. Write $X_r = \{z_1, z_2\}$. In view of Lemma 6(iii), we may assume $y_0 z_i \in E(G)$ for each $1 \le i \le 2$. Let u be a vertex in X_1 which is on a shortest $z - x_0$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le r-3$, $d_G(u, v) \le d_G(u, z) + d_G(z, v) = 1 + i \le 1 + (r-3) = r-2$. If $v = x_1$ or x_2, $d_G(u, v) \le d_G(u, z) + d_G(z, v) = 1 + (r-2) = r-1$. If $v = y_1$ or y_2, $d_G(u, v) \le d_G(u, z) + d_G(z, v) = 1 + (r-1) = r$. If $v = x_0$, $d_G(u, v) = (r-2) - 1 = r-3$. If $v = y_0$, $d_G(u, v) = (r-1) - 1 = r-2$. If $v = z_1$ or z_2, $d_G(u, v) = r-1$. By the assumption that

z is chosen so that $|X_r|$ is minimum, this implies $d_G(u,y_1)=d_G(u,y_2)=r$. Now by the assumption that z is chosen so that $|X_{r-1}|$ is maximum, we have $d_G(u,x_1)\le r-2$ or $d_G(u,x_2)\le r-2$. But since $x_1y_1, x_2y_2\in E(G)$, this implies $d_G(u,y_1)\le r-1$ or $d_G(u,y_2)\le r-1$, which contradicts the earlier assertion that $d_G(u,y_1)=d_G(u,y_2)=r$.

Thus $|X_r|=1$. Write $X_r=\{z_0\}$. Then by Lemma 3, $y_iz_0\in E(G)$ for each $0\le i\le 2$. If $d_G(u,v)\le 3$ for every $u,v\in X_{r-2}$, the desired conclusion holds with any $x\in X_{r-2}$. Thus we may assume $d_G(x_1,x_2)\ge 4$. Then for each $1\le i\le 2$, we have $N(y_i)\cap\{x_0,y_0\}\ne\phi$ because $|N(y_i)|\ge 3$. Therefore the desired conclusion holds with $x=x_0$.

Claim 6.2. $a\le r-3$.

Proof. Suppose that $a=r-2$. We derive a contradiction by showing that there exists $u\in V(G)$ such that $d_G(u,v)<r$ for every $v\in V(G)$. Let x be as in Claim 6.1, and let u be a vertex in X_2 which is on a shortest $z-x$ path. Take $v\in V(G)$, and let $v\in X_i$. If $0\le i\le r-3$, $d_G(u,v)\le d_G(u,z)+d_G(z,v)=2+i\le 2+(r-3)=r-1<r$. If $r-2\le i\le r$, then $d_G(x,v)\le 3$ by Claim 6.1, and hence $d_G(u,v)\le d_G(u,x)+d_G(x,v)\le \{(r-2)-2\}+3=r-1<r$.

We now introduce a graph structure $\mathscr{G}$ on X_{a+1} by joining v and w if and only if $d_G(v,w)\le 2$ and $v\ne w$. Let α denote the independence number of $\mathscr{G}$. We can prove the following claim by arguing as in the proof of Claim 3.1.

Claim 6.3. $\alpha\le 2$.

We consider two cases separately according as $\mathscr{G}$ is connected or not.

Case 1. $\mathscr{G}$ is connected.

We derive a contradiction by showing that there exists $u\in V(G)$ such that $d_G(u,v)<r$ for every $v\in V(G)$. Recall that $|X_{a+1}|=3$. Thus by the definition of $\mathscr{G}$, there exists $w_0\in X_{a+1}$ such that $d_G(w_0,w)\le 2$ for every $w\in X_{a+1}$. Let u be a vertex in X_3 which is on a shortest $z-w_0$ path. Take $v\in V(G)$, and let $v\in X_i$. Recall that we have $a\le r-3$ by Claim 6.2. If $0\le i\le a-1$, $d_G(u,v)\le d_G(u,z)+d_G(z,v)=3+i\le 3+(a-1)\le 3+\{(r-3)-1\}=r-1<r$. If $a+1\le i\le r$, then letting w be a vertex in X_{a+1} which is on a shortest $z-v$ path, we get $d_G(w_0,w)\le 2$, and hence $d_G(u,v)\le d_G(u,w_0)+d_G(w_0,w)+d_G(w,v)\le\{(a+1)-3\}+2+\{i-(a+1)\}=i-1\le r-1<r$. If $i=a$, then taking $w\in N(v)\cap X_{a+1}$ (see Lemma 3), we obtain $d_G(u,v)\le d_G(u,w_0)+d_G(w_0,w)+d_G(w,v)\le\{(a+1)-3\}+2+1\le\{\{(r-3)+1\}-3\}+2+1=r-2<r$.

Case 2. $\mathscr{G}$ is disconnected.

By Claim 6.3, $\mathscr{G}$ consists of two components. Let S_{a+1} and T_{a+1} be the vertex sets of the components of $\mathscr{G}$. For j with $a\le j\le r$, set

$$S_j=(\cup_{v\in S_{a+1}}N_{|a+1-j|}(v))\cap X_j\,,$$

$$T_j = (\cup_{w \in T_{a+1}} N_{|a+1-j|}(w)) \cap X_j \,.$$

As in the proof of Proposition 1, we have $S_j \cup T_j = X_j$ for each $a \le j \le r$, and $S_j \cap T_j = \phi$ for each $a \le j \le a+2$. Also we may assume $|S_a| = 1$ and $|T_a| = 2$. We can prove the following claim by arguing as in the proof of Claim 4.1.

Claim 6.4. *Let $a+1 \le i \le r$, and suppose that for each h with $a+1 \le h \le i-1, d_G(v,w) \ge 3$ for every $v \in S_h$ and every $w \in T_h$. Then the following hold.*

(1) (a) $|S_i| \ge 1$. (b) *If* $i \ge a+2, |S_i| \ge 2$.

(2) $|T_i| \ge 2$.

We define an integer c as follows. Set

$Q := \{i | a+1 \le i \le r-1$, there exists $w_1 \in S_i$ and there exists $w_2 \in T_i$ such that $d_G(w_1, w_2) \le 2\}$.

We have $Q \ne \phi$ because if $Q = \phi$, then $|X_r| = |S_r| + |T_r| \ge 2+2 = 4$ by Claim 6.4, which contradicts the assumption that $|X_r| \le 3$. Now set

$$c = \min Q \,.$$

Then $a+2 \le c \le \max Q \le r-1$ by the definition of S_{a+1} and T_{a+1}.

The following two claims immediately follow from Claim 6.4.

Claim 6.5. $|S_{a+1}| = 1$, *and* $|S_i| = 2$ *for each* $a+2 \le i \le c$.

Claim 6.6. $|T_i| = 2$ *for each* $a+1 \le i \le c$.

We can prove the following six claims by arguing as in the proofs of Claims 4.4 through 4.9.

Claim 6.7. $|S_{i-1} \cup S_i \cup S_{i+1}| \le 6$ *for each* $a+1 \le i \le c-1$.

Claim 6.8. $|T_{i-1} \cup T_i \cup T_{i+1}| = 6$ *for each* $a+1 \le i \le c-1$.

Claim 6.9. *Let $a+1 \le i \le c-1$. Then the following hold.*

(1) $d_G(v, v') \le 2$ *for every* $v, v' \in S_i$.

(2) $d_G(w, w') \le 2$ *for every* $w, w' \in T_i$.

Claim 6.10. *Let $a \le i < j \le c$. Then the following hold.*

(1) $d_G(w, v) \le j - i + 2$ *for every* $w \in S_j$ *and every* $v \in S_i$.

(2) $d_G(w,v) \le j-i+2$ *for every* $w \in T_j$ *and every* $v \in T_i$ *.*

Claim 6.11.

(1) $d_G(u,v) \le 3$ *for every* $u \in S_{c-1}$ *and every* $v \in S_c$ *.*

(2) $d_G(u,v) \le 3$ *for every* $u \in T_{c-1}$ *and every* $v \in T_c$ *.*

Claim 6.12.

(1) $d_G(w,w') \le 4$ *for every* $w, w' \in S_c$ *.*

(2) $d_G(w,w') \le 4$ *for every* $w, w' \in T_c$ *.*

Claim 6.13. *Let* $a \le i \le c-1$ *. Then the following hold.*

(1) $N(y) \cap S_{i+1} \ne \phi$ *for every* $y \in S_i$ *.*

(2) $N(y) \cap T_{i+1} \ne \phi$ *for every* $y \in T_i$ *.*

Proof. If $a \le i \le a+1$, then $|X_i| = 3$, and hence (1) and (2) immediately follow from Lemma 3. Thus we may assume $a+2 \le i \le c-1$. Then $G-(S_a \cup \{y \in S_i | N(y) \cap S_{i+1} \ne \phi\})$ and $G-(S_a \cup \{y \in T_i | N(y) \cap T_{i+1} \ne \phi\})$ are disconnected, and hence (1) and (2) follow from Claims 6.5 and 6.6 and the assumption that G is 3-connected.

Throughout the rest of the discussion for Case 2, we fix two vertices w_{c1}, w_{c2} with $w_{c1} \in S_c$ and $w_{c2} \in T_c$ such that $d_G(w_{c1}, w_{c2}) \le 2$. If possible, we choose w_{c1} and w_{c2} so that $d_G(w_{c1}, w_{c2}) = 1$. Write $S_a = \{w_{a1}\}$ and $S_{a+1} = \{w_{a+1,1}\}$, and write $S_i = \{w_{i1}, w_{i3}\}$ for $a+2 \le i \le c$. Then $w_{a1}w_{a+1,1}, w_{a+1,1}w_{a+2,1}, w_{a+1,1}w_{a+2,3} \in E(G)$. In view of Claim 6.13, we may assume $w_{i1}w_{i+1,1}, w_{i3}w_{i+1,3} \in E(G)$ for each $a+2 \le i \le c-1$. Similarly write $T_i = \{w_{i2}, w_{i4}\}$ for $a \le i \le c$ so that we have $w_{i2}w_{i+1,2}, w_{i4}w_{i+1,4} \in E(G)$ for each $a \le i \le c-1$.

Claim 6.14. *Suppose that* $d_G(w_{c1}, w_{c2}) = 1$ *. Let* $a \le i \le c-1$ *, and let* $v \in X_i$ *.*

(1) *If* $v \in S_i$ *, then* $d_G(w_{c-1,2}, v) \le c-i+4$ *.*

(2) *If* $v \in T_i$ *, then* $d_G(w_{c-1,2}, v) \le c-i+1$ *.*

(3) *If* $i = a$ *or* $a+1$ *and* $v \in S_i$ *, then* $d_G(w_{c-1,2}, v) \le c-i+2$ *.*

Proof. Statement (2) follows from Claim 6.9(2) or 6.10(2) according as $i = c-1$ or $i < c-1$. Note that $d_G(w_{c-1,2}, w_{c1}) \le d_G(w_{c-1,2}, w_{c2}) + d_G(w_{c2}, w_{c1}) = 1+1 = 2$ by the assumption of the claim. Hence (1) follows from Claim 6.10(1). To prove (3), let $v \in S_a \cup S_{a+1}$. Then since $|S_a| = |S_{a+1}| = 1$, it follows from the definition of S_j that v is on a shortest $z - w_{c1}$ path. Thus (3) immediately follows from the earlier assertion that $d_G(w_{c-1,2}, w_{c1}) \le 2$.

We can prove the following two claims by arguing as in the proofs of Claims 4.11 and 4.12.

Claim 6.15. *Suppose that* $a+3 \leq c \leq r-3$ *and* $d_G(w_{c1}, w_{c2}) = 2$. *Then one of the following holds:*

(1) $d_G(w_{c1}, y) \leq 4$ *for every* $y \in X_c$ *; or*

(2) $d_G(w_{c2}, y) \leq 4$ *for every* $y \in X_c$ *;or*

(3) *there exists* $x \in X_{c-1}$ *such that* $d_G(x, y) \leq 7$ *for* $y \in X_{c-1} \cup X_c$ *and* $d_G(x, y') \leq 4$ *for every* $y' \in X_{c+1}$.

Claim 6.16. *Suppose that* $a+2 = c \leq r-3$. *Then one of the following holds:*

(1) $d_G(w_{c1}, w) \leq 3, d_G(w_{c2}, w) \leq 4$ *for every* $w \in S_{a+2}$ *and* $d_G(w_{c1}, w') \leq 4, d_G(w_{c2}, w') \leq 3$ *for every* $w' \in T_{a+2}$ *; or*

(2) $d_G(w_{c2}, w) \leq 3$ *for every* $w \in S_{a+2}$ *; or*

(3) *there exists* $x \in X_{a+1}$ *such that* $d_G(x, y) \leq 5$ *for every* $y \in X_{a+2}$ *and* $d_G(x, y') \leq 4$ *for every* $y' \in X_{a+3}$.

Claim 6.17. *Suppose that* $c = r-2$. *Then* $d_G(w_{r-2,2}, y) \leq 5$ *for every* $y \in X_{r-2} \cup X_{r-1} \cup X_r$.

Proof. We have $|X_{r-2}| = 4$. Also recall that $|X_{r-1}| \leq 4$ and $|X_{r-1}| + |X_r| \leq 6$ (see the paragraph preceding the statement of Claim 6.1). We first show that

$$d_G(w, y) \leq 3 \text{ for every } w \in X_{r-1} \text{ and every } y \in X_r. \tag{6.1}$$

Take $w \in X_{r-1}$ and $y \in X_r$. Then $(\{w\} \cup N(w)) \cup (\{y\} \cup N(y) \cup N_2(y)) \subseteq X_{r-2} \cup X_{r-1} \cup X_r$. Since G is 3-connected, we have $|\{w\} \cup N(w)| \geq 1+3 = 4$ and $|\{y\} \cup N(y) \cup N_2(y)| \geq 1+3+3 = 7$. Since $|X_{r-2}| = 4$ and $|X_{r-1}| + |X_r| \leq 6$, we get $(\{w\} \cup N(w)) \cap (\{y\} \cup N(y) \cup N_2(y)) \neq \phi$, which implies $d_G(w, y) \leq 3$, as desired. Now suppose that the claim is false, and fix $v \in X_{r-2} \cup X_{r-1} \cup X_r$ such that $d_G(w_{r-2,2}, v) \geq 6$. By Claim 6.12(2), we have $v \in S_{r-2} \cup X_{r-1} \cup X_r$.

Assume first that $N(w_{r-2,2}) \cap X_{r-1} \neq \phi$. Fix $u_1 \in N(w_{r-2,2}) \cap X_{r-1}$. By (6.1), $d_G(w_{r-2,2}, y) \leq 1 + d_G(u_1, y) \leq 4$ for every $y \in X_r$. This implies $v \notin X_r$ and $N(v) \cap X_r = \phi$. Suppose that $v \in X_{r-1}$. Then since $N(v) \cap X_r = \phi$, we get $|X_{r-1}| = 4$ by Lemma 1, and hence it follows from Lemma 6(i) that $d_G(u_1, u) \leq 3$ for every $u \in X_{r-1} - \{v\}$, which implies $N(v) \cap X_{r-1} = \phi$. Since $w_{r-2,4} \notin N(v)$ by Claim 6.12(2) and since we also have $w_{r-2,1}, w_{r-2,2} \notin N(v)$, this forces $N(v) \subseteq \{w_{r-2,3}\}$, which contradicts the assumption that G is 3-connected. Thus $v \notin X_{r-1}$. Consequently $v \in S_{r-2}$, which means $v = w_{r-2,3}$. In view of Claim 6.12(1), this implies $d_G(w_{r-2,1}, w_{r-2,2}) = 2$, and hence it follows from our choice of $w_{r-2,1}$ and $w_{r-2,2}$ that there is no edge between S_{r-2} and T_{r-2}. We also have $w_{r-2,1}w_{r-2,3} \notin E(G)$ and $N(w_{r-2,1}) \cap N(w_{r-2,3}) = \phi$, which implies $|S_{r-3}| = 2$, $N(w_{r-2,1}) \cap S_{r-3} = \{w_{r-3,1}\}$ and $N(w_{r-2,3}) \cap S_{r-3} = \{w_{r-3,3}\}$. Since

$|N(w_{r-2,1})| \geq 3$ and $|N(w_{r-2,3})| \geq 3$, we obtain $|X_{r-1}| = 4$ and $|N(w_{r-2,3}) \cap X_{r-1}| = 2$. Note that

$$N(w_{r-2,1}) \cap (N(w_{r-2,3}) \cap X_{r-1}) = N(w_{r-2,2}) \cap (N(w_{r-2,3}) \cap X_{r-1}) = \phi\,. \tag{6.2}$$

Write $X_{r-1} - (N(w_{r-2,3}) \cap X_{r-1}) = \{u_1, u_2\}$ (recall that $u_1 \in N(w_{r-2,2}) \cap X_{r-1}$) and $N(w_{r-2,3}) \cap X_{r-1} = \{u_3, u_4\}$. Suppose that $N(u_1) \cap X_r \neq \phi$. By Lemma 1, $(N(u_3) \cup N(u_4)) \cap X_r \neq \phi$. Since $d_G(w_{r-2,2}, w_{r-2,3}) \geq 6$, it follows that $|X_r| = 2$, and if we write $N(u_1) \cap X_r = \{z_1\}$, then $N(z_1) \cap \{u_3, u_4\} = N(z_1) \cap ((N(u_3) \cup N(u_4)) \cap X_r) = \phi$, and hence $N(z_1) \subseteq \{u_1, u_2\}$, which contradicts the assumption that G is 3-connected. Thus $N(u_1) \cap X_r = \phi$. Hence by Lemma 6(ii), u_3 or u_4, say u_3, satisfies $d_G(u_2, u_3) \leq 2$. Since $d_G(w_{r-2,2}, w_{r-2,3}) \geq 6$, this implies $u_1 u_2 \notin E(G)$. Since $N(u_1) \cap \{u_3, u_4, w_{r-2,3}\} = \phi$, this forces

$$N(u_1) = \{w_{r-2,1}, w_{r-2,2}, w_{r-2,4}\}\,. \tag{6.3}$$

In particular, $d_G(w_{r-2,2}, w_{r-2,4}) \leq 2$, and hence

$$N(w_{r-2,4}) \cap \{u_3, u_4\} = \phi\,. \tag{6.4}$$

Now (6.2), (6.3) and (6.4) imply that $G - \{w_{r-2,3}, u_2\}$ is disconnected, which contradicts the assumption that G is 3-connected. This concludes the discussion for the case where $N(w_{r-2,2}) \cap X_{r-1} \neq \phi$.

Assume now that $N(w_{r-2,2}) \cap X_{r-1} = \phi$. Then $d_G(w_{r-2,1}, w_{r-2,2}) = 1$. We also get $N(w_{r-2,1}) \cap X_{r-1} \neq \phi$ by Lemma 1, and hence it follows from (6.1) that $d_G(w_{r-2,2}, y) \leq 5$ for every $y \in X_r$. Furthermore it follows from Claim 6.12(1) that $d_G(w_{r-2,2}, y) \leq 5$ for every $y \in S_{r-2}$. Thus $v \in X_{r-1}$. By Claim 6.12(2), $N(v) \cap X_{r-2} = \{w_{r-2,3}\}$. Set $M = N_2(v) \cap (\cup_{x \in N(v) - \{w_{r-2,3}\}} N(x))$. Then $G - (M \cup \{w_{r-2,3}\})$ is disconnected. This implies $|M| \geq 2$, and hence $|\{v\} \cup N(v) \cup M| \geq 6$. Since $N(v) \cap X_{r-2} = \{w_{r-2,3}\}$ and $N(w_{r-2,2}) \cap X_{r-1} = \phi$, we also have $\{v\} \cup N(v) \cup M \subseteq (X_{r-2} \cup X_{r-1} \cup X_r) - \{w_{r-2,2}\}$. Now fix $u \in N(w_{r-2,1}) \cap X_{r-1}$. Then again since $N(w_{r-2,2}) \cap X_{r-1} = \phi$, $\{u\} \cup N(u) \subseteq (X_{r-2} \cup X_{r-1} \cup X_r) - \{w_{r-2,2}\}$. Since $|\{u\} \cup N(u)| \geq 4$ and $|(X_{r-2} \cup X_{r-1} \cup X_r) - \{w_{r-2,2}\}| \leq 4 + 6 - 1$, we obtain $(\{v\} \cup N(v) \cup M) \cap (\{u\} \cup N(u)) \neq \phi$. But this contradicts the assumption that $d_G(w_{r-2,2}, v) \geq 6$, which completes the proof of Claim 6.17.

Claim 6.18. *Suppose that* $c = r - 1$. *Then* $d_G(w_{r-1,1}, w_{r-1,2}) = 1$, *and* $d_G(w_{r-2,2}, y) \leq 4$ *for every* $y \in X_{r-1} \cup X_r$.

Proof. We have $|X_{r-1}| = 4$ and $|X_r| \leq 2$. Suppose that $d_G(w_{r-1,1}, w_{r-1,2}) = 2$. Then there is no edge between S_{r-1} and T_{r-1}. Since $G - S_a$ is 2-connected, it follows that $|X_r| = 2$, there are two independent edges joining S_{r-1} and X_r, and there are two independent edges joining T_{r-1} and X_r. Assume first that $w_{r-1,2} w_{r-1,4} \notin E(G)$. We choose $x \in T_{r-2}$ as follows. If $N(w_{r-1,2}) \cap N(w_{r-1,4}) \cap T_{r-2} \neq \phi$, let $x \in N(w_{r-1,2}) \cap N(w_{r-1,4}) \cap T_{r-2}$; if $N(w_{r-1,2}) \cap N(w_{r-1,4}) \cap T_{r-2} = \phi$, take any $x \in T_{r-2}$. Note that if $N(w_{r-1,2}) \cap N(w_{r-1,4}) \cap T_{r-2} = \phi$, then $N(w_{r-1,2}) \cap N(w_{r-1,4}) \cap X_r = X_r$ because $|N(w_{r-1,2})| \geq 3$ and $|N(w_{r-1,4})| \geq 3$. Thus in either case, we have $d_G(x,w) = 2$ for each $w \in X_r$. Let u be a vertex in X_{r-a} which is on a shortest $z-x$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq a-1$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = (r-a)+i \leq (r-a)+(a-1) = r-1 < r$. If $v \in \cup_{a \leq j \leq r-2} T_j$, then $d_G(x,v) \leq \{(r-2)-a\}+2 = r-a$ by Claim 6.9(2) or 6.10(2), and hence $d_G(u,v) \leq d_G(u,x)+d_G(x,v) \leq \{(r-2)-(r-a)\}+(r-a) = r-2 < r$. If $v \in T_{r-1}$, then $d_G(x,v) \leq 3$ by Claim 6.11(2), and hence $d_G(u,v) \leq d_G(u,x)+d_G(x,v) \leq \{(r-2)-(r-a)\}+3 = a+1 \leq (r-3)+1 = r-2 < r$. If $v \in \cup_{a+1 \leq j \leq r} S_j$ (note that $S_r = X_r$), then letting w be a vertex in X_r such that v is on a shortest $z-w$ path (the existence of such a vertex is guaranteed by Claim 6.13(1) and the fact that there are two independent edges joining S_{r-1} and X_r), we have $d_G(x,w) = 2$, and hence $d_G(u,v) \leq d_G(u,x)+d_G(x,w)+d_G(w,v) = \{(r-2)-(r-a)\}+2+\{r-(a+1)\} = r-1 < r$. If $v = w_{a1}$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = (r-a)+a = r$. This contradicts the assumption that z is chosen so that $|X_r|$ is minimum. Assume now that $w_{r-1,2}w_{r-1,4} \in E(G)$. Let u be a vertex in X_{r-a} which is on a shortest $z-w_{r-1,2}$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq a-1$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = (r-a)+i \leq (r-a)+(a-1) = r-1 < r$. If $v \in \cup_{a \leq j \leq r-1} T_j$, then letting w be a vertex in T_{r-1} such that v is on a shortest $z-w$ path, we have $d_G(w_{r-1,2},w) \leq 1$, and hence $d_G(u,v) \leq d_G(u,w_{r-1,2})+d_G(w_{r-1,2},w)+d_G(w,v) \leq \{(r-1)-(r-a)\}+1+\{(r-1)-a\} = r-1 < r$. If $v \in \cup_{a+2 \leq j \leq r} S_j$, then letting w be a vertex in X_r such that v is on a shortest $z-w$ path, we have $d_G(w_{r-1,2},w) \leq 2$, and hence $d_G(u,v) \leq d_G(u,w_{r-1,2}) + d_G(w_{r-1,2},w)+d_G(w,v) \leq \{(r-1)-(r-a)\}+2+\{r-(a+2)\} = r-1 < r$. If $v = w_{a+1,1}$, $d_G(u,v) \leq d_G(u,w_{r-1,2})+d_G(w_{r-1,2},w_{r-1,1})+d_G(w_{r-1,1},w_{a+1,1}) = \{(r-1)-(r-a)\}+2+\{(r-1)-(a+1)\} = r-1 < r$. If $v = w_{a1}$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = (r-a)+a = r$. This contradicts the assumption that z is chosen so that $|X_r|$ is minimum.

Thus $d_G(w_{r-1,1},w_{r-1,2}) = 1$. Note that $d_G(w_{r-2,2},w_{r-1,4}) \leq 3$ by Claim 6.11(2), and hence $d_G(w_{r-2,2},y) \leq 3$ for every $y \in \{w_{r-1,1}\} \cup T_{r-1}$. Let $v \in X_r$. Since $|N(v)| \geq 3$ and $|X_r| \leq 2$, we have $N(v) \cap (\{w_{r-1,1}\} \cup T_{r-1}) \neq \phi$. Hence $d_G(w_{r-2,2},v) \leq 4$. Thus $d_G(w_{r-2,2},v) \leq 4$ for every $v \in X_r$. If $N(w_{r-1,3}) \cap (\{w_{r-1,1},w_{r-2,1}\} \cup T_{r-1}) \neq \phi$, then $d_G(w_{r-2,2},w_{r-1,3}) \leq 4$. Thus we may assume $N(w_{r-1,3}) \cap (\{w_{r-1,1},w_{r-2,1}\} \cup T_{r-1}) = \phi$. Since $|N(w_{r-1,3})| \geq 3$, this implies $N(w_{r-1,3}) \supseteq X_r$. On the other hand, we have $N(w_{r-1,1}) \cap X_r \neq \phi$ or $N(w_{r-1,2}) \cap X_r \neq \phi$ by Lemma 1. Since $d_G(w_{r-2,2},w_{r-1,2}) = 1$ and $d_G(w_{r-2,2},w_{r-1,1}) = 2$, we now obtain $d_G(w_{r-2,2},w_{r-1,3}) \leq 4$, which completes the proof of Claim 6.18.

We divide the rest of the proof for Case 2 into two cases. In each case, we derive a contradiction by showing that there exists $u \in V(G)$ such that $d_G(u,v) < r$ for every $v \in V(G)$.

Case 2.1. $a+2 \leq c \leq r-2$.

In view of Claims 6.15 through 6.17, we can derive a contradiction by arguing as in

Cases 2.1 through 2.7 of Proposition 1.

Case 2.2. $c = r-1$.

Let u be a vertex in X_{r-a} which is on a shortest $z - w_{r-2,2}$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le a-1$, $d_G(u,v) \le d_G(u,z) + d_G(z,v) = (r-a)+i \le (r-a)+(a-1) = r-1 < r$. If $r-1 \le i \le r$, then we get $d_G(w_{r-2,2}, v) \le 4$ by Claim 6.18, and hence $d_G(u,v) \le d_G(u, w_{r-2,2}) + d_G(w_{r-2,2}, v) \le \{(r-2)-(r-a)\}+4 = a+2 \le (r-3)+2 = r-1 < r$. Thus we may assume that $a \le i \le r-2$. If $v \in T_i$, then $d_G(w_{r-2,2}, v) \le (r-1)-i+1 \le (r-1)-a+1 = r-a$ by Claim 6.14(2); if $v \in S_i$ and $a+2 \le i \le r-2$, then $d_G(w_{r-2,2}, v) \le (r-1)-i+4 \le (r-1)-(a+2)+4 = r-a+1$ by Claim 6.14(1); if $v \in S_i$ and $a \le i \le a+1$, then $d_G(w_{r-2,2}, v) \le (r-1)-i+2 \le (r-1)-a+2 = r-a+1$ by Claim 6.14(3). Hence $d_G(w_{r-2,2}, v) \le r-a+1$. Therefore $d_G(u,v) \le d_G(u, w_{r-2,2}) + d_G(w_{r-2,2}, v) \le \{(r-2)-(r-a)\} + (r-a+1) = r-1 < r$.

This completes the proof of Proposition 2.

7. Proof of the Theorem

We continue with the notation of Section 2, and complete the proof of the Theorem. We first prove two propositions.

Proposition 3. *Suppose that* $r \ge 6$. *Then* $\sum_{i=3}^{r-2} |X_i| \ge 4(r-4)-2$.

Proof. Let $I := \{i | 3 \le i \le r-2, |X_i| = 3\}$. We may assume $|I| \ge 3$. Note that this implies $r-2 \ge 5$, i.e., $r \ge 7$. Write $I = \{i_1, i_2, \dots, i_{|I|}\}$ with $i_1 < i_2 < \cdots < i_{|I|}$. From I, we define a new sequence $j_1 < j_2 < \cdots < j_s$ inductively as follows. Set $j_1 = i_1$. For $l \ge 2$, set $j_l = \min\{i | i \in I, i \ge j_{l-1}+2\}$ (if $\{i | i \in I, i \ge j_{l-1}+2\} = \phi$, then we set $s = l-1$ and terminate this procedure). We have $j_s = i_{|I|}$ or $i_{|I|-1}$ by definition. By Proposition 1(1), $\sum_{i=j_{h-1}}^{j_h-1} |X_i| \ge 4(j_h - j_{h-1})$ for all $2 \le h \le s$. Taking the sum of these inequalities, we get $\sum_{i=j_1}^{j_s-1} |X_i| = \sum_{h=2}^{s} \sum_{i=j_{h-1}}^{j_h-1} |X_i| \ge 4(j_s - j_1)$. Consequently $\sum_{i=3}^{r-2} |X_i| = \sum_{i=3}^{j_1-1} |X_i| + \sum_{i=j_1}^{j_s-1} |X_i| + \sum_{i=j_s}^{r-2} |X_i| \ge 4(j_1-3)+4(j_s-j_1)+4(r-1-j_s)-2 = 4(r-4)-2$, as desired.

Proposition 4. *Suppose that* $r \ge 6$ *and* $|X_{r-1}| = |X_r| = 3$. *Then* $\sum_{i=3}^{r-2} |X_i| \ge 4(r-4)-1$.

Proof. Let $I := \{i | 3 \le i \le r-2, |X_i| = 3\}$. We may assume $|I| \ge 2$. Assume $r = 6$. Then this forces $|X_3| = |X_4| = 3$. But this contradicts Proposition 2(1). Thus

$r \geq 7$. Write $I = \{i_1, i_2, \ldots, i_{|I|}\}$ with $i_1 < i_2 < \cdots < i_{|I|}$. From I, we define a new sequence $j_1 < j_2 < \cdots < j_s$ inductively as follows. Set $j_1 = i_1$. For $l \geq 2$, set $j_l = \min\{i | i \in I, i \geq j_{l-1} + 2\}$ (if $\{i | i \in I, i \geq j_{l-1} + 2\} = \phi$, then we set $s = l - 1$ and terminate this procedure). We have $j_s = i_{|I|}$ or $i_{|I|-1}$ by definition. By Proposition 1(1), $\sum_{i=j_{h-1}}^{j_h - 1} |X_i| \geq 4(j_h - j_{h-1})$ for all $2 \leq h \leq s$. Taking the sum of these inequalities, we get $\sum_{i=j_1}^{j_s - 1} |X_i| = \sum_{h=2}^{s} \sum_{i=j_{h-1}}^{j_h - 1} |X_i| \geq 4(j_s - j_1)$. If $j_s = i_{|I|-1}$, we obtain $\sum_{i=j_s}^{r-2} |X_i| \geq 4(r - 1 - j_s) - 1$ by Proposition 2(1); if $j_s = i_{|I|}$, we have $\sum_{i=j_s}^{r-2} |X_i| \geq 4(r - 1 - j_s) - 1$ by the definition of I. Consequently $\sum_{i=3}^{r-2} |X_i| = \sum_{i=3}^{j_1 - 1} |X_i| + \sum_{i=j_1}^{j_s - 1} |X_i| + \sum_{i=j_s}^{r-2} |X_i| \geq 4(j_1 - 3) + 4(j_s - j_1) + 4(r - 1 - j_s) - 1 = 4(r - 4) - 1$, as desired.

We can prove the following two propositions by arguing as in the proofs of Propositions 3 and 4.

Proposition 5. *Suppose that* $r \geq 5$ *and* $|X_r| = 2$. *Then* $\sum_{i=3}^{r-1} |X_i| \geq 4(r - 3) - 1$.

Proposition 6. *Suppose that* $r \geq 4$ *and* $|X_r| = 1$. *Then* $\sum_{i=2}^{r-1} |X_i| \geq 4(r - 2) - 1$.

We are now in a position to complete the proof of the Theorem. First we consider the case where $|X_r| \geq 3$. If $r \leq 5$, the desired conclusion follows from Lemma 5. Thus we may assume $r \geq 6$. Note that $|X_0| = 1$, and $|X_i| \geq 3$ for each $i \in \{1, 2, r - 1\}$ by Lemma 2. If $|X_{r-1} \cup X_r| \geq 7$, then $\sum_{i=3}^{r-2} |X_i| \geq 4(r - 4) - 2$ by Proposition 3, and hence we obtain $|V(G)| = \sum_{i=0}^{r} |X_i| \geq 1 + 3 \times 2 + \{4(r - 4) - 2\} + 7 = 4r - 4$; if $|X_{r-1} \cup X_r| = 6$, then $\sum_{i=3}^{r-2} |X_i| \geq 4(r - 4) - 1$ by Proposition 4, and hence we obtain $|V(G)| = \sum_{i=0}^{r} |X_i| \geq 1 + 3 \times 2 + \{4(r - 4) - 1\} + 6 = 4r - 4$.

Next we consider the case where $|X_r| = 2$. If $r \leq 4$, the desired conclusion follows from Lemma 5. Thus we may assume $r \geq 5$. By Proposition 5, $\sum_{i=3}^{r-1} |X_i| \geq 4(r - 3) - 1$. Therefore we obtain $|V(G)| = \sum_{i=0}^{r} |X_i| \geq 1 + 3 \times 2 + \{4(r - 3) - 1\} + 2 = 4r - 4$.

Finally we consider the case where $|X_r| = 1$. If $r \leq 3$, the desired conclusion follows from Lemma 5. Thus we may assume $r \geq 4$. By Proposition 6, $\sum_{i=2}^{r-1} |X_i| \geq 4(r-2) - 1$. Therefore we obtain $|V(G)| = \sum_{i=0}^{r} |X_i| \geq 1 + 3 + \{4(r-2) - 1\} + 1 = 4r - 4$.

This completes the proof of the Theorem.

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