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Total Reinforcement Number of a Graph

N. Sridharan
Department of Mathematics
Alagappa University, Karaikudi - 630 003, India
E-mail: *math_sridhar@yahoo.co.in*
MD. Elias
Department of Mathematics
B. U. E. T. Dhaka -1000, Bangladesh.
and
V.S.A. Subramanian
Department of Mathematics
A. P. S. A. College, Tiruppatur - 630 211, India.

Abstract

A set D of vertices in a graph $G = (V, E)$ is said to be a total dominating set of G if every vertex in V is adjacent to some vertex in D. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set. $E(\overline{G})$ denotes the edge set of $\overline{G}$, the complement of G. The minimum cardinality of a set $E_1 \subset E(\overline{G})$ for which $\gamma_t(G+E_1) < \gamma_t(G)$ is denoted by $r_t(G)$ and is called the total reinforcement number of G. The number $r_t(G)$ is well defined if $\gamma_t(G) > 2$. In this paper, we obtain some results on the total reinforcement number of a graph.

Keywords: Total domination number, total reinforcement number.

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1. Introduction

We consider only simple finite graphs without isolated vertices. For graph theoretic terminology we refer to Bondy and Murty [1]. If x is a positive real number, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote respectively the integral part of x, and the least integer not less than x. To each vertex v of a graph $G, N(v)$ denotes the set of all vertices of G which are adjacent to v. For any subset $S \subseteq V, N(S) = \cup\{N(v) | v \in S\}$.

A set D of vertices in a graph G is said to be a *dominating set* if every vertex in $V - D$ is adjacent to some vertex in D. We call D a *total dominating set* for G if every vertex in V is adjacent to some vertex in D. The minimum cardinality of a dominating set (a total dominating set) of G is denoted by $\gamma(G), (\gamma_t(G))$ and is called the *domination number* of G (the *total domination number* of G).

Kulli [4] introduced the concept of the cobondage number $cb(G)$ of a graph G with respect to the domination, which is the minimum number of edges to be added to reduce the domination number. The same concept has been independently studied earlier by others under the name 'reinforcement number'(refer Chapter 17 of Haynes et al. [3]). In this paper we introduce the concept of total reinforcement number of a graph.

2. Total Reinforcement Number of a Graph

Definition 2.1. *Let G be a graph with $\gamma_t(G) > 2$. The total reinforcement number of G is denoted by $r_t(G)$ and is given by*

$$r_t(G) = min\{|E_0| : E_0 \subset E(\overline{G}) \quad \text{and} \quad \gamma_t(G + E_0) < \gamma_t(G)\}.$$

Lemma 2.2. *For any graph G, with $\gamma_t(G) \geq 3$, $r_t(G) \leq \Delta(G)$.*

Proof. Let D be a minimum total dominating set for G. If the induced subgraph $\langle D \rangle$ contains a component with more than two vertices, select such a component $\{u_1, u_2, \ldots, u_k\}, (k \geq 3)$ of $\langle D \rangle$. This component contains at least two vertices which are not cut vertices of $\langle D \rangle$. Let u_1 be one such vertex. Let $E_0 = \{u_2v | u_1v \in E(G)$ and $u_2v \in E(\overline{G})\}$. Then $D - \{u_1\}$ is a total dominating set for $G + E_0$. If every component of $\langle D \rangle$ is K_2, select a component $\{u_1, u_2\}$ and select one $u \in D - \{u_1, u_2\}$. If $E_0 = \{uu_2\} \cup \{u_2v | u_1v \in E(G)$ and $u_2v \notin E(G)\}$, then $D - \{u_1\}$ is a total dominating set for $G + E_0$. Hence $r_t(G) \leq \Delta$. □

Example 2.3. *For the graph given in figure 1, $r_t(G) = \Delta = 2$ and for the graph given in figure 2, $r_t(G) = \Delta(G) = 3$.*

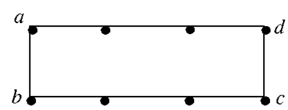

Figure 1

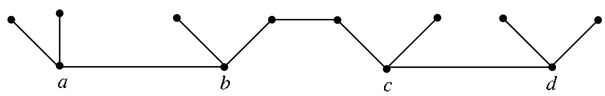

Figure 2

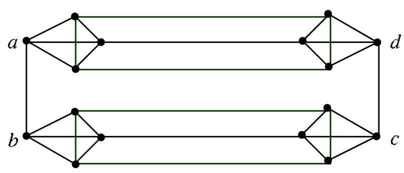

Figure 3

Now we show that for the graph given in figure 3, $r_t(G) = \Delta = 4$.

First we note that $n = 16, G$ *is regular with* $\Delta = \delta = 4$, *and* $\gamma_t(G) = 4$. *Let* $E_1 \subset E(\overline{G})$ *be such that* $\gamma_t(G+E_1) < \gamma_t(G)$. *Then* $\gamma_t(G+E_1) = 3$ *or 2.*

Consider the case $\gamma_t(G+E_1) = 3$. *Let* $D = \{x_1, x_2, x_3\}$ *be a* γ_t*-set for* $G+E_1$. *Then there is at least one edge from each vertex in* $\{V(G) - \{x_1, x_2, x_3\}\}$ *to* D. *As* D *is a total dominating set, there should be at least two edges of* $G+E_1$ *in* $\langle D \rangle$. *Hence there are at least 15 distinct edges which are incident with* $D = \{x_1, x_2, x_3\}$ *and* $(deg\{x_1\} + deg\{x_2\} + deg\{x_3\}) \geq 17$ *in* $G + E_1$.

We note that in G, $D = \{x_1, x_2, x_3\}$ *covers at most 11 or 10 or 9 distinct points of* $V - D$, *depending on whether* $\langle D \rangle$ *has no edge; one edge or two edges in* G *respectively. (If* $D = \{x_1, x_2, x_3\}$ *is an independent set in* G, *there exists* $y \in V - D$ *which is adjacent to at least two vertices of* D *). Hence* $|E_1| \geq 4$.

Now we consider the case $\gamma_t(G+E_1) = 2$. *Let* $D = \{x, y\}$ *be a* γ_t*-set for* $G+E_1$. *Again in* $G + E_1, deg\ x + deg\ y \geq 14 + 2 = 16$. *But in* $G, deg\ x + deg\ y = 8$. *Hence* $|E_1| \geq 4$.

Hence from Lemma 2.2, it follows that $r_t(G) = 4$.

Theorem 2.4. *Given any positive integer* $k \geq 2$, *there exists a graph* G *with* $\Delta(G) = k$, $\gamma_t(G) > 3$ *and* $r_t(G) = \Delta(G) = k$.

Proof. It is enough to consider the case $k \geq 5$ (in view of Example 2.3).

Let $V(G) = \{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k, c_1, c_2, \dots, c_k, d_1, d_2, \dots, d_k\}$ and $E(G) = \{a_i a_j, b_i b_j, c_i c_j, d_i d_j : i \neq j \in \{1, 2, 3, \dots, k\}\} \cup \{a_1 b_1, c_1 d_1\} \cup \{a_i d_i, b_i c_i \ : i = 2, 3, \dots, k\}$. It can be easily verified that $\gamma_t(G) = 4$, and $r_t(G) = \Delta = k$. □

We now obtain a set of necessary conditions for a graph G to have $r_t(G) = \Delta$.

Theorem 2.5. *Let* $G \neq mK_2$ *and* $\gamma_t(G) \geq 3$. *If* $r_t(G) = \Delta$, *then every* γ_t*-set* D *of* G *satisfies the following conditions:*

(i) $\langle D \rangle = mK_2$ *and* $n = 2m\Delta$ *for some integer* $m \geq 2$,

(ii) every vertex in D *is of maximum degree and*

(iii) every vertex in $V - D$ *is adjacent to exactly one vertex in* D.

Proof. Let D be a γ_t-set for G. If C is a component of $\langle D \rangle$ which contains more than two vertices, C contains at least two vertices u and v which are not cut vertices in $\langle C \rangle$.

If $E_1 = \{vz : z \in V - D$ and $N(z) \cap D = \{u\}\}$, then $D - \{u\}$ is a total dominating set for $G+E_1$ and hence $r_t(G) \leq |E_1| \leq deg(u) - 1 < \Delta$ which is a contradiction. Hence $\langle D \rangle = mK_2$, for some $m \geq 2$. ($m \geq 2$ follows from the fact that $\gamma_t(G) \geq 3$).

Let $x \in D$. We claim that $deg(x) = \Delta$. Let $V_x = \{z : z \in V - D, N(z) \cap D = \{x\}\}$ and $E_1 = \{yz : z \in V_x\}$, where y is the unique vertex in D which is adjacent to x. Let y_2 be a vertex in $D - \{x, y\}$ and $e = yy_2$. Then $D - \{x\}$ is a total dominating set for $G + (E_1 + e)$ and hence $\Delta = r_t(G) \leq |E_1| + 1 \leq deg(x)$. Thus, we have $deg(x) = \Delta$ and $|V_x| = |E_1| = \Delta - 1$. As $|V_x| = \Delta - 1$ for all $x \in D$, it follows that $N(x) \cap (V - D) = V_x$. Thus every vertex in $V - D$ is adjacent to exactly one vertex in D and $V - D = \cup\{V_x | x \in D\}$. Therefore, $|V - D| = 2m(\Delta - 1)$ and $n = |D| + |V - D| = 2m + 2m(\Delta - 1) = 2m\Delta$. □

Conjecture 2.6. *The conditions given in Theorem 2.5 are also sufficient for* $r_t(G) = \Delta$.

We recall the following result [2].

Theorem 2.7. [2] *If G is a connected graph with order $n \geq 3$, then $\gamma_t(G) \leq \frac{2n}{3}$.*

Theorem 2.8. *If G is a graph of order n, in which $\gamma_t(G) \geq 3$, then $r_t(G) \leq min\{\Delta(G), [\frac{n-2}{2}]\}$.*

Proof. Let Y be a γ_t-set of G. By the minimality, each $y \in Y$ has one of the following two properties.

P_1: There exists $z \in V - Y$ such that $N(z) \cap Y = \{y\}$.

P_2: $\langle Y - \{y\} \rangle$ contains an isolated vertex.

To each $y \in Y$, let $V_y = \{z \in V - Y : N(z) \cap Y = \{y\}\}$ and $C_y = \{z \in Y : z$ is an isolate in $\langle Y - \{y\} \rangle\}$.

Let $A = \{y \in Y : y$ has the property $P_1\}$ and $B = Y - A$. Then $A = \{y \in Y : V_y \neq \emptyset\}$ and $B = \{y \in Y : V_y = \emptyset\}$.

Case i. $B \neq \emptyset$.

Let $y_0 \in B$. Then $|C_{y_0}| \leq deg(y_0) \leq \Delta$ and $|C_{y_0}| \leq |Y| - 1 = \gamma_t(G) - 1$. Let $C_{y_0} = \{x_1, x_2, \ldots, x_m\}$. We note that if $m \neq 1$, then $x_i \in A$ and hence $\gamma_t(G) \leq n - m$. We construct E_1 as follows:

(a) If $m = 1, E_1 = \{x_1, y\}$ for some $y \in Y - \{x_1, y_0\}$.

(b) If m is even, $E_1 = \{x_1x_2, x_3x_4, \ldots, x_{m-1}x_m\}$.

(c) If $m \geq 3$ and odd, $E_1 = \{x_1x_2, x_1x_3, x_4x_5, \ldots, x_{m-1}x_m\}$.

Then $1 \leq |E_1| \leq \left\lceil \frac{|C_{y_0}|}{2} \right\rceil \leq \lceil \frac{\Delta}{2} \rceil$ and $Y - \{y_0\}$ is a total dominating set for $G + E_1$. Thus, in this case $r_t(G) \leq |E_1| \leq \lceil \frac{\Delta}{2} \rceil$ and

$$r_t(G) \leq |E_1| \leq \frac{\gamma_t}{2} \leq \begin{cases} \frac{n}{3} & \text{if } G \text{ is connected} \\ \frac{(n-m)}{2} & \text{if } m \neq 1. \end{cases}$$

$$\therefore r_t(G) \leq min\left\{\left\lceil\frac{\Delta}{2}\right\rceil, \frac{\gamma_t(G)}{2}, \frac{(n-2)}{2}\right\}. \tag{1}$$

Case ii. $B = \emptyset$.

Select a vertex $y_0 \in Y$ such that the cardinality of V_{y_0} is minimum. Then

(a) $1 \leq |V_{y_0}| < deg(y_0)$,

(b) $|V_{y_0}| \leq \frac{(|V-Y|)}{|Y|} = \frac{(n-\gamma_t(G))}{\gamma_t(G)}$,

(c) $0 \leq |C_{y_0}| \leq deg(y_0) - |V_{y_0}|$.

If $C_{y_0} \neq \emptyset$, let $C_{y_0} = \{x_1, x_2, \ldots, x_m\}$ and construct E_1 as in Case i, and if $C_{y_0} = \emptyset$, let $E_1 = \emptyset$. Select some $y \in Y - \{y_0\}$ and put $E_2 = \{yz|z \in V_{y_0}\}$. Then $Y - \{y_0\}$ is total dominating set for $G + E_1 + E_2$ and hence, $r_t(G) \leq |E_1| + |E_2| \leq deg(y_0) \leq \Delta$. Thus, in this case $r_t(G) \leq \Delta$.

If $\langle Y \rangle$ is not connected, let $C_1, C_2, \ldots, C_k$ be the components of Y. If there is a component C_1 with $|C_1| \geq 3$, we can find at least two points $y_1, y_2 \in C_1$ such that $\langle C_1 - \{y_i\}\rangle$ has no isolated vertices for $i = 1, 2$. For at least one $y_i, i \in \{1, 2\}$, we have $|V_{yi}| \leq \frac{[(n-\gamma_t(G))-(\gamma_t(G)-2)]}{2}$.

Let $|V_{y_1}| \leq \frac{[(n-2\gamma_t(G)+2)]}{2} \leq \frac{(n-4)}{2}$ as $\gamma_t(G) \geq 3$. Let $E_3 = \{y_2 z : z \in V_{y_1}\}$. Then $Y - \{y_1\}$ is a total dominating set for $G + E_3$ and hence $r_t(G) \leq |E_3| \leq \frac{(n-4)}{2}$. Thus, in this case

$$r_t(G) \in min\left\{\Delta, \frac{(n-4)}{2}\right\} \tag{2}$$

If there is no component C_i of $\langle Y \rangle$ such that $|C_i| \geq 3$, then $Y = mK_2$ for some $m \geq 2$. Let $Y = \{a_i, b_i | i = 1, 2, \ldots, m\}$ where $a_i b_i \in E(\langle Y \rangle)$. Then there is a point, say a_1 in Y such that

$$|V_{a_1}| \leq \frac{|V-Y|}{\gamma_t(G)} = \frac{(n-\gamma_t(G))}{\gamma_t(G)}.$$

Let $E_4 = \{b_1 z : z \in V_{a_1}\} \cup \{a_1 a_2\}$. Then $Y - \{z_1\}$ is a total dominating set for $G + E_4$ and $r_t(G) \leq |E_4| = 1 + |V_{a_1}| \leq 1 + \frac{(n-\gamma_t(G))}{\gamma_t(G)} = \frac{n}{\gamma_t(G)} \leq \frac{n}{4}$ (as $\gamma_t(G) \geq 4$ in this case).

So, in this case, we have

$$r_t(G) \leq min\left\{\Delta, \frac{n}{4}\right\}. \tag{3}$$

As $n \geq 4$, we have $\frac{n}{4} \leq \frac{(n-2)}{2}$.

Thus, in all the cases, we have $r_t(G) \leq min\left\{\Delta, \frac{(n-2)}{2}\right\}$. □

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