AKCE J. Graphs. Combin., 4, No. 1 (2007), pp. 83-98

Global alliances in planar graphs

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Communicated by: Odile Favaron
Received 16 July 2006; accepted 17 November 2006

Abstract

In this paper we study mathematical properties of alliances (defensive alliances, offensive alliances and dual alliances) in planar graphs. In particular, we obtain several tight bounds on different types of alliance numbers of a planar graph.

Keywords: Defensive alliance, offensive alliance, dual alliance, domination, planar graphs, trees.

2000 Mathematics Subject Classification: 05C69; 05C05

1. Introduction

The study of defensive alliances in graphs, together with a variety of other kinds of alliances, was introduced in [6]. In the cited paper there was initiated the study of the mathematical properties of alliances. In particular, several bounds on the defensive alliance number were given. The particular case of global (strong) defensive alliance was investigated in [5] where several bounds on the global (strong) defensive alliance number were obtained. In [7] there were obtained several tight bounds on different types of alliance numbers of a graph, namely the (global) defensive alliance number, (global) offensive alliance number and (global) dual alliance number. In particular, there was investigated the relationship between the alliance numbers of a graph and its algebraic connectivity, its spectral radius, and its Laplacian spectral radius. The dual alliances were introduced as powerful alliances in [1]. For the study of offensive alliances we cite [3, 4, 8, 13] and for the study of alliances in trees we cite [2, 3]. For the study of alliance free sets and alliance cover sets we cite [10, 11, 9] and, finally, for the study of defensive alliances in the line graph of a simple graph we cite [12]. The aim of this work is to study mathematical properties of alliances (defensive alliances, offensive alliances and dual alliances) in planar graphs. In particular, we obtain several tight bounds on different types of alliance numbers of a planar graph. The case of strong alliances is also investigated.

We begin by stating some notation and terminology. In this paper $\Gamma=(V,E)$ denotes a simple graph of order n and size m. For a non-empty subset $X\subset V$, and any vertex $v\in V$, we denote by $N_X(v)$ the set of neighbors v has in X, $N_X(v):=\{u\in X: u\sim v\}$. The subgraph induced by the set X will be denoted by $\langle X\rangle$.

It is well-known that the size of a planar graph Γ of girth at least g, $3\le g<\infty$, and order n is bounded by

$$m\le \frac{g}{g-2}(n-2). \tag{1}$$

This inequality allows us to obtain tight bounds for the studied parameters.

2. Defensive alliances

A nonempty set of vertices $S\subset V$ is called a *defensive alliance* if for every $v\in S$, $|N_S(v)|+1\ge |N_{V\setminus S}(v)|$. In this case, by strength of numbers, every vertex in S is *defended* from possible attack by vertices in $V\setminus S$. A defensive alliance S is called *strong* if for every $v\in S$, $|N_S(v)|\ge |N_{V\setminus S}(v)|$. In this case every vertex in S is *strongly defended.*

A defensive alliance S is called *global* if it affects every vertex in $V\setminus S$, that is, every vertex in $V\setminus S$ is adjacent to at least one member of the alliance S. Note that, in this case, S is a dominating set. The *global defensive alliance number* $\gamma_a(\Gamma)$ (respectively, *global strong defensive alliance number* $\gamma_{\hat{a}}(\Gamma)$) is the minimum cardinality of any global defensive alliance (respectively, global strong defensive alliance) in Γ.

2.1. Global defensive alliances in planar graphs

Theorem 2.1. *Let $\Gamma=(V,E)$ be a graph of order n. Let S be a global defensive alliance in Γ such that the subgraph $\langle S\rangle$ is planar.*

(a) *If $n>6$, then $|S|\ge\left\lceil\frac{n+12}{8}\right\rceil$.*

(b) *If $n>6$ and $\langle S\rangle$ is a triangle-free graph, then $|S|\ge\left\lceil\frac{n+8}{6}\right\rceil$.*

(c) *If $n>4$ and the alliance S is strong, then $|S|\ge\left\lceil\frac{n+12}{7}\right\rceil$.*

(d) *If $n>4$, the alliance S is strong, and $\langle S\rangle$ is a triangle-free graph, then $|S|\ge\left\lceil\frac{n+8}{5}\right\rceil$.*

Proof.

(a) As S is a defensive alliance in Γ,

$$\sum_{v\in S}|N_{V\setminus S}(v)| \le \sum_{v\in S}|N_S(v)| + |S|. \quad (2)$$

Moreover, if the defensive alliance S is global, we have

$$n - |S| \le \sum_{v\in S}|N_{V\setminus S}(v)|. \quad (3)$$

Thus, by (2) and (3) we obtain

$$n - 2|S| \le \sum_{v\in S}|N_S(v)|. \quad (4)$$

If $|S| \le 2$, for each $v \in S$ we have $|N_{V\setminus S}(v)| \le 2$. Thus, $n \le 6$. Therefore, $n > 6 \Rightarrow |S| > 2$.

As $\langle S\rangle$ is planar and $|S| > 2$, the size of $\langle S\rangle$ is bounded above by

$$\frac{1}{2}\sum_{v\in S}|N_S(v)| \le 3(|S| - 2). \quad (5)$$

The result follows from (5) and (4), considering that $|S|$ is an integer.

(b) If $\langle S\rangle$ is a triangle-free graph, then

$$\frac{1}{2}\sum_{v\in S}|N_S(v)| \le 2(|S| - 2). \quad (6)$$

The result follows by (6) and (4), considering that $|S|$ is an integer.

(c) As S is a global strong defensive alliance in Γ,

$$\sum_{v\in S}|N_{V\setminus S}(v)| \le \sum_{v\in S}|N_S(v)|, \quad (7)$$

and so by (3)

$$n - |S| \le \sum_{v\in S}|N_S(v)|. \quad (8)$$

If $|S| \le 2$, for each $v \in S$ we have $|N_{V\setminus S}(v)| \le 1$. Thus, $n \le 4$. Therefore, $n > 4 \Rightarrow |S| > 2$. The result follows by (8) and (5), considering that $|S|$ is an integer.

(d) The result follows by (8) and (6).

□

Corollary 2.2. *Let* Γ *be a planar graph of order* n *.*

(a) *If* $n > 6$ *, then* $\gamma_a(\Gamma) \geq \left\lceil \frac{n+12}{8} \right\rceil$.

(b) *If* $n > 6$ *and* Γ *is a triangle-free graph, then* $\gamma_a(\Gamma) \geq \left\lceil \frac{n+8}{6} \right\rceil$.

(c) *If* $n > 4$ *, then* $\gamma_{\hat{a}}(\Gamma) \geq \left\lceil \frac{n+12}{7} \right\rceil$.

(d) *If* $n > 4$ *and* Γ *is a triangle-free graph, then* $\gamma_{\hat{a}}(\Gamma) \geq \left\lceil \frac{n+8}{5} \right\rceil$.

The above bounds are attained for the right hand side graph of Figure 1. Moreover, the bounds (a) and (c) are attained for the left hand side graph Figure 1, the bound (c) is attained for the graph $\Gamma = K_2 \times K_3$ and the bound (d) is attained for the 3-cube graph ($\Gamma = K_2 \times K_2 \times K_2$).

Theorem 2.3. *Let* Γ *be a graph of order* n *. Let* S *be a global defensive alliance in* Γ *such that the subgraph* $\langle S \rangle$ *is planar connected with* f *faces. Then,*

(a) $|S| \geq \left\lceil \frac{n-2f+4}{4} \right\rceil$.

(b) *If the alliance* S *is strong,* $|S| \geq \left\lceil \frac{n-2f+4}{3} \right\rceil$.

Proof. By Euler's formula, $\sum_{v \in S} |N_S(v)| = 2(|S| + f - 2)$, and so the proof of (a) concludes by (4). The proof of (b) is analogous by using (8) instead of (4). □

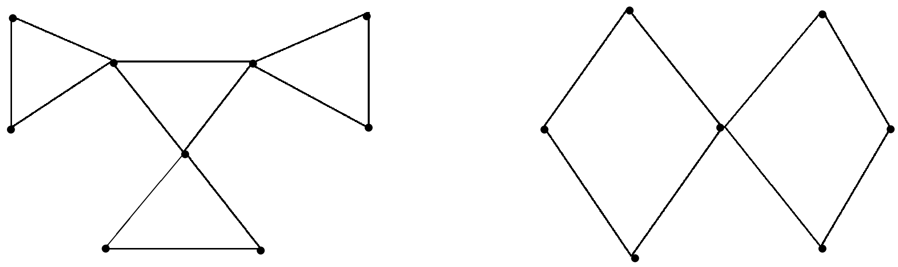

Figure 1:

The above bounds are attained, for instance, in the case of the left hand side graph of Figure 1 where S is the set of vertices of degree four.

Theorem 2.4. *Let* Γ *be a graph of order* n *. Let* S *be a global defensive alliance in* Γ *such that* $|S| > 2$ *. If* $\langle S \rangle$ *is planar and its minimum degree is at least* k *, then*

$$|S| \geq \left\lceil \frac{k-7+\sqrt{(k-7)^2+4(12+n)}}{2} \right\rceil,$$

moreover, if $\langle S \rangle$ *is also a triangle-free graph, then*

$$|S| \geq \left\lceil \frac{k-5+\sqrt{(k-5)^2+4(8+n)}}{2} \right\rceil.$$

Proof. Since S is a defensive alliance in Γ,

$$\sum_{v\in S} |N_{V\setminus S}(v)| \leq |S|^2. \tag{9}$$

Moreover, as the alliance S is global, by (3) and (9) we have

$$n - |S| \leq |S|^2. \tag{10}$$

As $\langle S \rangle$ is planar and its minimum degree is at least k, its size is bounded by

$$\frac{|S|k}{2} \leq \frac{1}{2}\sum_{v\in S} |N_S(v)| \leq 3(|S|-2). \tag{11}$$

By (10) and (11) we deduce the first bound. The second bound is obtained as before by using the fact that if $\langle S \rangle$ is a planar triangle-free graph and its minimum degree is at least k, then its size is bounded by $\frac{|S|k}{2} \leq \frac{1}{2}\sum_{v\in S} |N_S(v)| \leq 2(|S|-2)$. □

Figure 2:

The above bounds are attained, for instance, for the graphs of Figure 2. In this case $S = \{1, 2, 3, 4\}$.

2.2. Global defensive alliances in trees

In this section we study global alliances in trees but we impose a condition on the number of connected components of the subgraphs induced by the alliances.

Theorem 2.5. *Let T be a tree of order n. Let S be a global defensive alliance in T such that the subgraph $\langle S\rangle$ has c connected components. Then,*

(a) $|S| \geq \left\lceil \frac{n+2c}{4} \right\rceil$.

(b) *If the alliance S is strong,* $|S| \geq \left\lceil \frac{n+2c}{3} \right\rceil$.

Proof.

(a) As the subgraph $\langle S\rangle$ is a forest with c connected components,

$$\sum_{v\in S} |N_S(v)| = 2(|S| - c). \tag{12}$$

The bound of $|S|$ follows from (12) and (4), considering that $|S|$ is an integer.

(b) Analogously to the case (a), the bound of $|S|$ follows from (8) and (12).

□

Figure 3:

In the case of the left hand side graph of Figure 3, Theorem 2.5 (a) gives the best result for $c = 2$, $c = 3$ and $c = 4$. Theorem 2.5 (b) gives the best result for the right hand side graph of Figure 3 taking $c = 2$.

The following result, showed in [5], is a consequence of Theorem 2.5.

Corollary 2.6. [5] *For any tree T of order n,*

$$\gamma_a(T) \geq \left\lceil \frac{n+2}{4} \right\rceil \quad \textit{and} \quad \gamma_{\hat{a}}(T) \geq \left\lceil \frac{n+2}{3} \right\rceil.$$

3. Offensive alliances

The boundary of a set $S \subset V$ is defined as $\partial(S) := \bigcup_{v\in S} N_{V\setminus S}(v)$. A non-empty set of vertices $S \subset V$ is called *offensive alliance* if and only if for every $v \in \partial(S)$, $|N_S(v)| \geq |N_{V\setminus S}(v)| + 1$. An offensive alliance S is called *strong* if for every vertex $v \in \partial(S)$, $|N_S(v)| \geq |N_{V\setminus S}(v)| + 2$.

A non-empty set of vertices $S \subset V$ is a *global offensive alliance* if for every vertex $v \in V \setminus S$, $|N_S(v)| \geq |N_{V\setminus S}(v)| + 1$. Thus, global offensive alliances are also dominating sets, and one can define the *global offensive alliance number*, denoted $\gamma_{a_o}(\Gamma)$, to equal the minimum cardinality of a global offensive alliance in Γ. Analogously, $S \subset V$ is a *global strong offensive alliance* if for every vertex $v \in V \setminus S$, $|N_S(v)| \geq |N_{V\setminus S}(v)| + 2$, and the *global strong offensive alliance number*, denoted $\gamma_{\hat{a}_o}(\Gamma)$, is defined as the minimum cardinality of a global strong offensive alliance in Γ.

In this section we will use the following obvious but useful claim:

Claim 3.1. *Let $\Gamma = (V, E)$ be a graph of size m. If $S \subset V$, then*

$$2m = \sum_{v\in S} |N_S(v)| + 2\sum_{v\in V\setminus S} |N_S(v)| + \sum_{v\in V\setminus S} |N_{V\setminus S}(v)|.$$

3.1. Global offensive alliances in planar graphs

Theorem 3.2. *Let $\Gamma = (V, E)$ be a planar graph of order $n > 2$. If S is a global offensive alliance in Γ such that the subgraph $\langle V\setminus S\rangle$ has c connected components, then*

(a) $|S| \geq \left\lceil \frac{n+4-2c}{3} \right\rceil$.

(b) *If the alliance S is strong,* $|S| \geq \left\lceil \frac{n-c+2}{2} \right\rceil$.

Proof. The subgraph $\Gamma_s = (V, E')$, where $E' = \{\{x, y\} \in E : x \in S, y \notin S\}$, is planar and bipartite. So Γ_s is a triangle free graph, then we have

$$2(n-2) \geq \sum_{v\in V\setminus S} |N_S(v)|. \tag{13}$$

(a) If S is a global offensive alliance in Γ, then

$$\sum_{v\in V\setminus S} |N_S(v)| \geq \sum_{v\in V\setminus S} |N_{V\setminus S}(v)| + (n - |S|). \tag{14}$$

By (13) and (14) we have

$$|S| - n + 2(n-2) \geq \sum_{v\in V\setminus S} |N_{V\setminus S}(v)|. \tag{15}$$

As the subgraph $\langle V \backslash S \rangle$ has c connected components, we have

$$|S| - n + 2(n-2) \geq \sum_{v \in V \backslash S} |N_{V \backslash S}(v)| \geq 2(n - |S| - c).$$

Thus, the result follows.

(b) If S is a global strong offensive alliance in Γ, then

$$\sum_{v \in V \backslash S} |N_S(v)| \geq \sum_{v \in V \backslash S} |N_{V \backslash S}(v)| + 2(n - |S|). \tag{16}$$

By (13) and (16), we have

$$2(|S| - n) + 2(n-2) \geq \sum_{v \in V \backslash S} |N_{V \backslash S}(v)| \geq 2(n - |S| - c).$$

Hence, the result follows.

□

Figure 4:

Notice that (14) and (16) hold even if the graph is not planar.

In the following instances the above theorem gives the optimum result: case (a), the right hand side graph of Figure 4 taking $S = \{5, 6\}$; case (b), the left hand side graph of Figure 4 taking $S = \{5, 6, 7, 8\}$.

Theorem 3.3. *Let Γ be a planar graph of order n . If S is a global offensive alliance in Γ such that the minimum degree of $\langle V \backslash S \rangle$ is at least k , then*

(a) $|S| \geq \left\lceil \frac{n(k-1)+4}{k+1} \right\rceil$.

(b) *If the alliance S is strong,* $|S| \geq \left\lceil \frac{nk+4}{k+2} \right\rceil$.

Proof. The proof is basically as in Theorem 3.2. In this case we use that the size of $\langle V\backslash S\rangle$ is bounded below by $\frac{(n-|S|)k}{2}$. □

Figure 5:

In the following instances the above theorem gives the optimum result: case (a), the right hand side graph of Figure 5; case (b), the left hand side graph of Figure 5. In both cases we take $V\backslash S$ as the set of vertices of degree 5.

We say that a graph $\Gamma = (V, E)$ is a *4-book* if V is the union of subsets $X_1, X_2, ..., X_r$ such that $3 \leq |X_i| \leq 4$ and

i. $\exists\{v_1, v_2\} \in E$ such that $\forall i \neq j \;\; X_i \cap X_j = \{v_1, v_2\}$;

ii. For $i \neq j$, there are no edges connecting vertices in $X_i\backslash\{v_1, v_2\}$ to vertices in $X_j\backslash\{v_1, v_2\}$;

iii. If $y \in V\backslash\{v_1, v_2\}$, then y is adjacent to v_1 or is adjacent to v_2 and, if $deg(y) \geq 2$, then y is adjacent to both v_1 and v_2.

Each of these r induced subgraphs, $\langle X_i\rangle$, is called *page* of the 4-book Γ.

It is easy to check the following assertions.

- $\gamma_{a_o}(\Gamma) = 1$ if and only if $\Gamma = S_{n-1}$ (the star graph).
- $\gamma_{a_o}(\Gamma) = 2$ if and only if Γ is a 4-book graph.

Theorem 3.4. *Let Γ be a planar graph of order n and size m. Then the following assertions hold.*

(a) *If $\Gamma \neq S_{n-1}$, then Γ is a 4-book graph or $\gamma_{a_o}(\Gamma) \geq \left\lceil \frac{2m-5n+24}{7} \right\rceil$.*

(b) *Γ is a 4-book graph or $\gamma_{\hat{a}_o}(\Gamma) \geq \left\lceil \frac{m-2n+12}{4} \right\rceil$.*

Proof.

(a) Suppose $\gamma_{a_o}(\Gamma) > 2$. Let S be a global offensive alliance in Γ. By Claim 3.1 and (14) we deduce

$$\sum_{v\in S}|N_S(v)| + 3\sum_{v\in V\setminus S}|N_S(v)| \geq 2m + n - |S|. \qquad (17)$$

Hence, by (5), (17) and (13) we deduce the result.

(b) The proof is basically as in case (a). In this case we use (16) instead of (14).

□

Figure 6:

The bound (a) is attained for the left hand side graph of Figure 6 where a minimal offensive alliance is $S = \{1, 2, 3\}$. The bound (b) is attained for the right hand side graph of Figure 6 where a minimal strong offensive alliance is $S = \{1, 2, 3, 4\}$.

Theorem 3.5. *Let $\Gamma = (V, E)$ be a planar graph of order n. If S is a global offensive alliance in Γ such that the subgraph $\langle V\setminus S\rangle$ is connected and has f faces, then*

(a) $|S| \geq \left\lceil \frac{n+2f}{3} \right\rceil$.

(b) *If the alliance S is strong,* $|S| \geq \left\lceil \frac{n+f}{2} \right\rceil$.

Proof. By Euler's formula, $\sum_{v\in V\setminus S}|N_{V\setminus S}(v)| = 2(n - |S| + f - 2)$, and so the proof of (a) concludes by (15). The proof of (b) is analogous. □

3.2. Global offensive alliances in trees

As a direct consequence of the definition of global strong offensive alliance we obtain the following remark.

Remark 3.6. *For any tree T, the subgraph induced by a global strong offensive alliance different from T is not connected.*

Theorem 3.7. *If S is a global offensive alliance in a tree $T=(V,E)$ of order n such that the subgraph $\langle S\rangle$ has c connected components, then*

- *The size of $\langle V\setminus S\rangle$ is bounded above by $\frac{c-1}{3}$.*
- *If the alliance S is strong, the size of $\langle V\setminus S\rangle$ is bounded above by $\frac{1}{3}(c+|S|-n-1)$.*

Proof. If $\langle S\rangle$ has c connected components, then

$$n-1=(|S|-c)+\sum_{v\in V\setminus S}|N_S(v)|+\frac{1}{2}\sum_{v\in V\setminus S}|N_{V\setminus S}(v)|. \tag{18}$$

Moreover, if S is a global offensive alliance, then (14) and (18) lead to

$$c-1\geq\frac{3}{2}\sum_{v\in V\setminus S}|N_{V\setminus S}(v)|. \tag{19}$$

Thus, the size of $\langle V\setminus S\rangle$ is bounded above by $\frac{c-1}{3}$. Hence, the first result follows. The second result follows from (16) and (18). □

From above theorem we deduce the following result.

Corollary 3.8. *Let S be a global offensive alliance in a tree $T=(V,E)$ of order n. If the subgraph $\langle S\rangle$ has one, two or three connected components, then $V\setminus S$ is an independent set.*

We remark that if S is a global offensive alliance in a tree and the subgraph $\langle S\rangle$ is a tree, then every vertex of $V\setminus S$ is a leaf.

Corollary 3.9. *If S is a global strong offensive alliance in a tree of order n such that the subgraph $\langle S\rangle$ has c connected components, then $|S|\geq n-c+1$.*

Theorem 3.10. *Let $T=(V,E)$ be a tree of order n. If S is a global offensive alliance in T such that $\langle V\setminus S\rangle$ is a forest with c connected components, then*

(a) $|S|\geq\left\lceil\frac{3(n-c)+1}{4}\right\rceil$.

(b) *If the alliance* S *is strong,* $|S| \geq \left\lceil \frac{4n-3c+1}{5} \right\rceil$.

Proof. If $\langle V \backslash S \rangle$ is a forest with c connected components, then we have

$$\sum_{v \in V \backslash S} |N_S(v)| \leq |S| + c - 1.$$

Thus, by (14) we have

$$2(n - |S| - c) = \sum_{v \in V \backslash S} |N_{V \backslash S}(v)| \leq 2|S| + c - n - 1.$$

Hence, the result (a) follows. The proof of (b) is analogous by using (16) instead of (14). □

Figure 7:

The bound (a) is attained, for instance, for the left hand side graph of Figure 3. The bound (b) is attained for the left hand side graph of Figure 7. In both cases $V \backslash S = \{1, 2, 3, 4\}$.

4. Dual alliances

An alliance is called *dual* if it is both defensive and offensive. The *global dual alliance number* of a graph Γ, denoted by $\gamma_{a_d}(\Gamma)$, is defined as the minimum cardinality of any global dual alliance in Γ. In the case of *strong* alliances we denote the global dual alliance number by $\gamma_{\hat{a}_d}(\Gamma)$.

4.1. Global dual alliances in planar graphs

Theorem 4.1. *Let* $\Gamma = (V, E)$ *be a graph of order* n *and size* m *. Let* S *be a global dual alliance in* Γ *such that the subgraph* $\langle S \rangle$ *is planar. Then the following assertions hold.*

(a) *If* $n > 6$ *, then* $|S| \geq \left\lceil \frac{2m+n+48}{28} \right\rceil$.

(b) *If* $n > 6$ *and* $\langle S \rangle$ *is a triangle-free graph, then* $|S| \geq \left\lceil \frac{2m+n+32}{20} \right\rceil$.

(c) *If* $n > 4$ *and the alliance* S *is strong, then* $|S| \geq \left\lceil \frac{m+n+24}{13} \right\rceil$.

(d) *If* $n > 4$ *, the alliance* S *is strong, and* $\langle S \rangle$ *is a triangle-free graph, then* $|S| \geq \left\lceil \frac{m+n+16}{9} \right\rceil$.

Proof.

(a) Since $\sum_{v\in V\setminus S} |N_S(v)| = \sum_{v\in S} |N_{V\setminus S}(v)|$, by (14) and Claim 3.1 we obtain

$$\sum_{v\in V\setminus S} |N_S(v)| \geq \left(2m - \sum_{v\in S} |N_S(v)| - 2\sum_{v\in V\setminus S} |N_S(v)|\right) + n - |S|$$

Moreover, since the alliance S is defensive, by (2) we have

$$4|S| + 4\sum_{v\in S} |N_S(v)| \geq 2m + n. \tag{20}$$

From (5) and (20) the result immediately follows.

(b) Since S satisfies (6) and (20), the result immediately follows.

(c) If S is a global strong offensive alliance in Γ , then, by Claim 3.1 and (16) we have

$$\sum_{v\in V\setminus S} |N_S(v)| \geq \left(2m - \sum_{v\in S} |N_S(v)| - 2\sum_{v\in V\setminus S} |N_S(v)|\right) + 2(n - |S|),$$

and so we have

$$\sum_{v\in S} |N_S(v)| + 3\sum_{v\in V\setminus S} |N_S(v)| \geq 2m + 2(n - |S|). \tag{21}$$

Moreover, as the strong alliance S is defensive, by (21) and (7) we deduce

$$2\sum_{v\in S} |N_S(v)| \geq m + n - |S|. \tag{22}$$

From (5) and (22) the result immediately follows.

(d) Since S satisfies (6) and (22), the result immediately follows. □

Corollary 4.2. *Let $\Gamma = (V, E)$ be a planar graph of order n and size m.*

(a) *If $n > 6$, then $\gamma_{a_d}(\Gamma) \geq \left\lceil \frac{2m+n+48}{28} \right\rceil$.*

(b) *If $n > 6$ and Γ is a triangle-free graph, then $\gamma_{a_d}(\Gamma) \geq \left\lceil \frac{2m+n+32}{20} \right\rceil$.*

(c) *If $n > 4$, $\gamma_{\hat{a}_d}(\Gamma) \geq \left\lceil \frac{m+n+24}{13} \right\rceil$.*

(d) *If $n > 4$ and Γ is a triangle-free graph, then $\gamma_{\hat{a}_d}(\Gamma) \geq \left\lceil \frac{m+n+16}{9} \right\rceil$.*

Figure 8:

The bounds (a) is attained, for instance, for the right hand side graph of Figure 5; the bound (b) is attained for for the 3-cube; the bound (c) is attained for $\Gamma = K_1 * (K_2 \cup K_2)$, and for the right hand side graph of Figure 8; finally, the bound (d) is attained for the right hand side graph of Figure 7.

Theorem 4.3. *Let $\Gamma = (V, E)$ be a graph of order n. If S is a global dual alliance in Γ such that the subgraph $\langle S \rangle$ is planar connected with f faces, then*

(a) $|S| \geq \left\lceil \frac{2m+n-8f+16}{12} \right\rceil$.

(b) *If the alliance S is strong,* $|S| \geq \left\lceil \frac{m+n-4f+8}{5} \right\rceil$.

Proof. By Euler's formula, $\sum_{v \in S} |N_S(v)| = 2(|S|+f-2)$, and so the proof of (a) concludes by (20). The proof of (b) is analogous, by using (22) instead of (20). □

4.2. Global dual alliances in trees

Theorem 4.4. *Let T be a tree of order n. If S is a global dual alliance in T and the subgraph induced by S has c connected components, then*

(a) $|S| \geq \left\lceil \frac{3n+8c-2}{12} \right\rceil$.

(b) *If the alliance S is strong,* $|S| \geq \left\lceil \frac{2n+4c-1}{5} \right\rceil$.

Proof.

(a) As the subgraph induced by S is a forest with c connected components, then by (12) and (20) the result follows.

(b) Analogously to the case (a), the bound of $|S|$ follows from (12) and (22).

□

The bound (a) is attained, for instance, in the case of the left hand side graph of Figure 3 for $S = \{1,2,3,4\}$. The bound (b) is attained, for instance, in the case of the path graphs $\Gamma = P_n$, where $n = 2+3k$ $(k \in \mathbb{N})$. In this case $|S| = n-k$ and $c = k+1$.

By Corollary 3.9 we have that if S is a global strong dual alliance in T and the subgraph induced by S has c connected components, then $c \geq 2$.

Corollary 4.5. *For any tree T of order n,*

$$\gamma_{a_d}(T) \geq \left\lceil \frac{n+2}{4} \right\rceil \quad \textit{and} \quad \gamma_{\hat{a_d}}(T) \geq \left\lceil \frac{2n+7}{5} \right\rceil .$$

The above bounds are tight: the bound on γ_{a_d} is attained for the left hand side graph of Figure 8 and the bound on $\gamma_{\hat{a_d}}$ is attained for the path graph P_5.

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