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Partially Balanced Incomplete Block Designs Arising From Minimal Dominating sets of a Graph

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Abstract

In this paper, we determine the number of minimum dominating sets of paths and cycles and prove that the set of all minimum dominating sets of a cycle forms a partially balanced incomplete block design. We also determine all cubic graphs on ten vertices in which the set of all minimum dominating sets forms a partially balanced incomplete block design.

Keywords: PBIBD, domination, paths, cycles.

2000 Mathematics Subject Classification: 05C

1. Introduction

The relation between graph theory and partially balanced incomplete block designs (PBIBD) is not a new one and R. C. Bose, in his pioneering paper [1], established the relation between PBIBDs and strongly regular graphs. R.C. Bose [2] has shown that strongly regular graphs emerge from PBIBD; with 2-association schemes. Harary et al. [4, 5] considered the relation between isomorphic factorization of regular graphs and PBIBD with 2-association scheme.

In this paper, we establish a link between PBIBD and graphs through the collection of minimum dominating sets. We prove that the set of all minimum dominating sets of a cycle forms a PBIBD. We also determine all cubic graphs on ten vertices in which the set of all minimum dominating sets forms a PBIBD.

Throughout this paper, $G = (V, E)$ stands for a finite, connected, undirected graph with neither loops nor multiple edges. Terms not defined here are used in the sense of Harary[3]. A subset D of V is called a *dominating set* in G, if every vertex in $V - D$ is adjacent to a vertex in D. The minimum cardinality of a dominating set in G is called the *domination number* of G and is denoted by $\gamma(G)$ or simply γ. A dominating set of cardinality γ is called a *minimum dominating set* or a γ*-set*. An excellent treatment of fundamentals of domination in graphs and several advanced topics in domination are given in Haynes et al. [6, 7].

Definition 1.1. [1] *Given a set* $\{1, 2, \ldots, v\}$, *a relation satisfying the following conditions is said to be an association scheme with* m *classes.*

- (i) *Any two symbols* α, β *are* i^{th} *associates for some* i, *with* $1 \leq i \leq m$ *and this relation of being* i^{th} *associates is symmetric.*
- (ii) *The number of* i^{th} *associates of each symbol is* n_i.
- (iii) *If* α *and* β *are two symbols which are* i^{th} *associates, then the number of symbols which are* j^{th} *associates of* α *and* k^{th} *associates of* β *is* p^i_{jk}, *and is independent of the pair of* i^{th} *associates* α *and* β.

Definition 1.2. [1] *Consider a set of symbols* $S = \{1, 2, \ldots, v\}$ *and an association scheme with* m *classes. A partially balanced incomplete block design (PBIBD) is a collection of* b *subsets of* S, *each of cardinality* k $(k < v)$, *such that every symbol occurs in exactly* r *subsets and two symbols* α *and* β *which are* i^{th} *associates occur together in* λ_i *sets, the number* λ_i *being independent of the choice of the pair* α, β.

The numbers v, b, r, k, λ_i $(i = 1, 2, \ldots, m)$ *are called the parameters of the first kind and the numbers* n_i *'s and* p^i_{jk} *'s of the first definition are called the parameters of the second kind.*

2. Minimum Dominating sets in Paths and Cycles

Let $M^o_\gamma(G)$ denote the number of minimum dominating sets in G. We proceed to determine $M^o_\gamma(G)$ for paths and cycles.

Theorem 2.1.

$$M^o_\gamma(P_n) = \begin{cases} 1 & \text{if } n = 3k, k \geq 1 \\ \frac{k^2+5k+2}{2} & \text{if } n = 3k+1, k \geq 0 \\ k+2 & \text{if } n = 3k+2, k \geq 0. \end{cases}$$

Proof. Let $P_n = (v_1, v_2, \dots, v_n)$.

Case (i). $n = 3k$, where $k \geq 1$.

Then $\gamma(P_n) = k$ and $\{v_i / i \equiv 2(mod\ 3)\}$ is the unique γ-set of P_n, so that $M_\gamma^o(P_n) = 1$.

Case (ii). $n = 3k+1, k \geq 0$.

Claim 1. There exists a unique γ-set of P_n containing v_1 and v_{3k+1}.

Clearly $D = \{v_1, v_4, \dots, v_{3k+1}\}$ is a γ-set of P_n. Further, D is a γ-set of P_n containing v_1 and v_{3k+1} if and only if $D_1 = D - \{v_1, v_{3k+1}\}$ is a γ-set of the path $(v_3, v_4, v_5, \dots, v_{3k-3})$. By Case (i), D_1 is unique and hence D is unique.

Claim 2. The number of γ-sets of P_n containing v_1 and v_{3k} is k.

We prove this by induction on k. When $k = 1$, $\{v_1, v_3\}$ is the required unique γ-set of P_4 and hence the result is true. Assume that the result is true for $k-1$. Let $D_1, D_2, \dots, D_{k-1}$ be the γ-sets of P_{3k-2} containing the vertices v_1 and v_{3k-3}. Then $D_i \cup \{v_{3k}\}$ is a γ-set of P_{3k+1} containing v_1 and v_{3k}. Further by Claim 1, there exists a unique γ-set D_k of P_{3k-2} containing v_1 and v_{3k-2} and $D_k \cup \{v_{3k}\}$ is another γ-set of P_{3k+1} containing v_1 and v_{3k}.

Also any γ-set D of P_{k+1} containing v_1 and v_{3k} is one of the sets $D_i \cup \{v_{3k}\}$. Hence Claim 2 follows.

Claim 3. The number of γ-sets of P_{3k+1} containing v_2 and v_{3k+1} is k.

Proof of Claim 3 is similar to the proof of Claim 2.

Claim 4. $M_\gamma^o(P_{3k+4}) = M_\gamma^o(P_{3k+1}) + k + 3$ for all $k \geq 0$.

By Claim 1, there exists exactly one γ-set D of P_{3k+1} such that $v_1, v_{3k+1} \in D$ and D gives rise to two γ-sets of P_{3k+4}, namely, $D \cup \{v_{3k+3}\}$ and $D \cup \{v_{3k+4}\}$.

Also by Claim 2, there exist exactly k γ-sets $D_1, D_2, \dots, D_k$ of P_{3k+1} such that $v_1, v_{3k} \in D_i$ and each D_i gives rise to one γ-set of P_{3k+4}, namely $D_i \cup \{v_{3k+3}\}$. Now, by Claim 3, there exist exactly k γ-sets $S_1, S_2, \dots, S_k$ of P_{3k+1} such that $v_2, v_{3k+1} \in S_i$ and each S_i gives rise to two γ-sets of P_{3k+4}, namely, $S_i \cup \{v_{3k+3}\}$ and $S_i \cup \{v_{3k+4}\}$. Also, any γ-set of P_{3k+1} containing v_2 and v_{3k} gives rise to exactly one γ-set of P_{3k+4}. Further, there exist two more γ-sets of P_{3k+4}, namely, $\{v_2, v_5, v_8, \dots, v_{3k-1}, v_{3k+2}, v_{3k+3}\}$ and $\{v_2, v_5, v_8, \dots, v_{3k-1}, v_{3k+2}, v_{3k+4}\}$. Hence $M_\gamma^o(P_{3k+4}) = M_\gamma^o(P_{3k+1}) + k + 3$.

By solving this recurrence relation, using the condition $M_\gamma^o(P_4) = 4$, we get $M_\gamma^o(P_{3k+1}) = \frac{k^2+5k+2}{2}$.

Case iii. $n = 3k+2, k \geq 0$.

It can be easily verified that there is no γ-set of P_{3k+2} containing both v_1 and v_{3k+2}. Also, there exists exactly one γ-set of P_{3k+2} containing v_2 and v_{3k+2} and exactly one γ-set containing v_1 and v_{3k+1}. Also, it can be proved by induction on k that the number of γ-sets of P_{3k+2} containing v_2 and v_{3k+1} is k. Since every γ-set of P_{3k+2} contains exactly one of the vertices v_1, v_2 and exactly one of the vertices v_{3k+1}, v_{3k+2}, it follows that $M_\gamma^o(P_{3k+2}) = k+1$. □

We now proceed to determine the number of γ-sets of a cycle.

Theorem 2.2.

$$M_\gamma^o(C_n) = \begin{cases} 3 & \text{if } n = 3k, k \geq 1 \\ \frac{(3k+1)(k+2)}{2} & \text{if } n = 3k+1, k \geq 1 \\ 3k+2 & \text{if } n = 3k+2, k \geq 1. \end{cases}$$

Proof. Let $C_n = (v_1, v_2, v_3, \ldots, v_{n-1}, v_n, v_1)$.

Case i. $n = 3k, k \geq 1$.

Then $D_i = \{v_j / j \equiv i(mod\ 3)\}, 1 \leq i \leq 3$, are the only γ-sets of C_n so that $M^o_\gamma(C_n) = 3$.

Case ii. $n = 3k + 1$.

If D is a γ-set of C_{3k+1} containing two adjacent vertices, say, v_1 and v_2, then $D - \{v_1, v_2\}$ is a γ-set of the path $P_{3k-3} = (v_4, v_5, v_6, \ldots, v_{3k})$ and hence it follows from Theorem 2.1, that there is exactly one γ-set of C_{3k+1} containing two adjacent vertices.

We now claim that the number of γ-sets of C_n containing a given vertex, say v_1, is $\frac{(k+1)(k+2)}{2}$. For any γ-set D of the path $P_{3k-2} = (v_2, v_3, \ldots, v_{3k})$, $D \cup \{v_1\}$ is a γ-set of C_{3k+1}. Also there exists one γ-set of C_{3k+1} containing the adjacent vertices v_1 and v_2 and another γ-set containing the adjacent vertices v_1 and v_i. Hence the number of γ-sets of C_{3k+1} containing v_1 is $M^o_\gamma(P_{3k-2}) + 2$.

Also, by Theorem 2.1, $M^o_\gamma(P_{3k-2}) = \frac{(k-1)^2+5(k-1)+2}{2}$ and hence the number of γ-sets of C_{3k+1} containing v_1 is $\frac{(k+1)(k+2)}{2}$.

Since the cardinality of any γ-set of C_{3k+1} is $k+1$ and each vertex of C_{3k+1} lies in exactly $\frac{(k+1)(k+2)}{2}$ γ-sets, it follows that $\frac{(3k+1)(k+1)(k+2)}{2} = M^o_\gamma(C_{3k+1})(k+1)$, so that $M^o_\gamma(C_{3k+1}) = \frac{(3k+1)(k+2)}{2}$.

Case iii. $n = 3k + 2$.

We first prove by induction on k that the number of γ-sets of C_{3k+2} containing v_1 is $k+1$. When $k = 1$, $\{v_1, v_3\}$ and $\{v_1, v_4\}$ are the γ-sets of C_5 containing v_1. We now assume that the result is true for $k-1$, and let $D_1, D_2, \ldots, D_k$ be the γ-sets of C_{3k-1} containing v_1. Then $D_i \cup \{v_{3k}\}$ is a γ-set of C_{3k+2}. Further $D = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k+1}\}$ is another γ-set of C_{3k+2} containing v_1. Also $D, D_1, D_2, \ldots, D_k$ are the only γ-sets of C_{3k+2} containing v_1, and hence every vertex of C_{3k+2} lies in exactly $(k+1)$ γ-sets. Further, $\gamma(C_{3k+2}) = k+1$ and hence it follows that $M^o_\gamma(C_{3k+2}) = 3k+2$. □

3. Minimum Dominating sets and PBIBDs

We now proceed to establish a relation between the set of all minimum dominating sets and PBIBDs for cycles and some cubic graphs on ten vertices.

Definition 3.1. *A graph G is called a PBIB-graph if the set of all minimum dominating sets of G forms a PBIBD with a suitable m-association scheme.*

Theorem 3.2. *The collection of all minimum dominating sets of the cycle C_n, where $n \equiv 0(mod\ 3)$ and $n \geq 6$ are the blocks of a PBIBD with 2-association scheme and parameters $v = n$, $b = \frac{n}{3}$, $r = 1, k = 3, \lambda_1 = 0, \lambda_2 = 1$.*

Proof. Let $C_n = (v_1\ v_2 \ldots v_n\ v_1)$.

By Theorem 2.2, $M^o_\gamma(C_n) = 3$ and the three γ-sets are given by $D_i = \{v_j / j \equiv i(mod\ 3)\}$, $i = 1, 2, 3$. Two distinct vertices u and v are said to be first associates if $d(u, v) \equiv 0(mod\ 3)$; otherwise they are said to be second associates. Clearly, the parameters of the second kind are given by $n_1 = \frac{n}{3} - 1, n_2 = \frac{2n}{3}$ and

$$P^1 = \begin{pmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{pmatrix} = \begin{pmatrix} \frac{n}{3} - 2 & 0 \\ 0 & \frac{2n}{3} \end{pmatrix}$$

$$P^2 = \begin{pmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{n}{3} - 1 \\ \frac{n}{3} - 1 & 0 \end{pmatrix}$$

The three γ-sets D_1, D_2 and D_3 are the blocks of a PBIBD with parameters $v = n$, $b = 3, k = \frac{n}{3}, r = 1, \lambda_1 = 1$ and $\lambda_2 = 0$. □

Remark 3.3. *If G is a PBIB-graph, then the number of γ-sets containing any particular vertex v is the same for all vertices in G.*

Theorem 3.4. *Any distance transitive graph G is a PBIB-graph.*

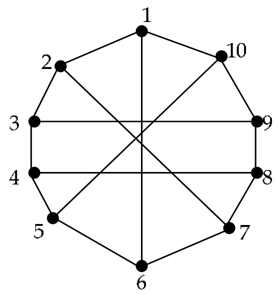
Proof. Let $m = diam\ G$.

We define two vertices u and v to be i^{th} associates if $d(u,v) = i$. Since G is vertex transitive this defines an association scheme with m classes. Further every vertex of G lies in exactly r minimum dominating sets. Now let $u, v, x, y \in V(G)$ and $d(u,v) = i = d(x,y)$. Since G is distance transitive there exists an automorphism α such that $\alpha(u) = x$ and $\alpha(v) = y$. Hence if $\{D_1, D_2, \ldots, D_{\lambda_1}\}$ is the collection of γ-sets containing u and v, then $\{\alpha(D_1), \alpha(D_2), \ldots, \alpha(D_{\lambda_i})\}$ is the collection of γ-sets containing x and y. Hence the set of all γ-sets of G forms a PBIBD. □

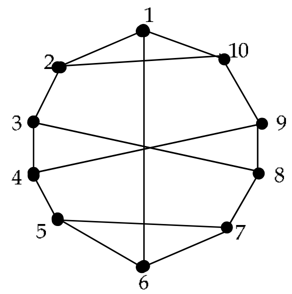
Corollary 3.5. *The collection of all minimum dominating sets of the cycle C_n, where $n \equiv 1$ or $2 (mod\ 3)$ are the blocks of a PBIBD with m association schemes where $m = diam\ (C_n)$.*

Example 3.6. *Consider the cycle $C_8 = (0, 1, 2, \ldots, 7)$. By Corollary 3.5 the minimum dominating sets of C_i are the blocks of a PBIBD with 4-association scheme. The minimum dominating sets are given by* $\{0,3,6\}, \{1,4,7\}, \{2,5,0\}, \{3,6,1\}, \{4,7,2\}, \{5,0,3\}, \{6,1,4\}$ and $\{7,2,5\}$. *The parameters $(v, b, r, k, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are given by $v = 8, b = 8, r = 3, k = 3, \lambda_1 = 0, \lambda_2 = 1$,* $\lambda_3 = 2$ and $\lambda_4 = 0$.

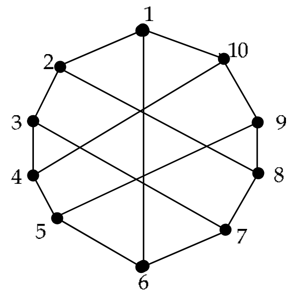
We now proceed to determine all cubic graphs with ten vertices in which the set of all minimum dominating sets forms a PBIBD. We observe that if G is a cubic graph on ten vertices then $\gamma(G) = 3$. There are 21 cubic graphs on ten vertices which are given below.



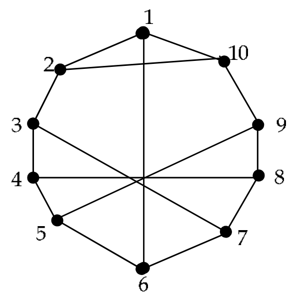
G_1



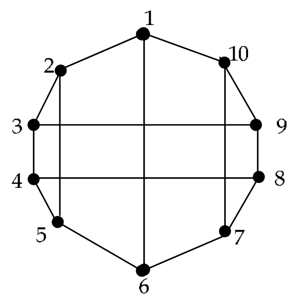
G_2



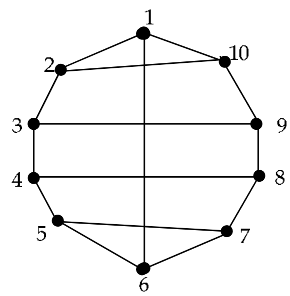
G_3



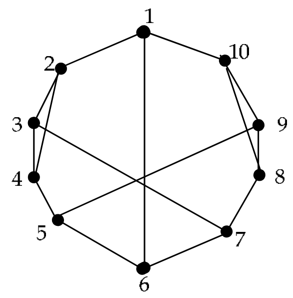
G_4



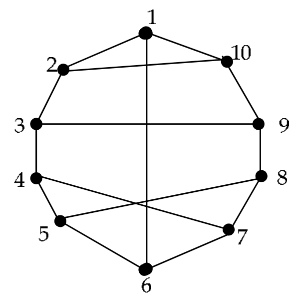
G_5



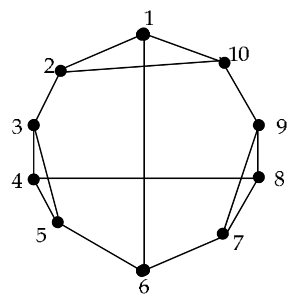
G_6



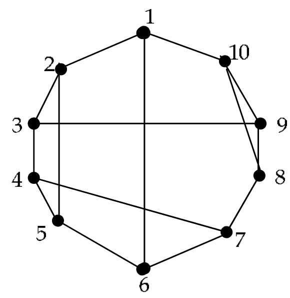
G_7



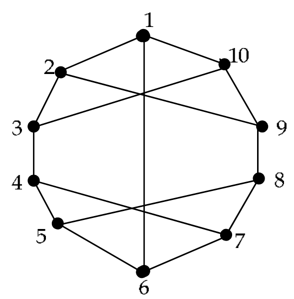
G_8



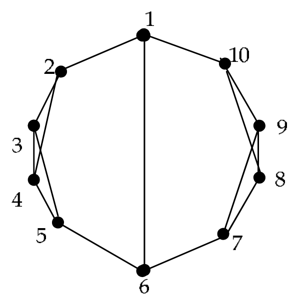
G_9



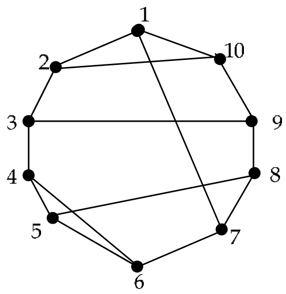
G_{10}



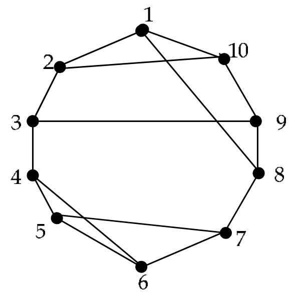
G_{11}



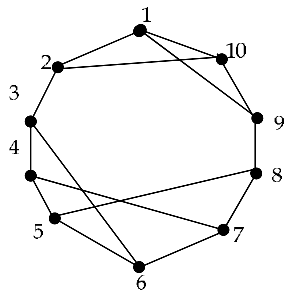
G_{12}



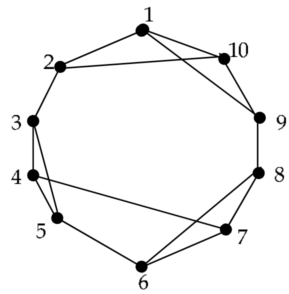
G_{13}



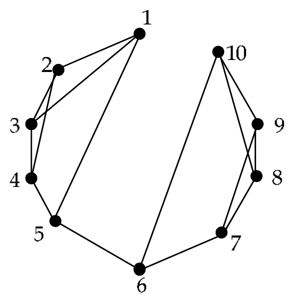
G_{14}



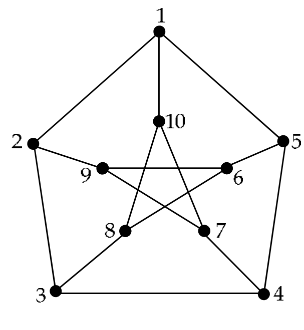
G_{15}



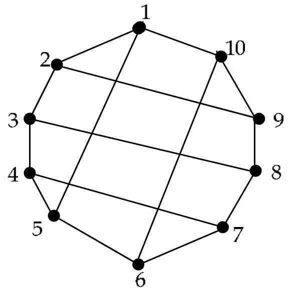
G_{16}



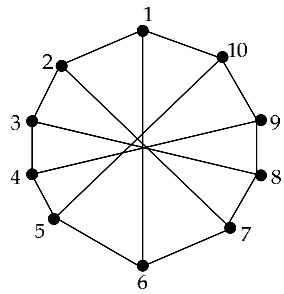
G_{17}



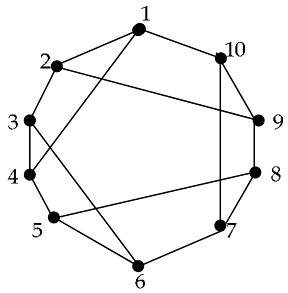
G_{18}



G_{19}



G_{20}



G_{21}

We prove that only four of them namely G_{18}, G_{19}, G_{20} and G_{21}, are PBIB-graphs.

Theorem 3.7. *The graphs* G_i, $1 \leq i \leq 17$ *are not PBIB-graphs.*

Proof. We prove the theorem for the graphs G_1, G_5, G_{11} and the proofs are similar for the remaining cases. In G_1, there is exactly one γ-set containing 1 and there are two γ-sets containing 5 and hence G_1 is not a PBIB-graph. In G_5, there are only three γ-sets namely $\{1,2,8\}, \{1,4,8\}$ and $\{3,6,9\}$ and hence is not a PBIB-graph. In G_{11}, the vertex 2 is not in any γ-set and hence G_{11} is not a PBIB-graph. □

Theorem 3.8. *The graphs* G_{18}, G_{19}, G_{20} *and* G_{21} *are PBIB-graphs*

Proof. It can be easily verified that the four graphs given in the theorem are distance transitive and hence the results follow from Theorem 3.4. □

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