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Very Excellent Graphs and Rigid Very Excellent Graphs

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Abstract

A graph G is said to be γ-excellent if given any vertex x of G, there is a γ-set of G containing x. A γ-excellent graph G is said to be γ-very excellent, if there is a γ-set S of G such that to each vertex $u \in V-S$, there exists a vertex $v \in S$ such that $(S-\{v\})\cup\{u\}$ is a γ-set of G. A γ-set S of G satisfying this property is called a very excellent γ-set of G. We provide methods of obtaining new γ-very excellent graphs from a given γ-very excellent graph. We also prove that if a tree is γ-very excellent, then every end vertex is a down vertex. We define a new class of γ-very excellent graphs called γ-rigid very excellent (RVE) graphs and prove that every γ-very excellent tree is RVE.

Keywords: Domination, γ-excellent graph, γ-very excellent graph, γ-rigid very excellent graph.

2000 Mathematics Subject Classification: 05C

1. Introduction

The graphs considered here are finite, undirected, non - trivial without loops or multiple edges. Let $G=(V,E)$ be a graph. For graph theoretic terminology, we refer to [1] or [4]. A subset D of V is a dominating set of G if every vertex in $V-D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A dominating set with minimum cardinality is said to be a $\gamma(G)$-set. An exhaustive treatment of fundamentals of domination and several advanced topics in domination are given in Haynes et al. [5,6].

G. H. Fricke et al. [3] call a vertex of a graph G to be *good* if it is contained in some $\gamma(G)$-set, and *bad* if it is not. They call a graph G to be γ-*excellent* if every vertex of G is good. In [7], new classes of *excellent graphs*, such as γ-*just excellent graphs* and γ-*very excellent graphs*, have been defined.

If D is a γ-set of G and $u \in D$, then let $pn[u, D] = \{v \in V(G) : v$ is not dominated by $D - u\} = N[u] - N[D - u]$, and $pn(u, D) = \{v \in V(G) - \{u\} : v$ is not dominated by $D - u\} = N(u) - N[D - u]$. A vertex u of G is said to be a *level vertex* of G, if $\gamma(G - u) = \gamma(G)$ and a vertex u of G is said to be a *down vertex* if $\gamma(G - u) < \gamma(G)$. In this paper we define a new class of γ-excellent graphs called γ-very excellent graphs and initiate a study on them.

2. γ-Very Excellent Graphs

Sometimes, a social group in a social network may have a 'feeling' that any one outside it might be exchanged with someone inside it for attaining a better status in the form of a new group. Such a situation can be modelled as a set S of vertices in the graph G representing the social network such that for every $y \in V(G) - S$ there exists $x \in S$ such that the new social group $S' = (S - \{x\}) \cup \{y\}$ has the same property as that of S and is possibly better in terms of external connections as well as its internal organizaion. This motivates the following definition.

Definition 2.1. *A graph G is said to be γ-very excellent, if there is a γ-set S of G such that to each vertex $u \in V - S$, there exists a vertex $v \in S$ such that $(S - v) \cup \{u\}$ is a γ-set of G. A γ-set S of G satisfying this property is called a very excellent γ-set of G. This property satisfied by a very excellent γ-set is called the exchange property.*

For example, paths P_2, P_4, P_7, the corona $H \circ K_1$ of any connected graph H, complete graphs K_n for all positive integers n, cycles C_3, C_4, C_7 are γ-very excellent.

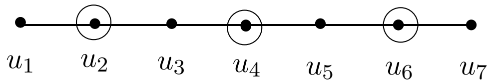


Fig. 1

The encircled vertices in Figure 1 form a very excellent γ-set for P_7.

Remark 2.2. *We observe that for every positive integer k, there are very excellent graphs G with $\gamma(G) = k$. Let H be any connected graph with k-vertices. Then its corona $G = H \circ K_1$ is a very excellent graph. Here $\gamma(G) = k$. If H is a tree then G is also a tree. So, for a given positive integer k, there are very excellent trees T with $\gamma(T) = k$.*

Let D be a dominating set of a graph G. A vertex in $V - D$ is said to be k-dominated if it is dominated by at least k vertices of D. The set D is said to be a k-dominating set if every vertex in $V - D$ is k-dominated. The minimum cardinality of a k-dominating set is called the k-domination number $\gamma_k(G)$. Fink and Jacobson [2] had proved that if D is a γ-set of G, then it is not a k-dominating set for any $k \geq 3$. In other words, a γ-set may be at the most a γ_2-set. For a graph G, if $\gamma_2(G) = \gamma(G)$ then G is very excellent. The generalized Petersen graph $P_{7,2}$ is an example of a very excellent graph with $\gamma_2(G) \neq \gamma(G)$.

We observe that a graph G is γ-very excellent if and only if there exists a γ-set D of G such that to each $u \notin D$, there exists $v \in D$ such that $pn[v, D] \subset N[u]$. Hereafter, by a very excellent graph we mean a γ-very excellent graph.

Further if D is a very excellent γ-set of a graph G and $u \notin D$, then there exists $v \in D$ such that the set $D' = (D - v) \cup \{u\}$ is a γ-set of G. In general the γ-set D' need not be a very excellent γ-set of G. For example for the graph in Figure 2(a), the encircled vertices form a very excellent γ-set D. In Figure 2(b), $D' = (D - v) \cup \{u\}$ is not a very excellent γ-set, since there exists no $w' \in D'$ such that $(D' - w') \cup \{w\}$ is a γ-set. But if $deg(v) = 1$ in G, then $D' = (D - v) \cup \{u\}$ is also a very excellent γ-set of D.

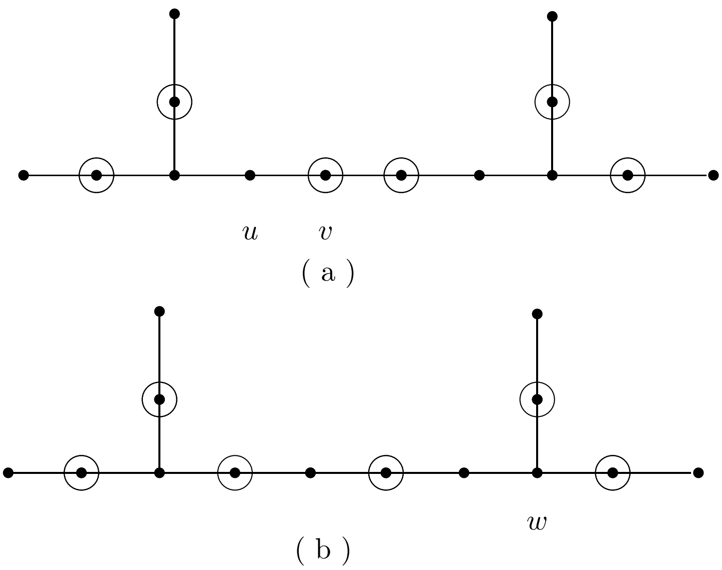


Fig. 2

Let G be a very excellent graph, u be a pendant vertex and let w be the vertex of G adjacent to u. For every γ-set S of G, $|S \cap \{u, w\}| = 1$. Let D_1 and D_2 be γ-sets of G such that $u \in D_1, w \in D_2$ and $D_1 - u = D_2 - w$. Then $D_1 = (D_2 - w) \cup \{u\}$ and $D_2 = (D_1 - u) \cup \{w\}$. Hence D_1 is a very excellent γ-set of G if and only if D_2 is a very excellent γ-set of G. Thus we have the following.

Proposition 2.3. *If $G \neq K_2$ is a connected very excellent graph, there exists a very excellent γ-set D of G not containing any pendant vertex of G.*

Lemma 2.4. *The paths P_2, P_4 and P_7 are the only very excellent paths.*

Proof. As the path P_n with $n \geq 3$ vertices is excellent if and only if $n \equiv 1(mod\ 3)$, it is enough to prove that P_n is not very excellent for all $n = 3k+1$, where $k \geq 3$. Let $n = 3k+1$ and $(u_1 u_2 ... u_n)$ be the path P_n. Assume that P_n is very excellent. Then there exists a (very excellent) γ-set D of P_n such that to each $u \notin D$, there exists $v \in D$ such that $pn[v, D] \subset N[u]$. The number of components of $P_n - D$ is at most $k+2$, and each component of $P_n - D$ is either K_1 or K_2. We can assume that the pendant vertices u_1 and $u_n \notin D$. Then both u_1 and u_n are isolated vertices in $P_n - D$. So the number of components of $P_n - D$ is either $k+1$ or $k+2$. If $P_n - D$ has $k+1$ components then $k-1$ vertices of D are isolated in $\langle D \rangle$ and the remaining two vertices are adjacent to each other. Thus either u_2 or u_{n-1} is isolated in D. If u_2 is isolated in D, then $u_3, u_4 \notin D$, (as each component of $P_n - D$ other than $\langle u_1 \rangle$ and $\langle u_n \rangle$ is K_2). Now there exists no $v \in D$ such that $pn[v, D] \subseteq N[u_1]$, a contradiction. Similarly we get a contradiction if u_{n-1} is isolated in D. If $P_n - D$ has $k+2$ components, then D is independent in P_n, and except four components, all other components of $P_n - D$ are K_2. As $u_1 \notin D$, we have $u_2 \in D$ and $u_3 \notin D$. If $u_4 \notin D$, then $u_3 \in pn[u_2, D]$ and $u_4 \in pn[u_5, D]$ and there is no $v \in D$ such that $pn[v, D] \subseteq N[u_1]$, a contradiction. So $u_4 \in D$. Similarly $u_{n-3} \in D$. Then u_1, u_3, u_{n-2}, u_n are isolated vertices in $P_n - D$ and other vertices of $P_n - D$ are not isolated. Hence $u_5, u_6 \notin D$ (note that $k \geq 3$) and $u_1 \in pn[u_2, D], u_5 \in pn[u_4, D]$ and hence there exists no $v \in D$ such that $pn[v, D] \subseteq N[u_3]$, a contradiction. Thus in all possible cases, we obtain a contradiction. Hence P_n is not very excellent. □

Analogous to Lemma 2.4 we have the following result.

Theorem 2.5. *C_3, C_4, C_7 are the only cycles which are very excellent.*

We now proceed to characterize caterpillars which are very excellent.

A caterpillar is a tree T such that the removal of all pendant vertices leaves a path, which is called the *spine* of T. Let T be a caterpillar and let the spine of T be the path $P_k : (u_1, u_2, ..., u_k)$. For each $i (1 \leq i \leq k)$, let a_i be the number of pendant vertices of T which are adjacent to the vertex u_i. Then the caterpillar T can be represented by the finite sequence $(a_1, a_2, ..., a_k)$. Note that each a_i is a non-negative integer and both $a_1 > 0$ and $a_k > 0$.

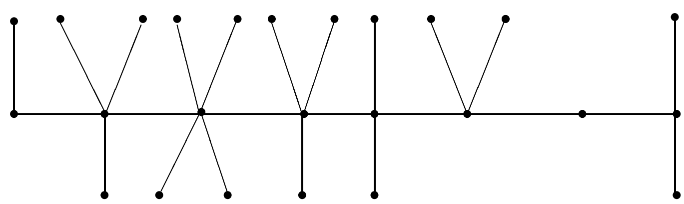


Figure 3

For example the sequence $(1, 3, 4, 3, 2, 2, 0, 2)$ represents the caterpillar given in Figure 3. We now characterize caterpillars which are very excellent.

Theorem 2.6. *A caterpillar* $T = (a_1, a_2, ..., a_k)$, $T \neq K_2$ *is very excellent if and only if the following hold.*

(i) $a_i \in \{0,1\}, 1 \leq i \leq k$.

(ii) $a_1 = a_k = 1$.

(iii) *If* $a_i = 1$ *and* $a_{i+1} = 0$ *for some* i, *then* $i < k-3$, $a_{i+2} = a_{i+3} = 0$ *and* $a_{i+4} = 1$.

Proof. Suppose T is very excellent. If $a_i \geq 2$ for some i, then v_i is in every γ-set of T and no pendant vertex adjacent to v_i is in any γ-set of T, so that T is not very excellent. Thus $a_i = 0$ or 1 for all i.

Now, let D be a very excellent γ-set of T. To each i for which $a_i \neq 0$, let v_i be the pendant vertex adjacent to u_i. Then $|D \cap \{u_i, v_i\}| = 1$. Let us assume that $u_i \in D$ whenever $a_i \neq 0$ (the very excellent γ-set property of D will not be affected by this assumption). So $D \subset \{u_i |\ 1 \leq i \leq k\}$. Let $a_i \neq 0$ and $a_{i+1} = 0$ for some i. Clearly as $a_k = 1$, we get $i < k-1$. Let s be the first positive integer such that $a_{i+s} \neq 0$. [So $a_{i+t} = 0$ for all $t, 1 \leq t \leq s-1$].

Let $Q = (v_i, u_i, u_{i+1}, u_{i+2}, ..., u_{i+s}, v_{i+s})$ which is a path on $s+3$ vertices. As $D \subset \{u_i |\ 1 \leq i \leq k\}$, and v_i, v_{i+s} are end vertices of T, both u_i and $u_{i+s} \in D$ and hence $D \cap Q$ is a very excellent γ-set for the path Q. Thus Q is a very excellent path. So by lemma 2.4, Q is either P_2, P_4, or P_7.

As $a_i = 1, a_{i+1} = 0$ and $a_{i+s} = 1, s+3 \geq 5$, it follows that $Q = P_7$. In fact Q is $(v_i, u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}, v_{i+4})$ and thus $a_i = 1, a_{i+1} = 0 \Rightarrow i < k-3; a_{i+2} = a_{i+3} = 0$ and $a_{i+4} = 1$.

Conversely if a caterpillar T satisfies the conditions given in the theorem, then the set $D = A \cup B$, where $A = \{u_i |\ a_i \neq 0\}$ and $B = \{u_i |\ a_{i-1} = a_i = a_{i+1} = 0\}$ is a very excellent γ-set of T. □

3 Construction of new very excellent graphs

In this section we present several methods of constructing very excellent graphs from given very excellent graphs.

Theorem 3.1. *If* G *is very excellent and* u *is a level vertex of* G, *then the graph* H *obtained from* G *by attaching a path* P_2 *at* u *is very excellent.*

Proof. Let (u, w_2, w_1) be the path P_2 attached at u of G to obtain H. Then $\gamma(H) \leq \gamma(G)+1$, as $S \cup \{w_2\}$ is a dominating set of H, whenever S is a γ-set of G. Now, let D be a γ-set of H. Then $D \cap \{w_1, w_2\} \neq \emptyset$ and $D \cap V(G)$ dominates $G-u$. As u is a level vertex in G, $\gamma(H) - 1 = |D \cap V(G)| \geq \gamma(G)$. (if $u \in D, |D \cap V(G)| \geq \gamma(G)$. If $u \notin D, |D \cap V(G)| \geq \gamma(G-u) = \gamma(G)$). Hence, $\gamma(H) = \gamma(G)+1$.

If S is a very excellent γ-set of G, then obviously $S \cup \{w_2\}$ is a very excellent γ-set of H. □

Theorem 3.2. *Let u be a vertex of a graph G and suppose that a graph H is obtained from G by attaching a path P_3 at u. Then H is very excellent if and only if G is very excellent and there exists a very excellent γ-set D of G such that $u \in D$ and $D - u$ dominates $G - u$.*

Proof. Let $(u\ w_3\ w_2\ w_1)$ be the path attached to u to obtain H from G. Clearly $\gamma(H) = \gamma(G)+1$. Assume that H is very excellent. We claim that there is a very excellent γ-set D of H containing u and w_2. Let S be a very excellent γ-set of H. We can assume that $w_2 \in S$. Then $w_1 \notin S$ and there exists $y \in S$ such that $(S - y) \cup \{w_1\}$ is a γ-set of H. As no γ-set of H contains both w_1 and w_2, it follows that $y = w_2$. In order to dominate the vertex $w_3, (S - w_2) \cup \{w_1\}$ contains either w_3 or u. So $S \cap \{w_3, u\} \neq \emptyset$. If $u \in S$, take $D = S$ and if $w_3 \in S$, take $D = (S - w_3) \cup \{u\}$. Then D is a very excellent γ-set of H containing both u and w_2.

Let $D_0 = D - w_2$. Then D_0 is a γ-set of G. Given any vertex $v \in V(G)$ such that $v \notin D_0$, we have $v \notin D$. So there exists $y \in D$ such that $(D - y) \cup \{v\}$ is a γ-set of H. Clearly $y \neq w_2$ and hence $y \in D_0$ and $(D_0 - y) \cup \{v\}$ is a γ-set of G. Thus D_0 is a very excellent γ-set of G and hence G is very excellent. We now claim that $D_0 - u$ is a dominating set of $G - u$.

As $w_3 \notin D$, there exists $z \in D$ such that $(D - z) \cup \{w_3\}$ is a γ-set of H. Clearly $z \neq w_2$. As any γ-set of H does not contain all the three vertices u, w_3 and w_2, it follows that $z = u$. Then $(D-u) \cup \{w_3\}$ is a γ-set of H and $D_0 - u = ((D-u) \cup \{w_3\}) - \{w_2, w_3\}$ dominates $G - u$.

Conversely let G be very excellent and D be a very excellent γ-set of G such that $u \in D$ and $D - u$ dominates $G - u$. Then $D \cup \{w_2\}$ is a very excellent γ-set of H. □

Theorem 3.3. *Let G be a very excellent graph and u be a vertex of G. Assume that there exists a very excellent γ-set D of G such that for some $v \in D, (D - v)$ is a γ-set for $G - u$. Then the graph H obtained from G and a disjoint path $w_1\ w_2\ w_3\ w_4\ w_5$ of length five by joining w_3 and u by an edge is very excellent.*

Proof. Let $P_5 : (w_1\ w_2\ w_3\ w_4\ w_5)$ be a path of length five with $V(P_5) \cap V(G) = \emptyset$. Let $H = (G \cup P_5) + e$ where $e = uw_3$. Clearly $\gamma(H) = \gamma(G) + 2$ and $S \cup \{w_2, w_4\}$ is a γ-set of H, whenever S is a γ-set of G.

By our assumption, there exists a very excellent γ-set D of G such that for some $v \in D, (D - v)$ is a γ-set for $G - u$. We claim that $S' = D \cup \{w_2, w_4\}$ is a very excellent dominating set of H and H is very excellent. Clearly S' is a γ-set of H.

The sets $(S' - w_2) \cup \{w_1\}, (S' - w_4) \cup \{w_5\}, (S' - v) \cup \{w_3\}$ are γ-sets of H containing w_1, w_5, w_3 respectively. Now, let $x \in V(G) - S$. Then $x \notin D$ and there exists one $y \in D$

such that $(D-y)\cup\{x\}$ is a γ-set of G and hence $(S'-y)\cup\{x\}$ is a γ-set of H. Thus S' is a very excellent γ-set of H and hence H is very excellent. □

Theorem 3.4. *Let G_1 and G_2 be very excellent graphs and let u_1 and u_2 be level vertices in G_1 and G_2 respectively. Then the graph H obtained from $G_1\cup G_2$ by joining the vertices u_1 and u_2 by an edge is very excellent.*

Proof. If D is a γ-set of H, then $D\cap V(G_i)$ dominates G_i-u_i, for $i=1,2$ and hence $|D|\geq\gamma(G_1-u_1)+\gamma(G_2-u_2)=\gamma(G_1)+\gamma(G_2)$ as u_i is a level vertex in G_i, $i=1,2$. So $\gamma(H)=\gamma(G_1)+\gamma(G_2)$. For each i, $i=1,2$, let S_i be a very excellent γ-set of G_i. Obviously $S_1\cup S_2$ is a γ-set for H and hence H is very excellent. □

Remark 3.5. *In Theorem 3.4, we regard both the vertices u_1 and u_2 to be level vertices. Even if one of them is not a level vertex, then the graph H need not be very excellent, not even excellent. For the graph G_1 given in Figure 4 u_2 is not a level vertex of G_1 and the resulting graph H is not even excellent.*

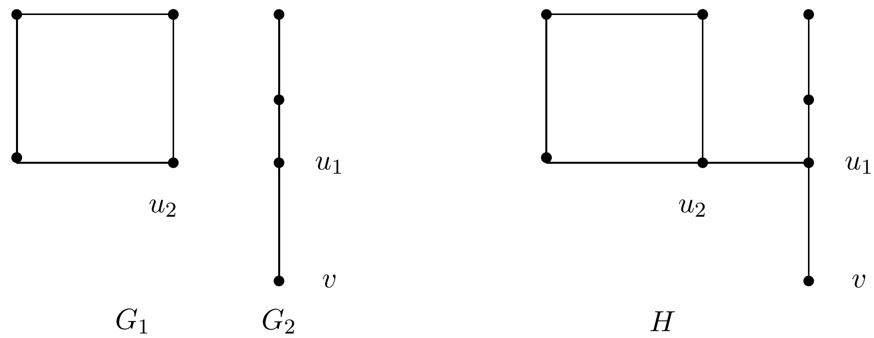


Fig. 4

Theorem 3.6. *Let G_1 and G_2 be two very excellent graphs with $V(G_1)\cap V(G_2)=\emptyset$. Let $u_i\in V(G_i), i=1,2$. Suppose there exists a very excellent γ-set S_i of G_i, such that $u_i\in S_i$ and S_i-u_i dominates G_i-u_i. Then the graph H obtained from $G_1\cup G_2$ by identifying the vertices u_1 and u_2, is very excellent.*

Proof. Let u be the vertex of H obtained by identifying u_1 and u_2. Clearly $S=((S_1\cup S_2)-\{u_1,u_2\})\cup\{u\}$ is a dominating set of H and hence $\gamma(H)\leq\gamma(G_1)+\gamma(G_2)-1$. Now, let D be any dominating set of H. Let $D_1=D\cap V(G_1)$ and $D_2=D\cap V(G_2)$. If $u\in D$, then $D_i\cup\{u_i\}$ is a dominating set of G_i for $i=1,2$ and hence $|D|\geq\gamma(G_1)+\gamma(G_2)-1$. Suppose $u\notin D$. Without loss of generality let $v_1\in D_1$ be such that v_1 dominates u. Then D_1 is a dominating set of G_1 and D_2 is a dominating set of G_2-u_2 and hence $|D|\geq\gamma(G_1)+\gamma(G_2)-1$. Thus $\gamma(H)=\gamma(G_1)+\gamma(G_2)-1$ and S is a γ-set for H. Let $v\in V(H)$ and $v\notin S_1\cup S_2$. Then $v\neq u$. Let $v\in V(G_1)$. Then there exists $w\in S_1$ such that $(S_1 - w) \cup \{v\}$ is a γ-set of G_1. Hence $(S - \{w\}) \cup \{u\}$ is a γ-set for H. Thus S is a very excellent γ-set for H and H is very excellent. □

Theorem 3.7. *If u is a down vertex of a graph G, with $\delta(G) > 0$, then $G-u$ is not excellent (and hence not very excellent).*

Proof. Suppose $G-u$ is excellent. Let $v \in N(u)$. Then there exists a γ-set S of $G-u$ such that $v \in S$. Clearly, S is also a dominating set of G, which is a contradiction since u is a down vertex of G □

Theorem 3.8. *In a very excellent tree $T \neq P_2$, every end vertex is a down vertex.*

Proof. If an end vertex is not a down vertex then it is a level vertex. Assume that an end vertex w is a level vertex of T. Let u be the vertex of T adjacent to w. Let $T^{'} = T - w$. We first show that $\gamma(T) = \gamma(T^{'}) = \gamma(T^{'} - u) + 1$. Obtain T^* from T by attaching a path P_2 at w. Then by Theorem 3.1, T^* is very excellent. By Theorem 3.2, $T^{'}$ is very excellent and there exists a very excellent γ-set S of $T^{'}$ containing u and $S-u$ dominates $T^{'} - u$. Hence $\gamma(T^{'}) = \gamma(T^{'} - u) + 1$. As w is a level vertex of T, $\gamma(T^{'}) = \gamma(T)$.

Let D be a very excellent γ-set of T. As w is an end vertex of T, we can assume that $u \in D$. As D is a very excellent γ-set of T, $(D-u) \cup \{w\}$ is a γ-set of T and $D-u$ dominates $T^{'} - u$. As $\gamma(T) = \gamma(T^{'}) = \gamma(T^{'} - u) + 1$, the set $D-u$ is a γ-set of $T^{'} - u$, but not a dominating set of $T^{'}$. As $T \neq P_2$, we can select one vertex $z \in N(u)$ in $T^{'}$. As $z \notin D$, and as D is a very excellent γ-set of T, there exists $y \in D$ such that $(D-y) \cup \{z\}$ is a γ-set of T. Note that $pn[y, D] \subseteq N[z]$ and $y \notin \{z, u\}$. Let $A = (D - \{y, u\}) \cup \{z\}$. Then A is not a dominating set of $T^{'} - u$ [If it is a dominating set of $T^{'} - u$, it is also dominating set of $T^{'}$ as $z \in A \cap N(u)$, which is a contradiction as $|A| < \gamma(T^{'})$]. Hence there exists a vertex $y^{'}$ in $T^{'} - u$ which is not dominated by A. As $z \in A, y^{'} \neq z$. Since $A \cup \{u\} = (D-y) \cup \{z\}$ is a γ - set of T, it follows that $y^{'} \in N(u)$ in $T^{'}$. As $D - u = (A \cup \{y\}) - \{z\}$ is a γ-set of $T^{'} - u$ and as $y^{'}$ is not dominated by A, $y^{'}$ is dominated by y. So $y^{'} \in N[y]$. Thus $y^{'} \in N[y] \cap N(u)$.

As D is a γ-set of T and $y \in D, pn[y, D] \neq \emptyset$. Select a vertex $y^{''} \in pn[y, D] \subseteq N[z]$. (It may happen that $y^{''} = y$). So $y^{''} \in N[z]$. As $y^{''} \in N[z] \cap N[y]$ and $y^{'} \in N[y]$, the vertices $y, y^{'}, z$ and $y^{''}$ are all in the same component of $T-u$ (so of $T^{'} - u$).

But as both z and $y \in N(u)$ and $z \neq y$, and as T is a tree, z and y belong to different components of $T-u$, which is a contradiction. Thus our assumption that w is a level vertex of T is wrong. □

Remark 3.9. *Theorem 3.8 is not true if the graph is not a tree. For example the graph in Figure 5 is very excellent and its only end vertex is not a down vertex.*

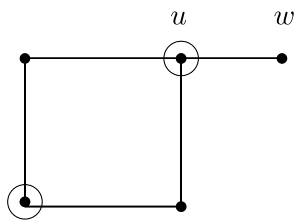
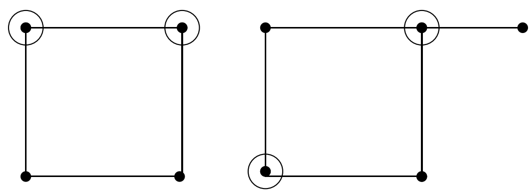


Fig. 5

4 Rigid Very Excellent Graphs

Consider the very excellent graphs G_1 and G_2 given in Figure 6



G_1 G_2

Fig. 6

For both G_1 and G_2, the encircled vertices form a very excellent γ-set D. For the graph G_1, to each vertex $u \notin D$, there are two possible ways to select a vertex $v \in$ D, such that $(D - v) \cup \{u\}$ is a γ-set of G_1. But for the graph G_2, to each vertex $u \notin D$, there is only one possible way to select a vertex $v \in D$ such that $(D - v) \cup \{u\}$ is a γ-set of G_2. We say that the very excellent γ-set D of G_2 is rigid.

Definition 4.1. *Let G be a very excellent graph and D be a very excellent γ-set of G. To each $u \notin D$, let $E(u, D) = \{v \in D \mid (D - v) \cup \{u\}$ is a γ-set of $G\}$ (i.e., $E(u, D)$ is the set of vertices of D which are exchangeable with u). If $|E(u, D)| = 1$, for all $u \notin D$, then D is said to be a rigid very excellent γ-set of G. If G has at least one rigid very excellent γ-set then G is said to be rigid very excellent (RVE).*

For example the paths P_2, P_4, P_7, cycles C_3, C_7, generalized Petersen's graph $P(7, 2)$ are all RVE.
The following two theorems are analogous to Theorems 3.1 and 3.2 (we omit the proofs).

Theorem 4.2. *If G is RVE and u is a level vertex of G, then the graph H obtained from G by attaching a path P_2 at u is RVE.*

Theorem 4.3. *Let u be a vertex of a graph G. A graph H is obtained from G by attaching a path P_3 at u. Then H is RVE if and only if G is RVE and there exists a RVE γ-set D of G such that $u \in D$ and $D - u$ dominates $G - u$.*

As for trees, we have the following theorem :

Theorem 4.4. *Every very excellent γ-set of a very excellent tree is RVE.*

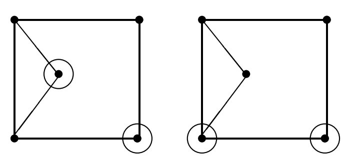
Proof. Let D be a very excellent γ-set of T. We claim that D is a RVE γ-set of T. If our claim is not true, then there exists a vertex $u \notin D$ such that $E(u, D)$ contains at least two distinct vertices, say u_1, u_2 of D. As $u_i \in E(u, D), (D - u_i) \cup \{u\}$ is a γ-set of T and hence $pn[u_i, D] \subseteq N[u]$. Also, as D is a γ-set of T, $pn[u_i, D] \neq \emptyset$. Let T_{u_i} be the component of $T - u$ that contains the vertex u_i. Let $T_{u_i} \cap N[u] = \{w_i\}$.

Suppose $T_{u_1} = T_{u_2}$. Then $w_1 = w_2$ and as $pn[u_1, D] \cup pn[u_2, D] \subset N[u]$, we have $pn[u_1, D] \cup pn[u_2, D] \subseteq \{u, w_1\}$. As $pn[u_1, D]$ and $pn[u_2, D]$ are non empty and disjoint sets, $pn[u_i, D] = \{u\}$ for one $i \in \{1, 2\}$ and $pn[u_j, D] = \{w_1\}$ for $j \neq i \in \{1, 2\}$. As $u_i \in D$, and $pn[u_i, D] = \{u\}$ we get $\{u_i\} \in N[u] \cap D \cap T_{u_i}$ and $w_1 = u_i \in D$. This is a contradiction as $u_i, u_j \in D, u_i \neq u_j$ and $u_i \notin pn[u_j, D]$. Thus $T_{u_1} \neq T_{u_2}$ and hence $w_1 \neq w_2$. Now $pn[u_i, D] \subseteq \{u, w_i\}$, for $i = 1, 2$.

Let $v \in T - \{u, w_1, w_2\}$. If $v \in T_{u_1}$, as $v \neq w_1$ and $v \notin pn[u_1, D], v$ is dominated by some $u' \neq u_1 \in D$. As u_2 and v are in different components of $T - u, u' \neq u_2$. So v is dominated by some $u' \in D - \{u_1, u_2\}$. Similarly if $v \in T_{u_2}$, then v is dominated by some $u' \in D - \{u_1, u_2\}$.

If $v \notin T_{u_1} \cup T_{u_2}$, then $v \notin N[u_1] \cup N[u_2]$ and hence v is dominated by some $u' \in D - \{u_1, u_2\}$. Thus $D - \{u_1, u_2\}$ is a dominating set of $T - \{u, w_1, w_2\}$ and hence $(D - \{u_1, u_2\}) \cup \{u\}$ is a dominating set of T, which is a contradiction as $|(D - \{u_1, u_2\}) \cup \{u\}| = |D| - 1 < \gamma(T)$. Thus $|E(u, D)| \leq 1$, for all $u \in D$. As D is a very excellent γ-set of T, $E(u, D) \neq \emptyset$, for all $u \notin D$. Thus $|E(u, D)| = 1$ for all $u \in D$ and hence D is RVE. □

Remark 4.5. *By the above theorem, every very excellent γ-set of a tree is rigid. But the result is not true in general for RVE graphs. For example for the graph in Figure 7(a) the encircled vertices form a RVE γ-set, while the encircled vertices in Figure 7(b) is a very excellent γ-set but not a RVE γ-set.*



(a) (b)
Figure 7

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