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Domination in Discrete Structures - New Directions

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As of today, the theory of domination has come to stay as one of the main streams of research, especially in the theory of graphs, and is getting stronger by numerous publications in internationally reputed research journals across the globe. Developments on the notion in the regime of hypergraphs has been reviewed for the first time in this volume, throwing open a very nascent field for further research with challenging open problems. In the world scenario, there has been significant contribution to the theory of domination in its mathematical programming formulation. In the recent past, five major surveys starting with the one by Cockayne and Hedetniemi [3], followed by a creative survey in the form of a monograph by Walikar et al. [7], a special issue of Discrete Mathematics entitled *Topics on Domination* edited by Hedetniemi and Laskar [6], a text book on *Fundamentals of Domination in Graphs* and a book on *Advanced Topics* both by Haynes et al. [4, 5], have enormously raised the tempo of research in this field worldwide. It is somewhat surprising that the algorithmic and computational aspects of domination and its applications are still to be reviewed even though there is huge amount of literature on this front. In such a situation we consider it prudent to confine the scope of this note to the specific problems raised in GDDDSA and discuss them in the global perspective.

There have been a number of contexts in the proof of specific results that one encounters transforming a given dominating set D by exchanging one of its vertices with a vertex from $V - D$. The first such instance in the literature appears in the work of C. Benzaken and P. L. Hammer [2]. In the subsequent published literature it resurfaces in the famous theorem of Allan and Laskar [1] that for any $K_{1,3}$-free graph G, $\gamma(G) = i(G)$. This prompts one to define a binary relation ρ on $V(G)$ by saying $u \ \rho \ v$ if either $u = v$ or there exists a dominating set D such that $u \in D$, $v \in V - D$ and $(D - \{u\}) \cup \{v\}$ is a dominating set. Clearly ρ is a symmetric and reflexive binary relation.

Problem 1. *For which graphs G is ρ transitive?*

Problem 2. *Study variations of ρ when D is a minimal or minimum dominating set in the above definition.*

A graph G is said to be *very excellent*, if there is a γ-set S of G such that to each vertex $u \in V - S$, there exists a vertex $v \in S$ such that $(S - v) \cup \{u\}$ is a γ-set of G.

Problem 3. *Given a graph theoretic property P and a graph G with property P, can G be embedded in a very excellent graph H with property P ?*

We next present several problems involving the number of dominating sets containing individual vertices.

Problem 4. *Let $d^\circ(v), d^m(v)$ and $d^u(v)$ denote respecting the number of minimum, minimal and in general any dominating set containing v. Clearly $d^\circ(v) \leq d^m(v) \leq d^u(v)$. Study the graphs in which $d^\circ(v)$ (respectively $d^m(v)$ and $d^u(v)$) is independent of the choice of v.*

Problem 5. *Characterize graphs for which $d^\circ(v) = d^m(v) = d^u(v)$ for all $v \in V$.*

Problem 6. *Let k be a positive integer with $\gamma \leq k \leq n$, where $n = |V(G)|$. Let $d_k(v) =$ number of dominating sets of cardinality k which contain v. Then $d_k(v)$ is called the k-dom-degree of v. The sequence $d_k(v_1), \ldots, d_k(v_n)$ arranged in non-descending order is called k-dom-degree sequence of G. Characterize k-dom-degree sequences.*

Next are some problems involving the relation of a dominating set to the rest of the graph.

Problem 7. *For any dominating set D, let $m(D, V - D)$ denote the number of edges with one end in D and other end in $V - D$. Let $\mu(G) = \min_{D \in \mathcal{D}(G)} m(D, V - D)$. Find either the exact value of $\mu(G)$ or 'good' bounds for it.*

Problem 8. *Characterize the class of dominating sets D for which $\mu(G) = m(D, V - D)$.*

Problem 9. *Investigate the analogous parameters when D is restricted to the class of minimal or minimum dominating sets.*

Problem 10. *Does there exist a domatic partition of a graph G in which at least one member is a γ-set?*

The seminal paper in the subject of colored problems appears in this issue. Some subsequent work has been done, but this is an essentially open area of research. A couple of sample problems involve the ratio $\gamma_{cpl}(G)/\gamma(G)$.

Problem 11. *Appearing in a paper of Suk Seo and P.J.Slater is a tree T of order 12 for which $\gamma_{cpl}(G)/\gamma(G) = 8/5$, and B. Stodolsky has shown that this is the minimum possible for all trees of order $n \geq 2$. Let t_n be the minimum value of $\gamma_{cpl}(T)/\gamma(T)$ for all trees of order n. What is $\liminf_{n\to\infty}(t_n)$?*

Problem 12. *For a rational number $r = p/q$ with $1 \leq r \leq 2$, what is the smallest order of a graph G with $\gamma_{cpl}(G)/\gamma(G) = r$?*

References

[1] R. B. Allan and R. C. Laskar, On domination and independent domination numbers of a graph, *Discrete Math.,* **23** (1978), 73 - 76.

[2] C. Benzaken and P. L. Hammer, Linear Separation of Dominating Sets in Graphs, *Annals of Discrete Mathematics,* **3** (1978) 1 - 10.

[3] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graphs, *Networks,* **7** (1977), 247 - 261.

[4] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[5] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Domination in Graphs - Advanced Topics*, Marcel Dekker, Inc., New York, 1998.

[6] S. T. Hedetniemi and R. C. Laskar, (Editors) *Topics on Domination,* Volume 48, North Holland, New York, 1990.

[7] H. B. Walikar, B. D. Acharya and E. Sampathkumar, Recent developments in the theory of domination in graphs, *MRI Lecture Notes in Mathematics Mehta Research Institute, Allahabad,* Volume 1, 1979.