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Total Bondage Number of a Graph

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Abstract

A set D of a vertices in a graph $G = (V, E)$ is said to be a total dominating set of G if every vertex in V is adjacent to some vertex in D. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set. If $\gamma_t(G) \neq |V(G)|$, the minimum cardinality of a set $E_0 \subseteq E(G)$, such that $G - E_0$ contains no isolated vertices and $\gamma_t(G - E_0) > \gamma_t(G)$, is called the total bondage number of G. In this paper, we improve the earlier known upper bounds for the total bondage number of a graph.

Keywords: Total domination number, bondage number.

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1. Introduction

We consider only simple finite graphs without isolated vertices. For graph theoretic terminology we refer to Bondy and Murty [1]. If x is a positive real number, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote respectively the integral part of x, and the least integer not less than x. To each vertex v of a graph $G, N(v)$ denotes the set of all vertices of G which are adjacent to v. For any subset $S \subseteq V, N(S) = \cup\{N(v)|v \in S\}$.

A set D of vertices in a graph G is said to be a *dominating set* if every vertex in $V - D$ is adjacent to some vertex in D. We call D a *total dominating set* for G if every vertex in V is adjacent to some vertex in D (i.e., $N(D) = V$). The minimum cardinality of a dominating set (a total dominating set) of G is denoted by $\gamma(G), (\gamma_t(G))$ and is called the *domination number* (the *total domination number*) of G.

We now define the total bondage number for a graph.

Definition 1.1. *Let G be a graph. If there exists $E_0 \subset E(G)$ such that (i) there is no isolated vertex in $G - E_0$ and (ii) $\gamma_t(G - E_0) > \gamma_t(G)$, then the edge set E_0 is called a total bondage edge set for G. If there is at least one total bondage edge set for G, we define $b_t(G) = min\{|E_0| : E_0$ is a total bondage edge set of $G\}$. Otherwise we put $b_t(G) = \infty$.*

We call $b_t(G)$ as total bondage number of the graph G.

Example 1.2. $b_t(G) = \infty$, *for the following graphs.*

(*i*) $K_{1,n}$

(*ii*) *Graphs for which each component is* $K_{1,n}$ *or* K_2

(*iii*) K_3.

In [5], Kulli and Patwari calculated the exact value of $b_t(G)$ for some standard graphs:

1. If P_p is the path with $p \geq 4$ vertices, then

$$b_t(P_p) = \begin{cases} 2 & \text{if } p \equiv 2(mod\ 4) \\ \infty & \text{Otherwise.} \end{cases}$$

2. If C_p is the cycle with $p \geq 4$ vertices

$$b_t(C_p) = \begin{cases} 3 & \text{if } p \equiv 2(mod\ 4) \\ 2 & \text{Otherwise.} \end{cases}$$

3. If $K_{m,n}$ is the complete bipartite graph with $2 \leq m \leq n$, then $b_t(K_{m,n}) = m$.
4. $b_t(K_4) = 4$.
5. If K_p is the complete graph with $p \geq 5$ vertices, then $b_t(K_p) = 2p - 5$.

In [5], an upper bound for $b_t(G)$ was obtained:

If G is any graph with $p \geq 5$ vertices, then $b_t(G) \leq 2p - 5$.

In the next section we improve the bounds given by Kulli for $b_t(G)$.

2. An upper bound for total bondage number of a tree

We know that for a nontrivial tree T, the bondage number $b(T) \leq 2$. But given any positive integer k, there are trees T for which the total bondage number $b_t(T) = k$.

For example, if k is any positive integer, let H_k be the tree obtained from the star $K_{1,k+1}$ by subdividing k edges twice. It can be easily verified that $b_t(H_k) = k$. The tree H_7 is shown in Figure 1.

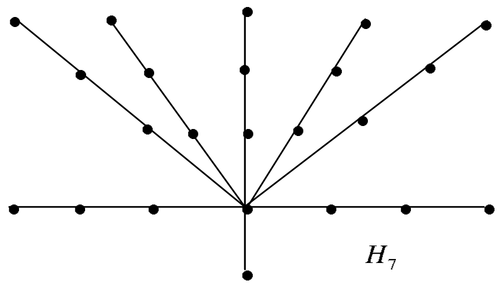

Figure 1

Theorem 2.1. *If T is a tree on p vertices and $T \neq K_{1,p-1}$, then $b_t(T) \leq min\{\Delta, \frac{(p-1)}{3}\}$.*

Proof. Let T be a tree which is not a star. Let $P = (u_1, u_2, u_3, \ldots, u_n)(n \geq 4)$ be a longest path in T. If $\gamma_t(T) \leq 3$ or $n = 5$, then $\gamma_t(T - e_1) > \gamma_t(T)$, where $e_1 = u_2u_3$. In this case $E_0 = \{e_1\}$ and $b_t(T) = 1$. Now we assume that $\gamma_t(T) \geq 4$ and $n \geq 6$. Let $e_1 = u_2u_3$. If $u_3 \in D$ for some minimum total dominating set D for T_3, where T_3 is the component of $T - e_1$ which contains the vertex u_3, then $D \cup \{u_2\}$ is a total dominating set for T and $\gamma_t(T - e_1) = |D| + 2$. In this case, $b_t(T) = 1$.

So assume that u_3 is not a vertex of any minimum total dominating set D for T_3. Then, clearly $deg(u_3) = 1$ in T_3 and $u_4 \in D$ for every minimum total dominating set D of T_3. If $deg(u_4) = 2$ in T_3, then $u_5 \in D$ for all minimum total dominating sets D of T_3. If $E_1 = \{u_2u_3, u_4u_5\}$, then $\gamma_t(T - E_1) > \gamma_t(T)$, and hence in this case $b_t(T) \leq 2$. Consider the case in which $deg(u_4) \geq 3$.

Let F be the set of all paths Q in T such that Q starts from the point u_4 and $Q \cap P = \{u_4\}$. Let $Q \in F$. If there is no $Q' \in F, Q' \neq Q$ such that Q is a part of Q', then we say that Q is maximal in F.

Let Q be maximal in F. Then the length of $Q \leq 3$. If the length of Q is 2, then let e_2 be the edge in Q incident with u_4. If e_2 is a part of a path Q' of length 3, for some $Q' \in F$, let e_3 be the edge of Q' which is incident with e_2. Then $\gamma_t(T - e_3) > \gamma_t(T)$. If e_2 is not a part of any path $Q' \in F$ of length 3, then $\gamma_t(T - e_1 - e_2) > \gamma_t(T)$ and $b_t(T) \leq 2$.

So assume that there is no maximal path Q of length 2 in F. If length of $Q = 1$ for all $Q \in F$, then as $u_3 \notin D$ for any minimum total dominating set D of T_3, we get $u_5 \in D$. In this case let $e_2 = u_4u_5$. Then $\gamma_t(T - e_1 - e_2) > \gamma_t(T)$ and hence $b_t(T) \leq 2$.

So, assume that there are maximal paths Q in F with length three. Let $Q_1, Q_2, \ldots, Q_k$ be a maximal collection of such paths with $Q_i \cap Q_j = \{u_4\}$, for $i \neq j$. If $u_5 \notin D$, for all minimum total dominating sets D of T_3, then let $E_0 = \{u_2u_3\} \cup \{$ central edges of $Q_j / j = 1, 2, \ldots, k\}$. If u_5 is in some minimum total dominating set for T_3, then take $E_0 = \{u_2u_3, u_4u_5\} \cup \{$ all central edges of $Q_j / j = 1, 2, \ldots, k\}$. Then $\gamma_t(T - E_0) > \gamma_t(T)$. Therefore $b_t(T) \leq |E_0| \leq \frac{(p-1)}{3}$. Since $|E_0| \leq deg(u_4)$, we have $b_t(T) \leq \Delta$. □

Remark 2.2. *Let G be a connected graph other than K_1, K_2, K_3 and $K_{1,n}$. Since there is a path of length 3 in G, we can find $E_1 \subseteq E$ such that $G - E_1$ is a spanning tree of T of G, containing a path of length 3. For the tree T we find $E_2 \subseteq E(G - E_1)$ such that $\gamma_t(G - E_1 - E_2) > \gamma_t(G - E_1) \geq \gamma_t(G)$. So $b_t(G) \leq |E_1| + |E_2|$. Thus, for every connected graph G other than K_1, K_2, K_3 and $K_{1,n}, b_t(G)$ is finite.*

3. Upper bounds for $b_t(G)$

In the following theorem we consider connected graphs G, other than K_1, K_2, K_3 and $K_{1,n}$. We consider various possible cases and obtain upper bounds for $b_t(G)$ in all the cases. Except in one case, in all other cases we get $b_t(G) \leq p - 1$.

Theorem 3.1. *Let G be a connected graph with order $p \geq 4$. Then*

(i) $b_t(G) \leq \frac{p-1}{3}$ *if G is a tree.*

(ii) $b_t(G) \leq p - 1$ *if the grith of $G \geq 5$.*

(iii) $b_t(G) \leq p - 2$ *if the grith of $G = 4$.*

(iv) $b_t(G) \leq p - 2$ *if there is a triangle $u_1u_2u_3$ in G such that at least one u_i is adjacent to a pendant vertex of G.*

(v) $b_t(G) \leq p - 1$ *if there is a triangle $u_1u_2u_3$ in G such that $deg(u_i) = 2$ in G for at least one i.*

(vi) $b_t(G) = 2p - 4$ *if $G = K_4$.*

(vii) $b_t(G) \leq 2p - 5$ *if $G \neq K_4$ and girth of $G = 3$.*

Proof. To each $u \in V(G)$, let $EV(u)$ be the set of all pendent vertices adjacent to u. i.e., $EV(u) = N(u) \cap \{v : deg(v) = 1 \text{ in } G\}$.

Case i. G contains a triangle. (i.e., the grith of $G = 3$).

Subcase (a) There is a triangle $(u_1u_2u_3u_1)$ in G with $EV(u_1) \neq \emptyset$. Let $A = \{u_1\} \cup EV(u_1)$ and $B = V(G) - A$. Clearly, $B \neq \emptyset$ as $u_2, u_3 \in B$. Let $E_1 = [A, B]$. (If A

and B are disjoint subsets of the vertex set $V(G)$, then $[A,B]$ denotes the set of all edges with one end vertex in A and the other in B). If $b_t(G) > |E_1|$, then $\gamma_t(G) = \gamma_t(G - E_1) = 2 + \gamma_t(\langle B\rangle)$. If D is a minimum total dominating set for $\langle B\rangle$, then $N(u_1) \cap D = \emptyset$, otherwise $D \cup \{u_1\}$ is a total dominating set with $\gamma_t(\langle B\rangle)+1$ vertices for G. Let S be the union of all minimum total dominating sets of $\langle B\rangle$. Let $E_2 = [u_2, S]$ in $\langle B\rangle$. Then $B - E_2$ will not contain isolated vertices and $\gamma_t(\langle B\rangle - E_2) \geq \gamma_t(\langle B\rangle) + 1$ and hence $\gamma_t(G) < \gamma_t(G - E_1 - E_2)$. Then $b_t(G) \leq |E_1 \cup E_2| = |E_1| + |E_2|$. Also $N(u_1) \cap S = \emptyset, |E_1| \leq (p-2) - |S|$ and $|E_2| \leq |S|$. In this case, $b_t(G) \leq p-2$. If $b_t(G) \leq |E_1|$, then also $b_t(G) \leq p-2$.

Subcase (b) Assume that $u_1u_2u_3$ is a triangle in $G, EV(u_i) = \emptyset$ for $i = 1,2,3$.

In G, if there is one triangle with one of its vertices has degree 2 in G, select one such triangle uvw with $deg(u) = 2$ in G. If there is no such triangle in G, select any triangle uvw. As G is not K_3, assume that $N(w) \neq \{u,v\}$. Let $A = \{u,v\}$ and $B = V(G) - A$. By our selection of the triangle uvw, there is no vertex x in $V(G)$, such that $N(x) = \{u,v\}$. Hence if $E_1 = [A,B]$, there is no isolated vertex in $G - E_1$. Let D be a minimum total dominating set for $\langle B\rangle$. There is at least one vertex $y \in D$ which is adjacent to w in $\langle B\rangle$. Hence $D \cup \{w\}$ is a total dominating set for G.

Thus $\gamma_t(G) \leq |D| + 1 = \gamma_t(\langle B\rangle) + 1 < \gamma_t(\langle B\rangle) + 2 = \gamma_t(G - E_1)$. Hence $b_t(G) \leq |E_1| = deg(u) + deg(v) - 2 \leq 2\Delta - 2 \leq 2p - 4$. So if there is a triangle uvw such that $deg(u) = 2$ in G, then $b_t(G) \leq deg(v) \leq p-1$. Assume that whenever uvw is a triangle in G, then $deg(u), deg(v), deg(w) > 2$. If there is a triangle uvw such that at least one of $deg(u), deg(v), deg(w) < p-1$, then $b_t(G) \leq 2p-5$.

If G has at least one triangle and for every triangle uvw in G, $deg(u) = deg(v) = deg(w) = p-1$ in G, then G is the complete graph K_p. But we know that

$$b_t(K_p) = \begin{cases} 2p-4 & \text{if } p = 4 \\ 2p-5 & p > 4. \end{cases}$$

Then in this case i, $b_t(G) = 2p-4$ if $G = K_4$ and $b_t(G) \leq 2p-5$ in all other cases. In all the following cases, we assume that there is no triangle in G.

Case ii. Assume that there exists an edge uv in G such that $EV(u) \neq \emptyset$ and $EV(v) \neq \emptyset$. Put $A = \{u,v\} \cup EV(u) \cup EV(v)$ and $B = X - A$. Let $E_0 = \{uv\}$ and $E_1 = [A,B]$. Then if $B = \emptyset, \gamma_t(G) = 2 \leq \gamma_t(G - E_0) = 4$ and if $B \neq \emptyset, \gamma_t(G) \leq \gamma_t(G - E_1) = 2 + \gamma_t(\langle B\rangle) < 4 + \gamma_t(\langle B\rangle) = \gamma_t(G - E_0 - E_1)$. Thus $b_t(G) \leq |E_0| + |E_1|$.

As there is no triangle in G, $|E_1| \leq p-4$. Thus in this case we have $b_t(G) \leq p-3$.

From now onwards, we further assume that if $e = uv$ is an edge in G, then either $EV(u) = \emptyset$ or $EV(v) = \emptyset$.

Case iii. Assume that there is a cycle $u_1u_2u_3u_4$ (on four vertices) in G.

Subcase (a) Let $EV(u_1) = \emptyset = EV(u_2)$.

Let $A = \{u_1, u_2\}$ and $B = V(G) - A$. As there is no triangle in G, it follows that $G - [A, B]$ has no isolated vertex. Let $E_1 = [A, B]$. If $b_t(G) > |E_1|$ then $\gamma_t(G) = \gamma_t(G - E_1) = 2 + \gamma_t(\langle B \rangle)$. If D is any minimum total dominating set for $\langle B \rangle$, then $(N(u_1) \cup N(u_2)) \cap D = \emptyset$, otherwise $D \cup \{u_1\}$ or $D \cup \{u_2\}$ is a total dominating set for G and $\gamma_t(G) < \gamma_t(G - E_1)$.

Let S be the union of all minimum total dominating sets of $\langle B \rangle$. Let $E(u_3) = [\{u_3\}, S]$ in $\langle B \rangle$. In the subgraph $(\langle B \rangle - E(u_3))$, there is no isolated vertex and $\gamma_t(\langle B \rangle - E(u_3)) > \gamma_t(\langle B \rangle)$.

Thus $\gamma_t(G - E_1 - E(u_3)) = 2 + \gamma_t((\langle B \rangle) - E(u_3) > 2 + \gamma_t(\langle B \rangle) = \gamma_t(G)$. Hence $b_t(G) \leq |E_1| + |E(u_3)|$. But (i) $|N(u_1) \cup N(u_2)) \cap (\langle B \rangle)| = |E_1|$, (as there is no triangle in G) and (ii) $|(N(u_1) \cup N(u_2)) \cap (\langle B \rangle)| + |S| \leq p - 2$. i.e., $|E_1| + |S| \leq p - 2$ and hence $|E_1| + |E(u_3)| \leq p - 2$. Thus, in this case $b_t(G) \leq p - 2$.

Subcase (b) Let $u_1u_2u_3u_4$ be a cycle of length four in G such that $EV(u_1) \neq \emptyset$ and $EV(u_3) \neq \emptyset$ (and hence $EV(u_2) = \emptyset$ and $EV(u_4) = \emptyset$).

Let $A = \{u_1, u_4\} \cup EV(u_1), B = V(G) - A$ and $E_1 = [A, B]$. If $|E_1| < b_t(G), \gamma_t(G) = \gamma_t(G - E_1) = 2 + \gamma_t(\langle B \rangle)$.

Let D be a minimum total dominating set for $\langle B \rangle$. Thus $u_2 \notin D$ (if $u_2 \in D$, then $D \cup \{u_1\}$ is a total dominating set with $\gamma_t(\langle B \rangle) + 1$ vertices.

As $EV(u_3) \neq \emptyset, u_3 \in D$ and also $EV(u_3) \cap D = \emptyset$, for if $EV(u_3) \cap D \neq \emptyset$, then $(D \backslash EV(u_3) \cup \{u_2\}$ is a total dominating set for $\langle B \rangle$. Let S be the union of all minimum total dominating sets for $\langle B \rangle$. Let E_3 be the set of all edges joining u_3 with S. We claim that $\langle B \rangle - E_3$ has no isolated vertex. If $z \in \langle B \rangle$ is an isolated vertex in $(\langle B \rangle) - E_3$ then $z \in S$ and let D be a minimum total dominating set for $\langle B \rangle$ which contains z. Then $D - \{z\} \cup \{u_2\}$ is a minimum total dominating set for $\langle B \rangle$, which is a contradiction, as we have assumed that u_2 is not in any minimum total dominating set of $\langle B \rangle$. Now, $\gamma_t(\langle B \rangle - E_3) > \gamma_t(\langle B \rangle)$ and $\gamma_t(G - E_1 - E_3) = 2 + \gamma_t(\langle B \rangle - E_3) > 2 + \gamma_t(\langle B \rangle) > \gamma_t(G)$. Therefore, $b_t(G) \leq |E_1| + |E_3|$. We note that

(i) $N(u_1) \cap N(u_4) = \emptyset, N(u_4) \cap N(u_3) = \emptyset$ (as G contains no triangle) and

(ii) $|E_1| \leq p - 4 - |E_3|$. So $b_t(G) \leq p - 4$.

Case iv. We assume that grith of $G \geq 5$.

So we can find a path of length 3, $u_1u_2u_3u_4$ with $EV(u_1) = \emptyset$. Let $A = \{u_1, u_2, u_3, u_4\} \cup (EV(u_2) \cup EV(u_3) \cup EV(u_4)) \cup \{w/N(w) = \{u_1, u_4\}\}$ and $B = V(G) - A$. If $B = \emptyset$, then $b_t(G) \leq 2$. If $B \neq \emptyset$, let $E_1 = [A, B]$. Then $\gamma_t(G) \leq \gamma_t(G - E_1) \leq 3 + \gamma_t(\langle B \rangle)$. Take $E_2 = \{u_2u_3\}$, if there is no $w \in G$ with $N(w) = \{u_1, u_4\}$. Otherwise, take $E_2 = \{u_1u_2, u_3u_4\}$. Then $\gamma_t(G - E_1 - E_2) \geq 4 + \gamma_t(\langle B \rangle)$ and therefore $b_t(G) \leq |E_1| + |E_2|, |E_2| \leq 2$, we note that $N(u_1) \cap B, N(u_2) \cap B, N(u_3) \cap B$ are mutually disjoint. Also $N(u_4) \cap B, N(u_3) \cap B, N(u_2) \cap B$ are mutually disjoint.

As there is no cycle of length 4, $|N(u_1) \cap N(u_4) \cap B| \leq 1$. Therefore $|E_1| \leq p-3$ and $b_t(G) \leq p-1$. □

Remark 3.2. *Bounds given in (i), (iii), (iv),(v) and (vii) are attained for the graphs* P_4, $C_4, K_{1,3}+e, K_4-e$ *and* K_4-e *respectively.*

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