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Colored Problems in Graphs

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Abstract

This paper introduces the concept of colored problems for graph parameters. Specifically, assume that we have a partition $\{S_1, S_2, \ldots, S_t\}$ of vertex set $V(G)$. One seeks to optimize the order of a set S with a certain property such that S must also contain either none or all of the elements of each S_i. For example, one might seek a minimum cardinality dominating set D with the property that $D \cap S_i \neq \emptyset$ implies that $S_i \subseteq D$ for $1 \leq i \leq t$. As another example, one might seek a maximum cardinality independent set A with the property that either $A \cap S_i = \emptyset$ or $S_i \subseteq A$ for $1 \leq i \leq t$. The focus, in particular, will be on coupled parameters for which each S_i has cardinality at most two.

Keywords: Colored problems, coupled domination, coupled independence

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1. Introduction

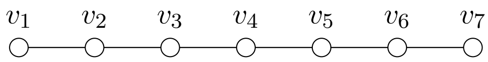
In this paper the general class of "colored problems" in graphs is introduced. For these problems the elements of the graph are partitioned into color classes. Then, when one is looking for something like a dominating set, an independent set of vertices or edges, or a maximum length cycle, the solution set is required to use all or none of the elements of each given color. Here colored-domination and colored-independence will be considered.

Given a graph $G = (V, E)$, a vertex set $S \subseteq V(G)$ is a *dominating set* if each vertex in $V(G) - S$ is adjacent to at least one vertex in S. The *neighborhood* of vertex v, denoted $N(v)$, is the set of vertices adjacent to v, and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. Thus $S \subseteq V(G)$ is a dominating set if and only if $\cup_{s \in S} N[s] = V(G)$. Suppose that the graph G models a facility or street network and that $S \subseteq V(G)$ is to be the set of locations for a guard force, such as a set of policemen, that will protect all of the vertex locations. We might further require that when a policeman responds to an emergency from a vertex v she have a backup policeman located close to her, for example, in $N(v)$. When we are required to assign police to vertex locations in pairs so that each can act as a backup to the assigned partner and partners must be located at adjacent vertices, we seek a *paired-dominating set*, a dominating set S with the property that the subgraph generated by S, $\langle S\rangle$, has a perfect matching. The *domination number* $\gamma(G)$ and the *paired-domination number* $\gamma_{pr}(G)$ are the minimum cardinalities of dominating sets and paired-dominating sets, respectively. Paired-dominating sets are considered in [1-4].

In forming a paired-dominating set, one is free to pair a vertex with any one of its neighbors. For a related problem one might have the paired locations pre-assigned.

In general, consider a surjection $\pi : V(G) \to \{1, 2, \ldots, t\}$, and let $\pi^{-1}(i) = V_i \subseteq V(G)$. Then $\mathcal{S} = \{V_1, V_2, \ldots, V_t\}$ is a partition of $V(G)$, and we consider V_i to be the color class i. The current problem is to find a minimum cardinality $\mathcal{S}$*-dominating set*, a dominating set S with the property that if S contains any one vertex of color i, then it contains all vertices of color i, that is $S \cap V_i \neq \emptyset \Rightarrow V_i \subseteq S$. We can denote the $\mathcal{S}$*-domination number* by $\gamma(G;\mathcal{S})$ or $\gamma(G;\pi)$ with $\gamma(G;\pi) = \gamma(G;\mathcal{S}) = \min\{|\cup_{i\in I} V_i| : I \subseteq \{1, 2, \ldots, t\}$ and $N[\cup_{i\in I} V_i] = V(G)\}$ where for a set $R \subseteq V(G)$ the closed neighborhood of R is $N[R] = \cup_{r\in R} N[r]$. We will also call an $\mathcal{S}$-dominating set a π*-dominating set* for the associated π.

If each color class is required to have at most two elements, we have the *coupled-domination number* $\gamma_{cpl}(G) = \max\{\gamma(G;\mathcal{S}) : |V_i| \leq 2$ for every $V_i \in \mathcal{S}\}$. Note that we are seeking the maximum value over all possible pairings where two vertices of the same color might or might not be adjacent. For the three colorings π_1, π_2 and π_3 illustrated in Figure 1, we have $\gamma(P_7;\pi_1) = |\{v_2, v_3\} \cup \{v_6, v_7\}| = 4, \gamma(P_7;\pi_2) = |\{v_1, v_7\} \cup \{v_4\}| = 3$, and $\gamma(P_7;\pi_3) = |\{v_1, v_2\} \cup \{v_5\} \cup \{v_6, v_7\}| = 5$. In fact, $\gamma_{cpl}(P_7) = 5$.

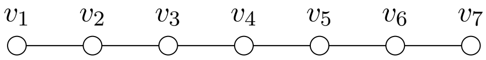


	v_1	v_2	v_3	v_4	v_5	v_6	v_7
π_1:	4	1	1	2	2	3	3
π_2:	3	1	1	4	2	2	3
π_3:	1	1	3	3	4	2	2

Figure 1: Colored labelling of path P_7

Vertex set $S \subseteq V(G)$ is an *independent set* if no two vertices of S are adjacent, that is, no edge in $E(G)$ has both end points in S. The independence number $\beta(G)$ is the maximum cardinality of an independent set. Again consider a surjection $\pi : V(G) \to \{1, 2, \ldots, t\}$ with $\pi^{-1}(i) = V_i \subseteq V(G)$ and the partition $\mathcal{S} = \{V_1, V_2, \ldots, V_t\}$ of $V(G)$ with V_i considered to be color class i. Now the problem might be to find the maximum cardinality of an independent set A with the property that either $A \cap V_i = \emptyset$ or $V_i \subseteq A$ for $1 \leq i \leq t$. We can denote the $\mathcal{S}$-independence number by $\beta(G;\mathcal{S})$ or $\beta(G;\pi)$ with $\beta(G;\mathcal{S}) = \beta(G;\pi) = \max\{|\cup_{i\in I} V_i| : I \subseteq \{1, 2, \ldots, t\}$ and $\cup_{i\in I} V_i$ is independent$\}$. Note that if any two vertices in V_i are adjacent then one must have $A \cap V_i = \emptyset$, so we assume that each V_i is an independent set, that is, we only consider proper colorings.

For the coupled-independence number, $\beta_{cpl}(G) = \min\{\beta(G;\mathcal{S}) : \mathcal{S}$ is a partition of $V(G)$ into independent sets V_i, with each $|V_i| \leq 2\}$. For the proper coloring π_4 and π_5 illustrated in Figure 2, we have $\beta(P_7;\pi_4) = 4$ and $\beta(P_7;\pi_5) = 3$. In fact, $\beta_{cpl}(P_7) = 3$.



π_4 :	1	2	3	2	3	4	1
π_5 :	1	2	1	3	2	3	4

Figure 2: Proper colorings of P_7

2. Coupled-domination and coupled-independence of paths

A more extensive study of coupled-domination will be available in Seo and Slater [5]. In this section attention is restricted to coupled-domination and coupled-independence for paths.

Theorem 2.1. *For every graph,* $\gamma(G) \leq \gamma_{cpl}(G) \leq 2\gamma(G)$.

Proof. For any coloring π with associated partition $\mathcal{S}$, any $\mathcal{S}$-dominating set S is a dominating set, and $\gamma(G;\mathcal{S}) \geq \gamma(G)$. Trivially then $\gamma(G) \leq \gamma_{cpl}(G)$. Assume that $\mathcal{S}$ is a coupled-partition of $V(G)$ with associated coloring π and $\gamma_{cpl}(G) = \gamma(G;\mathcal{S})$. Let D be a minimum dominating set, so $\gamma(G) = |D|$, and let $S = \{v \in V(G) : \exists w \in D$ with $\pi(v) = \pi(w)\}$. Then $\gamma_{cpl}(G) = \gamma(G;\mathcal{S}) \leq |S| \leq 2|D| = 2\gamma(G)$. □

Note that, in general, $\gamma(G;\mathcal{S}) \leq \gamma(G)\cdot \max\{|V_i| : V_i \in \mathcal{S}\}$.

Let $G_{2n} = K_n \times P_2$ be the graph on $2n \geq 6$ vertices obtained from two disjoint copies of complete graph K_n by adding a matching between them. We have $\gamma(G_{2n}) = 2, \gamma_{cpl}(G_{2n}) = 2 = \gamma(G_{2n})$ if n is odd, and $\gamma_{cpl}(G_{2n}) = 4 = 2\cdot\gamma(G_{2n})$ if n is even.

Theorem 2.2. *For the path P_n on n vertices,* $\gamma_{cpl}(P_{6t}) = 4t = 2\gamma(P_{6t}), \gamma_{cpl}(P_{6t+1}) = 4t+1 = 2\gamma(P_{6t+1}) - 1, \gamma_{cpl}(P_{6t+2}) = 4t+2 = 2\gamma(P_{6t+2}), \gamma_{cpl}(P_{6t+3}) = 4t+2 = 2\gamma(P_{6t+3}), \gamma_{cpl}(P_{6t+4}) = 4t+4 = 2\gamma(P_{6t+4}),$ *and* $\gamma_{cpl}(P_{6t+5}) = 4t+4 = 2\gamma(P_{6t+5})$.

Proof. Note that if six consecutive vertices of path P_n are labelled $abb'cc'a'$, then any coupled dominating set S must contain four of these vertices. For example, if $S \cap \{a, a', c, c'\} = \emptyset$, then the fifth vertex c' is not dominated by S. Note that S can contain any two of the couples, and the remaining two vertices will be dominated by S.

In particular, for $n = 6t$, we have $\gamma(P_{6t}) = 2t$ and so $\gamma_{cpl}(P_{6t}) \leq 4t$. We can repeat the pattern $abb'cc'a'$ so that the vertices of P_{6t} are labelled $a_1b_1b_1'c_1c_1'a_1'a_2b_2b_2'c_2c_2'a_2' \ldots a_tb_tb_t'c_tc_t'a_t'$. For this labelling π, we have $\gamma_{cpl}(P_{6t};\pi) = 4t$. Thus, $\gamma_{cpl}(P_{6t}) \geq 4t$. Hence $\gamma_{cpl}(P_{6t}) = 4t$.

For $n = 6t+2$, we have $\gamma(P_{6t+2}) = 2t+1$ and so $\gamma_{cpl}(P_{6t+2}) \leq 4t+2$. For the labelling $\pi = a_1b_1b_1'c_1c_1'a_1'a_2b_2b_2'c_2c_2'a_2' \ldots a_tb_tb_t'c_tc_t'a_t'a_{t+1}a_{t+1}'$, we have $\gamma_{cpl}(P_{6t+2};\pi) = 4t+2$. Hence $\gamma_{cpl}(P_{6t+2}) = 4t+2$.

For $n = 6t+4$, we have $\gamma(P_{6t+4}) = 2t+2$ an so $\gamma_{cpl}(P_{6t+4}) \leq 4t+4$. For the labelling $\pi = a_0a_0'a_1b_1b_1'c_1c_1'a_1'a_2b_2b_2'c_2c_2'a_2' \ldots a_tb_tb_t'c_tc_t'a_t'a_{t+1}a_{t+1}'$, we have $\gamma_{cpl}(P_{6t+4};\pi) = 4t+4$. Hence $\gamma_{cpl}(P_{6t+4}) = 4t+4$.

For $n = 6t+3$, we have $\gamma(P_{6t+3}) = 2t+1$ and so $\gamma_{cpl}(P_{6t+3}) \leq 4t+2$. For the labelling $\pi = a_0a_1b_1b_1'c_1c_1'a_1'a_2b_2b_2'c_2c_2'a_2' \ldots a_tb_tb_t'c_tc_t'a_t'a_{t+1}a_{t+1}'$, we have $\gamma_{cpl}(P_{6t+3};\pi) = 4t+2$. Hence, $\gamma_{cpl}(P_{6t+3}) = 4t+2$.

For $n = 6t+5$, we have $\gamma(P_{6t+5}) = 2t+2$ and so $\gamma_{cpl}(P_{6t+5}) \leq 4t+4$. For the labelling $\pi = a_0a_0'a_1b_1b_1'c_1c_1'a_1'a_2b_2b_2'c_2c_2'a_2' \ldots a_tb_tb_t'c_tc_t'a_t'b_{t+1}a_{t+1}a_{t+1}'$, we have $\gamma_{cpl}(P_{6t+5};\pi) = 4t+4$. Hence $\gamma_{cpl}(P_{6t+5}) = 4t+4$.

For $n = 6t+1$, let $P_{6t+1} = v_1v_2v_3 \ldots v_{6t}v_{6t+1}$. Assume that π is a coupling with $\gamma_{cpl}(P_{6t+1}) = \gamma_{cpl}(P_{6t+1};\pi)$, and let v_i be a vertex that is not coupled with another vertex. First, we will show that $\gamma_{cpl}(P_{6t+1}) \leq 4t+1$. If $i = 6j+1$ or $6j+4$, let $S = \{v_1, v_4, v_7, v_{10}, \ldots, v_{6(t-1)+1}, v_{6(t-1)+4}, v_{6t+1}\}$ and S' be the set of vertices that are coupled with the vertices in S. We observe that $|S \cup S'| \leq 4t+1$ and $S \cup S'$ is a dominating set of P_{6t+1}. Hence, $\gamma_{cpl}(P_{6t+1}) \leq 4t+1$. If $i = 6t+2$ or $6t+5$, let $S = \{v_2, v_5, v_8, v_{11}, \ldots, v_{6(t-1)+2}, v_{6(t-1)+5}, v_{6t+1}\}$ and S' be the set of vertices that are coupled with the vertices in S. We observe that $|S \cup S'| \leq 4t+1$ and $S \cup S'$ is a dominating set of P_{6t+1}. Hence $\gamma_{cpl}(P_{6t+1}) \leq 4t+1$. If $i = 6j+3$ or $6j$, let $S = \{v_1, v_3, v_6, v_9, v_{12}, \ldots, v_{6(t-1)+3}, v_{6(t-1)+6}\}$ and S' be the set of vertices that are coupled with the vertices in S. We observe that $|S \cup S'| \leq 4t+1$ and $S \cup S'$ is a dominating set of P_{6t+1}. Hence $\gamma_{cpl}(P_{6t+1}) \leq 4t+1$.

Now for the labelling $\pi = a_ta_t'a_1b_1b_1'c_1c_1'a_1'a_2b_2b_2'c_2c_2'a_2' \ldots a_{t-1}b_{t-1}b_{t-1}'c_{t-1}c_{t-1}'a_{t-1}'b_tb_t'$ $a_{t+1}c_tc_t'$, where a_{t+1} is the vertex that is not coupled, we have $\gamma_{cpl}(P_{6t+1};\pi) = 4t+1$. Hence $\gamma_{cpl}(P_{6t+1}) = 4t+1$, which concludes the proof. □

As noted in Theorem 2.1, it is obvious that $\gamma(G) \leq \gamma_{cpl}(G)$ because every $\gamma_{cpl}(G)$-set is a dominating set. Likewise, one always has $\beta_{cpl}(G) \leq \beta(G)$. However, unlike the case for $\gamma_{cpl}(G)$, we do not get a lower bound for $\beta_{cpl}(G)$ in terms of $\beta(G)$. Consider the graph $H_{2n} = K_n \circ K_1$ on $2n$ vertices obtained from a complete graph K_n with $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ by attaching a vertex v_i of degree one adjacent to u_i for $1 \leq i \leq n$. The proper coloring $\mathcal{S} = \{\{u_1, v_2\}, \{u_2, v_3\}, \ldots, \{u_{n-1}, v_n\}, \{u_n, v_1\}\}$ shows that $\beta_{cpl}(H_{2n}) = 2$.

Theorem 2.3. *For the path P_n on n vertices, $\beta_{cpl}(P_{8j}) = 2j$, $\beta_{cpl}(P_{8j+1}) = \beta_{cpl}(P_{8j+2}) = 2j+1$, $\beta_{cpl}(P_{8j+3}) = \beta_{cpl}(P_{8j+4}) = \beta_{cpl}(P_{8j+5}) = \beta_{cpl}(P_{8j+6}) = 2j+2$, and $\beta_{cpl}(P_{8j+7}) = 2j+3$.*

Proof. Let $n = 8j + r$ where $0 \le r \le 7$. Clearly $\beta_{cpl}(P_1) = 1$, and $\beta_{cpl}(P_2) = 1$. If $3 \le r \le 6$ we can choose the two vertices in any color class of order two or any two nonadjacent singleton color classes to get $\beta(P_r;\mathcal{S}) \ge 2$ for any coupled proper coloring, and it is easy to find a particular $\mathcal{S}$ for which $\beta(P_r;\mathcal{S}) = 2$, so $\beta_{cpl}(P_r) = 2$. For P_7 the proper coloring (1,2,3,4,1,3,2) (that is, $\{\{v_1,v_5\},\{v_2,v_7\},\{v_3,v_6\},\{v_4\}\}$) shows that $\beta_{cpl}(P_7) \le 3$. Let $\mathcal{S}$ be any coupled proper coloring of P_7. If $\mathcal{S}$ has one singleton color class and three pairs, let the singleton be v_i, then at least one of the colored pairs has neither vertex adjacent to v_i, so $\beta(P_7;\mathcal{S}) \ge 3$. Suppose $\mathcal{S}$ has at least three singleton color classes $\{v_i\},\{v_j\}$, and $\{v_k\}$ with $i < j < k$. If $i \le j-2$ and $k \ge j+2$, then $\{v_i,v_j,v_k\}$ shows that $\beta(P_7;\mathcal{S}) \ge 3$. If, for example, $i = j-1$ then there are at least three vertices whose colors are not those of v_{j-1}, v_j, and v_{j+1}, so one can use a color pair or two singletons from these three vertices along with v_j to see that $\beta(P_7;\mathcal{S}) \ge 3$. Hence, $\beta_{cpl}(P_7) = 3$.

For $n = 8j + r$ with $j \ge 1$, $8j$ consecutive vertices will be colored as L : $(1,2,4,3,2,3,1,4,5,6,8,7,6,7,5,8,\ldots,4i-3,4i-2,4i,4i-1,4i-2,4i-1,4i-3,4i,\ldots,4j-3,4j-2,4j,4j-1,4j-2,4j-1,4j-3,4j)$. For each group of four colors, at most one pair can be used in any independent set. Use L for P_{8j}, $(4j+1,L)$ for P_{8j+1}, $(4j+1,4j+2,L)$ for P_{8j+2}, $(4j+1,4j+2,L,4j+1)$ for P_{8j+3}, $(4j+2,4j+1,L,4j+1,4j+2)$ for P_{8j+4}, $(4j+1,4j+2,4j+3,4j+1,4j+2,L)$ for P_{8j+5}, $(4j+3,4j+2,4j+1,4j+3,4j+2,4j+1,L)$ for P_{8j+6}, and $(4j+1,4j+2,4j+3,4j+4,4j+1,4j+3,4j+2,L)$ for P_{8j+7} to see that $\beta_{cpl}(P_{8j}) \le 2j, \beta_{cpl}(P_{8j+1}) \le 2j+1, \ldots, \beta_{cpl}(P_{8j+7}) \le 2j+3$.

Let $\mathcal{S}$ be any proper coupled coloring of P_n with $n \ge 8$. We can find a sufficiently large $\mathcal{S}$-independent set as follows. Start with $S = \emptyset$ and repeat the following until fewer than eight vertices remain. Choose a vertex v of degree at most one. Put v in S and if there is another vertex v' with $\{v,v'\} \in \mathcal{S}$, then also put v' in S. Delete v, v', and any vertex in $N(v) \cup N(v')$ or of the same color as a vertex in $N(v) \cup N(v')$. If $\{v,v'\} \in \mathcal{S}$, then two vertices are added to S and at most eight deleted. If $\{v\} \in \mathcal{S}$, then one vertex in placed into S and at most three are deleted. At the first point where fewer than eight vertices remain, at least one-fourth of the deleted vertices are in S, that is $|S| \ge 2j$. From what remains we can add the required number of vertices to S in the same manner as we did for $P_1, P_2, \ldots, P_7$. Hence, $\beta_{cpl}(P_{8j}) \ge 2j, \beta_{cpl}(P_{8j+1}) \ge 2j+1, \ldots, \beta_{cpl}(P_{8j+7}) \ge 2j+3$, and the proof is complete. □

To conclude, it is noted that along with further study of colored-domination and colored-independence, we have a colored-problem associated with essentially every graph parameter.

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