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The Vertex Detour Number of a Graph

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Abstract

Let G be a connected graph of order $p \geq 2$. For any vertex x in G, a set S of vertices of G is an x-detour set if each vertex v of G lies on an x-y detour in G for some element y in S. The minimum cardinality of an x-detour set of G is defined as the x-detour number of G, denoted by $d_x(G)$ or simply d_x. An x-detour set of cardinality $d_x(G)$ is called a d_x-set of G. We determine bounds for it and characterize graphs which realize these bounds. We define an x-superior vertex of a graph and characterize graphs G for which $d_x(G) = 1$ in terms of x-superior vertices. Also, we find its relation with the detour number of a graph. It is shown that if G is a graph of order p, then $d_x(G) \leq p - e_D(x)$ for any vertex x in G. Connected graphs of order p with vertex detour numbers $p-1$ or $p-2$ for every vertex are characterized. For positive integers R, D and $n \geq 2$ with $R < D \leq 2R$, there exists a connected graph G with $rad_D\ G = R$, $diam_D\ G = D$ and $d_x(G) = n$ or $d_x(G) = n-1$ for every vertex x of G. For each triple D, n and p of integers with $1 \leq n \leq p - D + 1$ and $D \geq 4$, there is a connected graph G of order p, detour diameter D and $d_x(G) = n$ or $d_x(G) = n-1$ for every vertex x of G. Also, for an integer $p \geq 2$ and a number n with $1 \leq n \leq p-1$, there exists a connected graph G of order p and $d_x(G) = n$ or $d_x(G) = n-1$ for every vertex x of G.

Keywords: detour distance, detour number, vertex detour set, vertex detour number.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [6]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest x-y path in G. An x-y path of length $d(x, y)$ is called an x-y *geodesic.* The *diameter diam* G of a connected graph G is the length of any longest geodesic. For any vertex u of G, the *eccentricity of* u is $e(u) = \max\{d(u, v) : v \in V\}$. A vertex v of G such that $d(u, v) = e(u)$ is called an *eccentric vertex* of u. A *caterpillar* is a tree for which the removal of all the end vertices gives a path. A *double star* is a tree of diameter 3. For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at v. An *end-block* of G is a block containing exactly one cut-vertex of G. Thus every end-block is a branch of G.

The *closed interval* $I[x, y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a g*-set.* The geodetic number of a graph was introduced in [1, 7] and further studied in [3]. It was shown in [7] that determining the geodetic number of a graph is an NP-hard problem.

For vertices x and y in a connected graph G, the *detour distance* $D(x, y)$ is the length of a longest x-y path in G. For any vertex u of G, the *detour eccentricity* of u is $e_D(u) = \max\{D(u, v) : v \in V\}$. A vertex v of G such that $D(u, v) = e_D(u)$ is called a *detour eccentric vertex of* u. The *detour radius* R and *detour diameter* D of G are defined by $R = rad_D\ G = \min\{e(v) : v \in V\}$ and $D = diam_D\ G = \max\{e(v) : v \in V\}$ respectively. An x-y path of length $D(x, y)$ is called an x-y *detour.* The *closed interval* $I_D[x, y]$ consists of all vertices lying on some x-y detour of G, while for $S \subseteq V$, $I_D[S] = \bigcup_{x,y \in S} I_D[x, y]$. A set S of vertices is a *detour set* if $I_D[S] = V$, and the minimum cardinality of a detour set is the *detour number* $dn(G)$. A detour set of cardinality $dn(G)$ is called a *minimum detour set.* The detour number of a graph was introduced in [4] and further studied in [5]. The following theorem will be used in the sequel.

Theorem 1.1. [6] *Let v be a vertex of a connected graph G. The following statements are equivalent:*

(i) *v is a cut vertex of G.*

(ii) *There exist vertices u and w distinct from v such that v is on every u-w path.*

(iii) *There exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every u-w path.*

Theorem 1.2. [2] *For every connected graph* G, $rad_D\ G \leq\ diam_D\ G \leq 2\ \ rad_D\ G$.

Throughout the following G denotes a connected graph with at least two vertices.

2. The Vertex Detour Number

Definition 2.1. *Let* x *be a vertex of a connected graph* G. *A set* S *of vertices of* G *is an* x*-detour set if each vertex* v *of* G *lies on an* x - y *detour in* G *for some element* y *in* S. *The minimum cardinality of an* x*-detour set of* G *is defined as the* x*-detour number of* G *and is denoted by* $d_x(G)$ *or simply* d_x. *An* x*-detour set of cardinality* $d_x(G)$ *is called a* d_x*-set of* G.

Result 2.2. *For any vertex* x *in* G, x *does not belong to any* d_x*-set of* G.

Example 2.3.

(i) $d_x(K_p) = 1$ *for every vertex* x *in* K_p.

(ii) *For the graph* G *given in Figure 2.1, the minimum vertex detour sets and the vertex detour numbers are given in Table 2.1.*

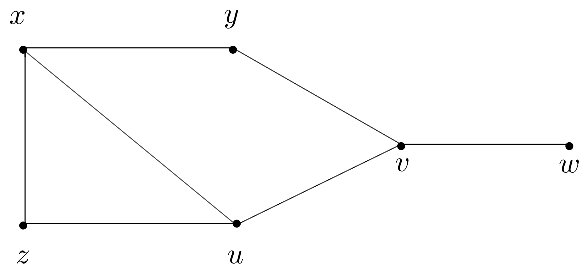

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Figure 2.1

Vertex	Minimum Vertex Detour Sets	Vertex Detour Number
x	$\{y,w\}$, $\{z,w\}$, $\{u,w\}$	2
y	$\{w\}$	1
z	$\{w\}$	1
u	$\{w\}$	1
v	$\{y,w\}$, $\{z,w\}$, $\{u,w\}$	2
w	$\{y\}$, $\{z\}$, $\{u\}$	1

Table 2.1.

Remark 2.4. *Let x be any vertex of G. Then for any vertex y belonging to a d_x-set S_x of G, the internal vertices of an x-y detour may belong to S_x. For the graph G given in Figure 2.1, $S_x = \{u, w\}$ is a d_x-set of G and u belonging to S_x is an internal vertex of the x-w detour : x, z, u, v, w. Also $S_x = \{y, w\}$ is a d_x-set of G such that y is not an internal vertex of any x-w detour and w is not an internal vertex of any x-y detour in G.*

Theorem 2.5. *Let x be any vertex of a connected graph G.*

(i) *Every end-vertex of G other than the vertex x (whether x is end-vertex or not)* belongs to every d_x-set.

(ii) No cut vertex of G belongs to any d_x-set.

Proof. (i) Let x be any vertex of G. By Result 2.2, x does not belong to any d_x-set. So let $v \neq x$ be an end-vertex of G. Then v is the terminal vertex of an x-v detour and v is not an internal vertex of any detour so that v belongs to every d_x-set of G.

(ii) Let y be a cut vertex of G. Then by Theorem 1.1, there exists a partition of the set of vertices $V - \{y\}$ into subsets U and W such that for any vertex $u \in U$ and $w \in W$, the vertex y is on every u-w path. Hence, if $x \in U$, then for any vertex w in W, y lies on every x-w path so that y is an internal vertex of an x-w detour. Let S_x be any d_x-set of G. Suppose $S_x \cap W = \Phi$. Let $w_1 \in W$. Since S_x is an x-detour set, there exists an element z in S_x such that w_1 lies in some x-z detour $P : x = z_0, z_1, \ldots, w_1, \ldots, z_n = z$ in G. Then the x-w_1 subpath of P and w_1-z subpath of P both contain y so that P is not a path in G. Hence $S_x \cap W \neq \Phi$. Let $w_2 \in S_x \cap W$. Then y is an internal vertex of an x-w_2 detour. If $y \in S_x$, let $S = S_x - \{y\}$. It is clear that every vertex that lies on an x-y detour also lies on an x-w_2 detour. Hence it follows that S is an x-detour set of G, which is a contradiction to S_x is a minimum x-detour set of G. Thus y does not belong to any d_x-set. Similarly if $x \in W$, y does not belong to any d_x-set. If $x = y$, then by Result 2.2, y does not belong to any d_x-set. □

Note 2.6. *Even if x is an end-vertex of G, x does not belong to any d_x-set by Result 2.2.*

Corollary 2.7. *Let T be a tree with t end-vertices. Then $d_x(T) = t - 1$ or $d_x(T) = t$ according to whether x is an end-vertex or not. In fact, if W is the set of all end-vertices of T, then $W - \{x\}$ is the unique d_x-set of T.*

Proof. Let W be the set of all end-vertices of T. It follows from Result 2.2 and Theorem 2.5 that $W - \{x\}$ is the unique d_x-set of T for any end-vertex x in T and W is the unique d_x-set of T for any cut vertex x in T. Thus $W - \{x\}$ is the unique d_x-set of T for any vertex x in T. □

Corollary 2.8. *Let P_n be a non-trivial path. Then $d_x(P_n) = 1$ or $d_x(P_n) = 2$ according as x is an end-vertex or not.*

Corollary 2.9. *For any star $K_{1,n}(n \geq 2)$, $d_x(K_{1,n}) = n-1$ or $d_x(K_{1,n}) = n$ according as x is an end-vertex or not.*

Theorem 2.10. *For any hamiltonian graph G, $d_x(G) = 1$ for every vertex x in G.*

Proof. Let C be a hamiltonian cycle of G. Let x be any vertex of G and let y be any adjacent vertex of x in C. Clearly every vertex of G lies on a detour joining x and y. Thus $d_x(G) = 1$ for every vertex x in G. □

Corollary 2.11. *For the n-cube $Q_n(n \geq 2)$, $d_x(Q_n) = 1$ for every vertex x in Q_n.*

Remark 2.12. *The converse of Theorem 2.10 is false. For the graph G given in Figure 2.2, $d_x(G) = 1$ for every vertex x in G. But G is not hamiltonian.*

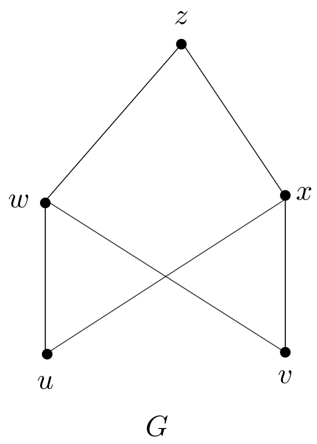

Figure 2.2

Corollary 2.13. *For any cycle C, $d_x(C) = 1$ for every vertex x in C.*

Corollary 2.14. *For the wheel $W_n = K_1 + C_{n-1}(n \geq 4)$, $d_x(W_n) = 1$ for every vertex x in W_n.*

Theorem 2.15. *If a connected graph G has a hamiltonian path, then $d_x(G) = 1$ for at least two vertices.*

Proof. Let P be a hamiltonian path with end-vertices x and y. Then it is clear that $d_x(G) = d_y(G) = 1$. □

The following theorem is an easy consequence of the definition of the vertex detour number.

Theorem 2.16.

(i) *For* $m = n = 1$, $d_x(K_{m,n}) = 1$ *for every vertex* x *in* G.

(ii) *For* $m, n \geq 2$, $d_x(K_{m,n}) = 1$ *for every vertex* x *in* G.

(iii) *For* $m = 1$ *and* $n \geq 2$, $d_x(K_{m,n}) = n$ *or* $d_x(K_{m,n}) = n - 1$ *for every vertex* x *of* G.

Theorem 2.17. *Let* G *be a connected graph with cut vertices and let* S_x *be an* x*-detour set of* G. *Then every branch of* G *contains an element of* $S_x \cup \{x\}$.

Proof. Suppose that there is a branch B of G at a cut vertex v such that B contains no vertex of $S_x \cup \{x\}$. Then clearly, $x \in V - (S_x \cup V(B))$. Let $u \in V(B) - \{v\}$. Since S_x is an x-detour set, there exists an element $y \in S_x$ such that u lies in some x-y detour $P : x = u_0, u_1, \ldots, u, \ldots, u_n = y$ in G. By Theorem 1.1 the x-u subpath of P and u-y subpath of P both contain v, and it follows that P is not a path, contrary to assumption. □

Since every end-block B is a branch of G at some cut-vertex, it follows by Theorems 2.5 and 2.17 that every d_x-set of G together with the vertex x contains at least one vertex from B that is not a cut-vertex. Thus the following corollaries are consequences of Theorem 2.17.

Corollary 2.18. *If* G *is a connected graph with* k *end-blocks, then* $d_x(G) \geq k - 1$ *for every vertex* x *in* G.

Corollary 2.19. *If* k *is the maximum number of blocks to which a vertex in a graph* G *belongs, then* $d_x(G) \geq k - 1$ *for every vertex* x *in* G.

Theorem 2.20. *For any vertex* x *in* G, $1 \leq d_x(G) \leq p - 1$.

Proof. It is clear from the definition of d_x-set that $d_x(G) \geq 1$. Also since the vertex x does not belong to any d_x-set, it follows that $d_x(G) \leq p - 1$. □

Remark 2.21. *The bounds for* $d_x(G)$ *in Theorem 2.20 are sharp. For the cycle* C_n, $d_x(C_n) = 1$ *for every vertex* x *in* C_n. *Also for any non-trivial path* P_n, $d_x(P_n) = 1$ *for any end-vertex* x *in* P_n. *For the graph* K_2, $d_x(K_2) = p - 1$ *for every vertex* x *in* K_2.

Now we proceed to characterize graphs for which the lower bound in Theorem 2.20 is attained. For this, we introduce the following definition.

Definition 2.22. *Let* x *be any vertex in* G. *A vertex* y *in* G *is said to be an* x*-detour superior vertex if for any vertex* z *with* $D(x, y) < D(x, z)$, z *lies on an* x-y *detour.*

Example 2.23.

(i) *In the even cycle* C_{2n} *, both eccentric and detour eccentric vertices of* x *are* x *-detour superior vertices.*

(ii) *For the graph* G *given in Figure 2.3, the vertex detour superior vertices are given in Table 2.2.*

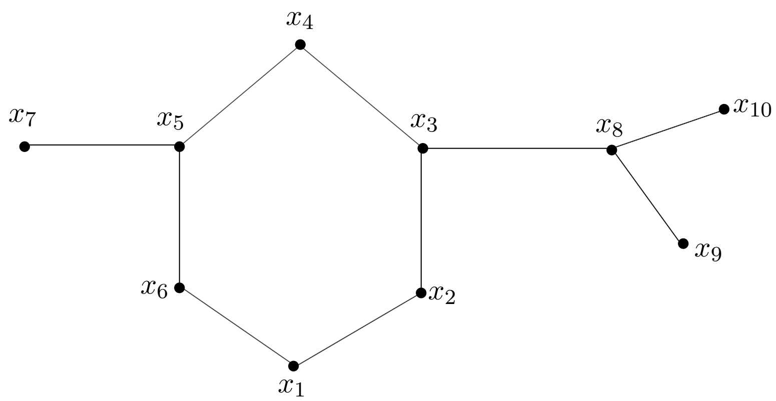

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Figure 2.3

Vertex	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
Vertex Detour Supeior Vertices	x_9 x_{10}	x_9 x_{10}	x_2 x_4 x_7	x_9 x_{10}	x_9 x_{10}	x_7	x_9 x_{10}	x_2 x_4 x_7	x_2 x_4 x_7	x_2 x_4 x_7

Table 2.2

We give below a property related with detour eccentric vertex of x and x-detour superior vertex in a graph G.

Theorem 2.24. *Let* x *be any vertex in* G*. Then every detour eccentric vertex of* x *is an* x *-detour superior vertex.*

Proof. Let y be a detour eccentric vertex of x so that $e_D(x) = D(x,y)$. If y is not an x-detour superior vertex, then there exists a vertex z in G such that $D(x,y) < D(x,z)$ and z does not lie on any x - y detour and hence $e_D(x) < D(x,z)$, which is a contradiction. □

Note 2.25. *The converse of Theorem 2.24 is not true. For the even cycle* C_{2n}, *the eccentric vertex of* x *is an* x*-detour superior vertex but it is not a detour eccentric vertex of* x.

Theorem 2.26. *Let* G *be a connected graph. For a vertex* x *in* G, $d_x(G) = 1$ *if and only if there exists an* x*-detour superior vertex* y *in* G *such that every vertex of* G *is on an* x-y *detour.*

Proof. Let $d_x(G) = 1$ and $S_x = \{y\}$ be a d_x-set of G. If y is not an x-detour superior vertex, then there is a vertex z in G with $D(x, y) < D(x, z)$ and z does not lie on any x-y detour. Thus S_x is not a d_x-set of G, which is a contradiction. The converse is clear from the definition. □

In the following theorem, we establish the relationship between the vertex detour number of a vertex and the detour number of a graph.

Theorem 2.27. *For any vertex* x *in* G, $dn(G) \leq d_x(G) + 1$.

Proof. Let x be any vertex of G and let S_x be a d_x-set of G. Then every vertex of G lies on an x-y detour for some y in S_x. Thus $S_x \cup \{x\}$ is a detour set of G. Since $dn(G)$ is the minimum cardinality of a detour set, it follows that $dn(G) \leq d_x(G) + 1$. □

Note 2.28. *The bound in Theorem 2.27 is sharp. For the complete graph* K_p, $dn(K_p) = d_x(K_p) + 1$ *for every vertex* x *in* K_p.

3. Bounds for the Vertex Detour Number of a Graph

We have seen that if G is a connected graph of order $p \geq 2$, then $1 \leq d_x(G) \leq p - 1$ for any vertex x in G. Also we have for a vertex x in G, $d_x(G) = 1$ if and only if there is an x-detour superior vertex y such that every vertex of G is on an x-y detour. In the following theorem we give an improved upper bound for the vertex detour number of a graph.

Theorem 3.1. *For any vertex* x *in a connected graph* G *of order* p, $d_x(G) \leq p - e_D(x)$.

Proof. Let x be any vertex of G and v a detour eccentric vertex of x. Then $D(x, v) = e_D(x)$. Let $P : x = x_0, x_1, \ldots, x_k = v$ be an x-v detour in G. Let $S = V(G) - \{x_0, x_1, \ldots, x_{k-1}\}$. Since each $x_i (0 \leq i \leq k - 1)$ lies on an x-v detour, S is an x-detour set of G so that $d_x(G) \leq p - e_D(x)$. □

Remark 3.2. *The bound in Theorem 3.1 is sharp. For the cycle* C_p, $d_x(C_p) = 1 = p - e_D(x)$ *for every vertex* x *in* C_p. *Also for the graph* G *given in Figure 2.3,* $p = 10$, $e_D(x_7) = 7$ *and* $S = \{x_4, x_9, x_{10}\}$ *is a* d_{x_7}*- set so that* $d_{x_7} = 3$. *Thus* $d_{x_7} = p -$

$e_D(x_7)$. The inequality in Theorem 3.1 can also be strict. For the same graph G given in Figure 2.3, $e_D(x_3) = 5$ and $S = \{x_4, x_7, x_9, x_{10}\}$ is a d_{x_3}-set so that $d_{x_3}(G) = 4$. Thus $d_{x_3}(G) < p - e_D(x_3)$.

Corollary 3.3. *If G is a connected graph of order p and detour diameter D, then $d_x(G) \leq p - D/2$ for every vertex x in G.*

Proof. Since $R \leq e_D(x)$ for every vertex x in G, it follows from Theorem 1.2 and Theorem 3.1 that $d_x(G) \leq p - D/2$. □

Remark 3.4. *The bound in Corollary 3.3 is sharp. For the star $K_{1,p-1}(p \geq 3)$, by Corollary 2.9, $d_x(K_{1,p-1}) = p - 1 = p - D/2$ for the cut vertex x in $K_{1,p-1}$. Also, the inequality in Corollary 3.3 can be strict. For the star $K_{1,p-1}(p \geq 3)$, by Corollary 2.9, $d_x(K_{1,p-1}) = p - 2 < p - D/2$ for an end vertex x in $K_{1,p-1}$.*

Theorem 3.5. *Let G be a connected graph of order $p \geq 2$. Then $G = K_2$ if and only if $d_x(G) = p - 1$ for every vertex x in G.*

Proof. If $G = K_2$, then $d_x(G) = 1 = p - 1$ for every vertex x in K_2. Conversely, let $d_x(G) = p - 1$ for every vertex x in G. If $D \geq 2$, then there exists a vertex x in G such that $e_D(x) \geq 2$. By Theorem 3.1, $d_x(G) \leq p - e_D(x) \leq p - 2$, which is a contradiction. Thus $D = 1$ so that $G = K_2$. □

Theorem 3.6. *Let G be a connected graph of order $p \geq 2$ and $G \neq K_3$. Then $G = K_{1,p-1}$ if and only if $d_x(G) = p - 1$ or $d_x(G) = p - 2$ for every vertex x of G.*

Proof. If $G = K_{1,p-1}$, then by Corollary 2.9, $d_x(G) = p - 1$ or $d_x(G) = p - 2$ for every vertex x of G. Conversely, suppose $d_x(G) = p - 1$ or $d_x(G) = p - 2$ for every vertex x of G. If $p = 2$, then $G = K_2 = K_{1,p-1}$. If $p = 3$, then $G = P_3 = K_{1,p-1}$. Let $p \geq 4$. We prove that G is a star. Suppose G is not a star. If G is a tree, then G has at most $p - 2$ end-vertices. By Corollary 2.7, $d_x(G) \leq p - 3$ if x is an end-vertex, which is a contradiction. Now, if G is not a tree. Let $c(G)$ be the length of a longest cycle, say C, in G. If $c(G) \geq 4$, then $D \geq 3$ so that $e_D(x) \geq 3$ for some vertex x in G. Hence by Theorem 3.1, $d_x(G) \leq p - 3$, which is a contradiction. If $c(G) = 3$, let u, v, w, u be a triangle in G. Since $p \geq 4$, there exists $x \in V(G) - \{u, v, w\}$ such that x is adjacent to at least one of u, v, w, say $xu \in E(G)$. Then x, u, v, w is a path in G so that $e_D(x) \geq 3$. Then by Theorem 3.1, $d_x(G) \leq p - 3$, which is a contradiction. Thus G is a star. □

Theorem 3.7. *Let G be a connected graph of order $p \geq 3$. Then $G = K_3$ if and only if $d_x(G) = p - 2$ for every vertex x in G.*

Proof. If $G = K_3$, then it is clear that $d_x(G) = 1 = p - 2$ for every vertex x in G. Conversely, let $d_x(G) = p - 2$ for every vertex x in G. If $D \geq 3$, then $e_D(x) \geq 3$ for some vertex x in G. Hence by Theorem 3.1, $d_x(G) \leq p - e_D(x) \leq p - 3$, which is a contradiction. If $D = 1$, then $G = K_2$ and so $d_x(G) = p-1$ for every vertex x in G, which is a contradiction. Hence $D = 2$. If $p \geq 4$, then $G = K_{1,p-1}$ and hence by Corollary 2.9, $d_x(G) = p-1$ for the cut vertex x in G, which is a contradiction. Thus $p = 3$ and so G is either P_3 or K_3. If $G = P_3$, then by Corollary 2.7 $d_x(G) = 2 = p-1$ for the cut vertex x in G, which is a contradiction. If $G = K_3$, then $d_x(G) = 1 = p-2$ for every vertex x in G. Thus $G = K_3$ is the only graph which satisfies the requirement of the theorem. □

Theorem 3.8. *Let G be a connected graph of order $p \geq 5$. Then $d_x(G) = p-2$ or $d_x(G) = p-3$ for every vertex x of G if and only if G is a double star or $K_{1,p-1}+e$.*

Proof. It is straightforward to verify that if G is a double star or $K_{1,p-1}+e$, then $d_x(G) = p-2$ or $d_x(G) = p-3$ for every vertex x of G. For the converse, let G be a connected graph of order $p \geq 5$ such that $d_x(G) = p-2$ or $d_x(G) = p-3$ for every vertex x of G. If $D \leq 2$, then G is the star $K_{1,p-1}$ and so by Corollary 2.9, $d_x(G) = p-1$ for the cut vertex x in G, which is a contradiction.

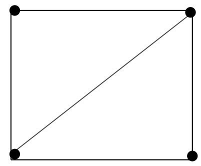
Let $D = 3$. If G is a tree, then G is a double star and the result follows from Corollary 2.7. Assume that G is not a tree. Let $c(G)$ denote the length of a longest cycle in G. Since $D = 3$, it follows that $c(G) \leq 4$. We consider two cases.

Case 1. Let $c(G) = 4$. Let $C_4 : v_1, v_2, v_3, v_4, v_1$ be a 4-cycle in G. Since $p \geq 5$ and G is connected, there exists a vertex x not on C_4 such that x is adjacent to some vertex, say v_1, of C_4. Then x, v_1, v_2, v_3, v_4 is a path of length 4 in G so that $D \geq 4$, which is a contradiction.

Case 2. Let $c(G) = 3$. If G contains two or more triangles, then $c(G) = 4$ or $D \geq 4$, which is a contradiction. Hence G contains an unique triangle $C_3 : v_1, v_2, v_3, v_1$. Now, we prove that there is exactly one vertex on C_3 of degree at least 3. If there are two or more vertices of C_3 having degree 3 or more, then $D \geq 4$, which is a contradiction. Thus exactly one vertex in C_3 has degree 3 or more. Since $D = 3$, it follows that $G = K_{1,p-1}+e$. Now, it follows from Theorems 2.5 and 2.17 that $d_x(G) = p-2$ or $d_x(G) = p-3$ according as x is a cut vertex or not.

If $D \geq 4$, then $e_D(x) \geq 4$ for some vertex x in G. Hence by Theorem 3.1, $d_x(G) \leq p - e_D(x) \leq p-4$, which is a contradiction. □

Remark 3.9. *Theorem 3.8 is not true for $p = 4$. For the graph G given in Figure 3.1, $p = 4$ and $d_x(G) = 1 = p-3$ for every vertex x in G. However, G is neither a double star nor $K_{1,p-1}+e$.*



G

Figure 3.1

Theorem 3.10. *For every non-trivial tree T, $d_x(T) = p - D$ or $d_x(T) = p - D + 1$ for every vertex x of T if and only if T is a caterpillar.*

Proof. Let T be any non-trivial tree. Let $P : u = v_0, v_1, \ldots, v_D = v$ be a diametral path. Let k be the number of end vertices of T and l be the number of internal vertices of T other than $v_1, v_2, \ldots, v_{D-1}$. Then $D-1+l+k = p$. By Corollary 2.7, $d_x(T) = k$ or $d_x(T) = k-1$ for every vertex x of T and so $d_x(T) = p-D-l+1$ or $d_x(T) = p-D-l$ for every vertex x of T. Hence $d_x(T) = p-D+1$ or $d_x(T) = p-D$ for every vertex x of T if and only if $l = 0$, if and only if all the internal vertices of T lie on the diametral path P, if and only if T is a caterpillar. □

For every connected graph G, $rad_D\ G \leq\ diam_D\ G \leq 2\, rad_D\ G$. Chartrand, Escuadro and Zhang [2] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the detour radius and detour diameter, respectively, of some connected graph. This theorem can also be extended so that the vertex detour number can be prescribed when $a < b \leq 2a$.

Theorem 3.11. *For positive integers R, D and $n \geq 2$ with $R < D \leq 2R$, there exists a connected graph G with $rad_D\ G = R$, $diam_D\ G = D$ and $d_x(G) = n$ or $d_x(G) = n-1$ for every vertex x of G.*

Proof. If $R = 1$, then $D = 2$. Take $G = K_{1,n}$. Then by Corollary 2.9, $d_x(G) = n$ or $d_x(G) = n-1$ for every vertex x of G. Now, let $R \geq 2$. We construct a graph G with the desired properties as follows.

Let $C_{R+1} : v_1, v_2, \ldots, v_{R+1}, v_1$ be a cycle of order $R+1$ and let $P_{D-R+1} : u_0, u_1, \ldots, u_{D-R}$ be a path of order $D-R+1$. Let H be a graph obtained from C_{R+1} and P_{D-R+1} by identifying v_1 in C_{R+1} and u_0 in P_{D-R+1}. Now, add $n-2$ new vertices $w_1, w_2, \ldots, w_{n-2}$ to H by joining each vertex $w_i (1 \leq i \leq n-2)$ to the vertex u_{D-R-1} and obtain the graph G of Figure 3.2. Now $rad_D\ G = R$, $diam_D\ G = D$ and G has $n-1$ end vertices.

Case 1. Let R be even. If $R = 2$, then $d_x(G) = n$ or $d_x(G) = n-1$ according as $x \in \{v_1, u_1, u_2, u_3, \ldots, u_{D-R-1}\}$ or $x \in \{v_2, v_3, u_{D-R}, w_1, w_2, w_3, \ldots, w_{n-2}\}$. If $R \geq 4$, then $d_x(G) = n$ or $d_x(G) = n-1$ according as $x \in \{v_1, v_3, v_4, \ldots, v_R, u_1, u_2, u_3, \ldots, u_{D-R-1}\}$ or $x \in \{v_2, v_{R+1}, u_{D-R}, w_1, w_2, w_3, \ldots, w_{n-2}\}$.

Case 2. Let R be odd. If $R = 3$, then $d_x(G) = n$ or $d_x(G) = n - 1$ according as $x \in \{v_1, u_1, u_2, u_3, \ldots, u_{D-R-1}\}$ or $x \in \{v_2, v_3, v_4, u_{D-R}, w_1, w_2, w_3, \ldots, w_{n-2}\}$. If $R \geq 5$, then $d_x(G) = n$ or $d_x(G) = n - 1$ according as $x \in \{v_1, v_3, v_4, \ldots, v_{(R+1)/2}, v_{(R+5)/2}, \ldots, v_R, u_1, u_2, u_3, \ldots, u_{D-R-1}\}$ or $x \in \{v_2, v_{(R+3)/2}, v_{R+1}, u_{D-R}, w_1, w_2, w_3, \ldots, w_{n-2}\}$. Thus $d_x(G) = n$ or $d_x(G) = n - 1$ for every vertex x of G. □

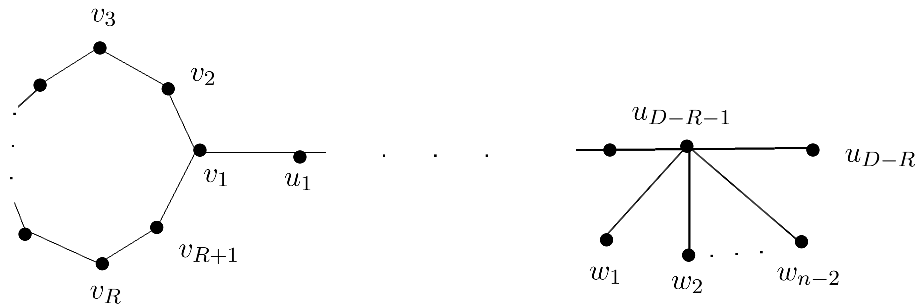


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Figure 3.2

The graph G of Figure 3.2 is the smallest graph with the properties described in Theorem 3.11. We leave the following problem as an open question.

Problem 3.12. *For positive integers R, D and $n \geq 2$ with $R = D$, does there exist a connected graph G with rad_D $G = R$, $diam_D$ $G = D$ and $d_x(G) = n$ or $d_x(G) = n - 1$ for every vertex x of G.*

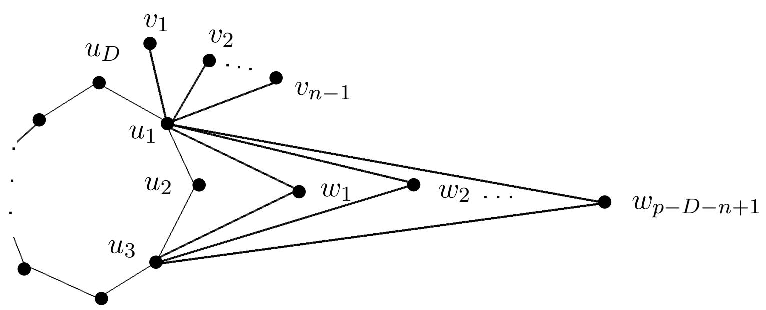
In the following, we construct a graph of prescribed order, detour diameter and vertex detour number under suitable conditions.

Theorem 3.13. *For each triple D, n and p of integers with $1 \leq n \leq p - D + 1$ and $D \geq 4$, there is a connected graph G of order p, detour diameter D and $d_x(G) = n$ or $d_x(G) = n - 1$ for every vertex x of G.*

Proof. Let G be a graph obtained from the cycle $C_D : u_1, u_2, \ldots, u_D, u_1$ of order D by (i) adding $n-1$ new vertices $v_1, v_2, \ldots, v_{n-1}$ and joining each vertex $v_i (1 \leq i \leq n-1)$ to u_1 and (ii) adding $p - D - n + 1$ new vertices $w_1, w_2, \ldots, w_{p-D-n+1}$ and joining each vertex $w_i (1 \leq i \leq p - D - n + 1)$ to both u_1 and u_3. The graph G has order p and detour diameter D and is shown in Figure 3.3. If $n = 1$, then $d_x(G) = n$ for every vertex x in G. If $n \geq 2$, then we consider two cases.

Case 1. Let D be even. If $D = 4$, then $d_x(G) = n$ or $d_x(G) = n-1$ according as $x = u_1$ or $x \in \{u_2, u_3, u_4, v_1, v_2, \ldots, v_{n-1}, w_1, \ldots, w_{p-D-n+1}\}$. If $D \geq 6$, then $d_x(G) = n$ or $d_x(G) = n - 1$ according as $x \in \{u_1, u_2, \ldots, u_{D/2}, u_{(D+4)/2}, \ldots, u_{D-1}, w_1, \ldots, w_{p-D-n+1}\}$ or $x \in \{u_{(D+2)/2}, u_D, v_1, v_2, \ldots, v_{n-1}\}$.

Case 2. Let D be odd. Clearly $d_x(G) = n$ or $d_x(G) = n-1$ according as $x \in \{u_1, u_2, \ldots, u_{D-1}, w_1, \ldots, w_{p-D-n+1}\}$ or $x \in \{u_D, v_1, v_2, \ldots, v_{n-1}\}$. Thus $d_x(G) = n$ or $d_x(G) = n-1$ for every vertex x of G. □



G

Figure 3.3

Theorem 3.14. *Let $p \geq 2$ be any integer. For $1 \leq n \leq p-1$ there exists a connected graph G with order p and $d_x(G) = n$ or $d_x(G) = n-1$ for every vertex x of G.*

Proof. For $p = 2$, $G = K_2$ has the desired properties. For $p = 3$, $G = C_3$ or P_3 has the desired properties according as $n = 1$ or $n = 2$. For $p \geq 4$, we consider three cases.

Case 1. Let $n = 1$. Then $G = C_p$ has the desired properties.

Case 2. Let $2 \leq n \leq p-2$. Then $p-n+1 \geq 3$. The graph G is obtained from the cycle $C_{p-n+1} : u_1, u_2, \ldots, u_{p-n+1}, u_1$ by adding the $n-1$ new vertices $v_1, v_2, \ldots, v_{n-1}$ and joining these to u_1. The graph G is shown in Figure 3.4.

Subcase a. Let $p-n+1$ be even. If $p-n+1 = 4$, then $d_x(G) = n$ or $d_x(G) = n-1$ according as $x = u_1$ or $x \in \{u_2, u_3, u_4, v_1, v_2, \ldots, v_{n-1}\}$. If $p-n+1 \geq 6$, then $d_x(G) = n$ or $d_x(G) = n-1$ according as $x \in \{u_1, u_3, u_4, \ldots, u_{(p-n+1)/2}, u_{(p-n+5)/2}, \ldots, u_{p-n}\}$ or $x \in \{u_2, u_{(p-n+3)/2}, u_{p-n+1}, v_1, v_2, \ldots, v_{n-1}\}$.

Subcase b. Let $p-n+1$ be odd. If $p-n+1 = 3$, then $d_x(G) = n$ or $d_x(G) = n-1$ according as $x = u_1$ or $x \in \{u_2, u_3, v_1, v_2, \ldots, v_{n-1}\}$. If $p-n+1 \geq 5$, then $d_x(G) = n$ or $d_x(G) = n-1$ according as $x \in \{u_1, u_3, u_4, \ldots, u_{p-n}\}$ or $x \in \{u_2, u_{p-n+1}, v_1, v_2, \ldots, v_{n-1}\}$. Thus $d_x(G) = n$ or $d_x(G) = n-1$ for every vertex x of G.

Case 3. Let $n = p-1$. Then $G = K_{1,p-1}$ has the desired properties. □

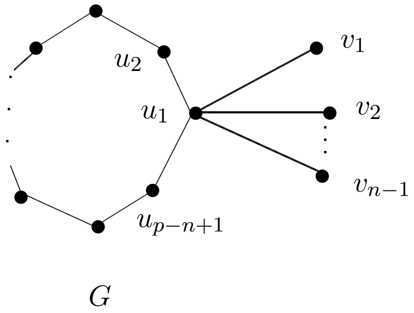


Figure 3.4

4. Acknowledgements

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