AKCE J. Graphs. Combin., 4, No. 2 (2007), pp. 127-138

Further Results on Domination in Graphoidally Covered Graphs

B. D. Acharya
Department of Science and Technology
Technology Bhawan, New Mehrauli Road
New Delhi-110 016, India.
E-mail: *bdacharya@yahoo.com*
and
Purnima Gupta
Department of Mathematics
Sri Venkateswara College
(University of Delhi), Rao Tula Ram Marg
Dhaula Kuan, New Delhi -110 021, India.
purnimachandni@rediffmail.com

Abstract

In any graph $G = (V, E)$ that is not necessarily finite, a *graphoidal cover* is a set ψ of nontrivial paths $P_1, P_2, \ldots,$ not necessarily open and called ψ *-edges*, such that **(GC-1)** no vertex of G is an internal vertex of more than one path in ψ, and **(GC-2)** every edge of G is in exactly one of the paths in ψ. A ψ *-dominating set* of G is then defined as a set D of vertices in G such that every vertex of G is either in D or is an end-vertex of a ψ-edge having its other end-vertex in D. In this note, we present some new results that facilitate having more insight into the notion of ψ-domination in graphs; particularly, we give a characterization of (i) finite connected graphs possessing a graphoidal cover ψ such that (G, ψ) is ψ-independent and (ii) trees and unicyclic graphs which possess a graphoidal cover ψ such that their ψ-domination numbers turn out to be one.

Keywords: Graphoidal cover, graphoidally covered graph, ψ-domination.

2000 Mathematics Subject Classification: 05C

1. Introduction

All graphs considered in this article are simple, self-loop-free and not necessarily finite (*cf.*: Ore [10]). For standard terminology and notations, however, we refer the reader to Harary [8] or West [13].

In any graph $G = (V, E)$ that is not necessarily finite, a *graphoidal cover* is a set ψ consisting of nontrivial paths $P_1, P_2, \ldots,$ each of which is not necessarily open and called a ψ *-edge*, such that **(GC-1)** no vertex of G is an internal vertex of more than one path in ψ, and **(GC-2)** every edge of G is in exactly one of the paths in ψ (cf. Acharya and Sampathkumar [4]; *see* Arumugam et al. [5] for a creative review and Acharya and Purnima Gupta [3] for extension of the notion to infinite graphs). The ordered pair (G, ψ) is then called a *graphoidally covered graph* in which we shall say that two vertices u and v are ψ*-adjacent* if they happen to be the end-vertices of some ψ-edge.

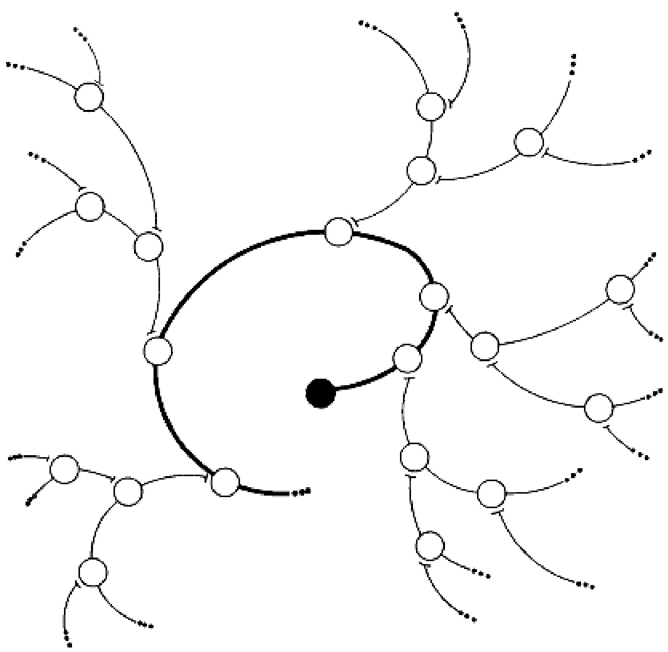

Figure 1. Example of an infinite graph G and a graphoidal cover ψ of G. Each of the ψ-edges is a one way infinite path and (G, ψ) is ψ-independent.

Unless mentioned or specified otherwise, we continue to use the terminology and notation developed in these authors earlier paper [3] except that what were called 'ψ-paths' therein will now on be called ψ*-edges*. We denote by $\mathcal{G}_G$ the set of all graphoidal covers of the given graph G.

Clearly, for any graph $G = (V, E)$, the set $E := E(G)$ of its edges is itself a graphoidal cover of G, which we shall refer to as the *trivial graphoidal cover* of G. In general, it may be gainful to note that the notion of a graphoidal cover of a graph is an instance of a more general discrete structure than that of a graph, called *polar configuration* introduced by the first author [2]. Further, graphoidal cover of a graph, as a discrete structure on its own right, resembles the *underlying graph* in many ways; for example, any vertex is internal to at most one ψ-edge as any *ordinary* edge of the underlying graph of the cover trivially has (since no edge of it has any 'internal vertex' as such). We shall point out such common features of graphoidal covers whenever found prominent or pertinent as we proceed with this article. Our focus however will be on the notion of *domination*.

Extending the standard notion of domination in graphs (Ore [10]), these authors [3] defined a ψ-*dominating set* (or, 'ψ-domset' in short) of a graphoidally covered graph (G,ψ) as a set D of vertices in G such that every vertex of G is either in D or is an end-vertex of a ψ-edge having its other end-vertex in D. We denote by $\mathcal{D}_\psi(G)$ the set of all ψ-domsets of (G,ψ). Clearly, the vertex set $V(G)$ of G is a ψ-domset for any $\psi \in \mathcal{G}_G$ so that $V(G) \in \mathcal{D}_\psi(G)$ for any $\psi \in \mathcal{G}_G$; therefore, we call $V(G)$ the *trivial* ψ-domset of (G,ψ); a graphoidal cover ψ such that $\mathcal{D}_\psi(G) = \{V(G)\}$ will be called a *totally disconnecting graphoidal cover* of G, in the sense that no two vertices in G are then ψ-adjacent, and the corresponding graphoidally covered graph (G,ψ) is said to be ψ-*independent*; *see* Figures 1, 2(a) and 2(b) for few typical instances of such graphoidal covers. A ψ-domset D of (G,ψ) is said to be *proper* if $V(G) - D \neq \emptyset$ and a set X of vertices in (G,ψ) is said to be ψ-*independent* if no two vertices in X are ψ-adjacent. The following characterizations of totally disconnecting graphoidal covers of G are known.

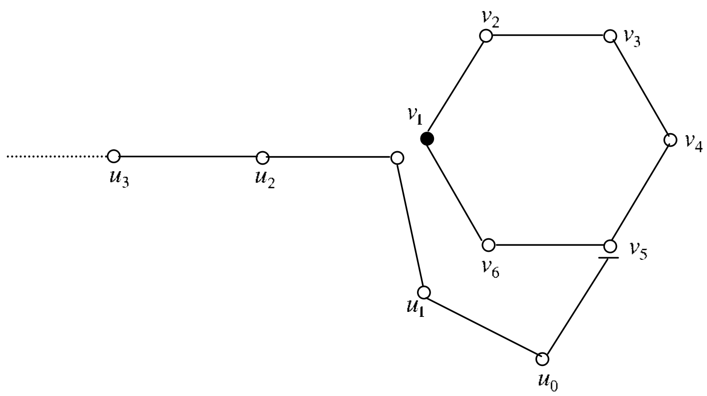

Figure 2(a) $\psi_1 = \{(v_1v_2v_3v_4v_5v_6v_1), (v_5u_0u_1v_1u_2u_3\dots)\}.(G_1,\psi_1)$ is ψ_1-independent.

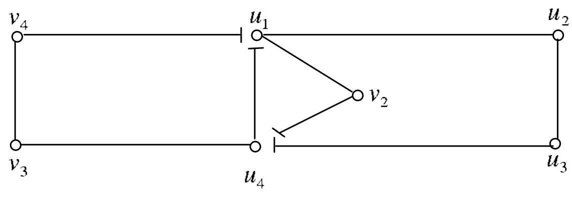

Figure 2(b) $\psi_2 = \{(u_1v_4v_3u_4u_1), (u_4u_3u_2u_1v_2u_4)\}.(G_2,\psi_2)$ is ψ_2-independent.

Theorem 1.1. [3] *Given a graphoidally covered graph (G,ψ), the following statements are equivalent:*

(a) $\mathcal{D}_\psi(G) = \{V(G)\}$

(b) ψ *contains no open path of finite length*

(c) *No two vertices of* G *are* ψ*-adjacent.*

Remark 1.2. *It is possible that* G *may not have a finite domset, but may possess a graphoidal cover* ψ *such that* (G,ψ) *has a finite* ψ*-domset as the following example shows. Let* $V = N$, *the set of all natural numbers and define a graph* $G = (V,E)$ *where an unordered pair* uv *of vertices* u *and* v *is in* E *whenever the greater of the two numbers* u *and* v *is odd. It may be easily seen that* G *is an infinite split graph with the set of all even integers together with the vertex* 1 *forming the maximum independent set and the set of all odd integers forming a maximum clique. Clearly,* G *has no finite domset. Consider the* ψ*-cover* $\psi = \psi_1 \bigcup (E(G) - \cup_{P\in\psi_1} E(P))$ *where* $\psi_1 = \{(2, 2n+3, 2n+2) : n \in N\}$. *We see that the graphoidally covered graph* (G,ψ) *has a finite* ψ*-domset, viz.,* $\{1,2\}$.

In this article, we bring out certain new aspects of graphoidally covered graphs and use them to make new observations on *graphoidal dominating sets* and the ψ-domination number of a graph.

2. Graphoidal Independence

Characterizing graphs G which admit graphoidal covers ψ such that $\mathcal{D}_\psi(G) = \{V(G)\}$ was posed as an open problem and to this date it continues to be so. However, some necessary conditions for a graph to admit such graphoidal covers have been found [3]. Recently, we have characerized finite graphs that possess totally disconnecting graphoidal covers. For stating this result, we need to recall some terminology and notation from our original paper [3]. Given a graph $G = (V,E)$, by a *closed hyperchain*, or a 'hypercycle', we mean a sequence $(x_1, P_1, x_2, P_2, \ldots, x_k, P_k, x_1)$, $k \geq 2$ where x_i's are all distinct vertices and each P_i is either a cycle or an infinite path in G such that the following conditions are satisfied with indices reduced modulo k:

(1) For each i, $x_i \in V(P_i)$

(2) $x_{i+1} \in V(P_i) - \{x_i\}$ for each i

(3) $V(P_i) \cap V(P_{i+1}) = \{x_{i+1}\}$ for every i

(4) $|i-j| \geq 2$ implies $V(P_i) \cap V(P_j) = \emptyset$.

Here, k is called the *length* of the hypercycle. Further, a hypercycle is called a *necklace block* if each path P_i in its description happens to be a cycle of G. A *cycle-cactus* is a graph in which every block is a cycle. By a *nontrivial noncycle block* we mean any block which is neither isomorphic to K_2 nor to any cycle C_n, $n \geq 3$. We can now state our new result.

Theorem 2.3. *Let G be a connected finite graph. Then, there exists a graphoidal cover ψ of G such that $\mathcal{D}_\psi(G) = \{V(G)\}$ if and only if G is a 2-edge connected graph satisfying exactly one of the following two conditions:*

(a) *G is a cycle-cactus*

(b) *G has exactly one nontrivial noncycle block which is a necklace block and every other block is a cycle in G.*

Proof. Let $\psi \in \mathcal{G}_G$ be such that $\mathcal{D}_\psi = \{V(G)\}$. First, we show that G must be 2-edge connected. Suppose it is not. Then, G must contain a *bridge* (that is, a trivial block of G) and hence, by **GC-2**, ψ contains an open path containing this bridge. Since G is finite, this ψ-edge has two ends, whence there exists a proper ψ-domset in (G, ψ), a contradiction to the hypothesis. Thus, G must be 2-edge connected. Next, the *necessity* part of the proof being already available in ([3], Lemma 2.12), it is enough to establish the *sufficiency* part of the proof. Toward this end, suppose first that the structure of G is as given in the statement (a) of the theorem. We will prove that there exists $\psi \in \mathcal{G}_G$ such that G is ψ-independent. The following algorithm constructs such a graphoidal cover ψ for G.

Step 1. Let $\mathcal{B}_0$ be any cycle block of G and let $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ be the cycle blocks of G such that $V(\mathcal{B}_i) \cap V(\mathcal{B}_0) \neq \emptyset$, $1 \leq i \leq k$. Condition (a), implies, that $|V(\mathcal{B}_i) \cap V(\mathcal{B}_0)| = 1$, $1 \leq i \leq k$ and $|V(\mathcal{B}_i) \cap V(\mathcal{B}_j)| = \phi$, for all distinct $i, j \in \{1, 2, \ldots, k\}$. If $V(G) = \bigcup_{0 \leq i \leq k} V(\mathcal{B}_i)$, then let $\psi = \{\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_k\}$ and for each $i \in \{1, 2, \ldots, k\}$ if $\{v_i\} = V(\mathcal{B}_i) \cap V(\mathcal{B}_0)$, let v_i be identified as the end-vertex of $\mathcal{B}_i$, and let any arbitrary vertex u of $V(\mathcal{B}_0)$ be the end-vertex of $\mathcal{B}_0$. Clearly, G is ψ-independent.

Step 2. Let $V(G) \neq \bigcup_{0 \leq \ell \leq k} V(\mathcal{B}_\ell)$. Then, for every $i \in \{1, 2, \ldots, k\}$ let $\mathcal{B}_{i\ell}$ be the cycle block such that $V(\mathcal{B}_i) \cap V(\mathcal{B}_{i\ell}) \neq \emptyset$, $1 \leq \ell \leq k_i$. We repeat the procedure as in Step 1, replacing $\mathcal{B}_1$ with $\mathcal{B}_0$ except that the end vertex of $\mathcal{B}_i$ has been already fixed as in Step 1.

If $V(G) = V(\mathcal{B}_0) \cup \bigcup_{i=1}^{k} (V(\mathcal{B}_i) \cup V(\mathcal{B}_{ik_i}))$ then we let $\psi = \{\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_{11}, \ldots, \mathcal{B}_{1k_1}, \ldots, \mathcal{B}_k, \mathcal{B}_{k1}, \ldots, \mathcal{B}_{kk_k}\}$ and fix the end-vertices as in Step 1.

Since the graph G is finite, connected and satisfies (a), the above procedure will end in a finite number of steps when all the edges of G are exhausted, yielding at the end a totally disconnecting graphoidal cover ψ of G.

Next, let G satisfy condition (b) of the theorem, *viz.*, G has a necklace block, $\mathcal{B} = \{x_1, P_1, x_2, \ldots, x_k, P_k, x_1\}$. Let x_i be the end-vertex of P_i for each i, $1 \leq i \leq k$. Thereafter, any cycle P that has a nonempty intersection with $\mathcal{B}$ has that vertex x_P for its both the ends since the vertex x_p in the intersection $V(P) \cap V(\mathcal{B})$ is unique. We continue in this manner and since G is finite, connected and satisfies (b), we exhaust all the cycles P by fixing their ends x_p successively. Let ψ consist of these cycles. It is easy to see that G is then ψ-independent and the proof is complete. □

Due to Theorem 2.3, some new perspectives are now in order. The following extension of the well known notion of *eulerian graphs* to the class of infinite graphs is fundamental to gain new insight into the problem of determining infinite graphs possessing totally disconnecting graphoidal covers.

Definition 2.4. [7] *A graph G, not necessarily finite, is* eulerian *if its edge set $E(G)$ can be partitioned into subsets $E_1, E_2, \ldots, E_i, \ldots,$ such that the subgraph spanned by each of these sets is either a cycle or a two-way infinite path in G.*

It may be easily shown that a necessary and sufficient condition for a graph to be eulerian is that the degree of every vertex in the graph is either finite and even or is infinite [7]. Hence, the following result is a step forward, whose first part has already been established in our earlier work [3].

Theorem 2.5. *Let G be a connected graph having a graphoidal cover ψ such that $\mathcal{D}_\psi(G) = \{V(G)\}$. Then,*

(i) *ψ contains at most one two-way infinite path;*

(ii) *if ψ does not contain any one-way infinite path then G is eulerian.*

Proof. As mentioned above, a proof of the first part is already known. Next, to prove (ii), note that part (b) of Theorem 1.1 and the hypothesis of condition (ii) together imply that all but possibly one member of ψ are cycles; if a member of ψ is not a cycle of G then it must be a two-way infinite path in G and then it must be unique due to the assertion (i). Then, since G is connected by hypothesis, the result follows by invoking the axioms of a graphoidal cover. □

We now raise few potential open problems. If (G, ψ) has a finite ψ-domset then, the infimum of the cardinalities of the ψ-domsets of (G, ψ), denoted $\gamma_\psi(G)$, is called the *ψ-domination number* of (G, ψ). Henceforth, whenever we mention of ψ-domination number, we shall mean implicitly that (G, ψ) has a finite ψ-domset (see Figure 3). Although the general problem of determining exactly under what necessary and sufficient conditions (G, ψ) has a finite ψ-domset still continues to be wide open, some necessary conditions for a graphoidally covered graph (G, ψ) to possess a finite ψ-domset is given in [3].

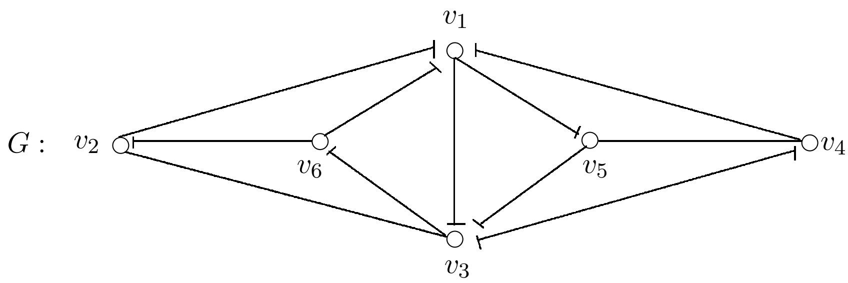

Figure 3(a) $\psi_0 = \{(v_1v_2v_3v_6), (v_1v_6v_2), (v_3v_1v_5), (v_1v_4v_5v_3), (v_3v_4)\}$ is a nonrivial graphoidal cover of G. $D = \{v_1, v_3\}$ is a minimum ψ_0-domset of (G, ψ_0). $\gamma_{\psi_0} = 2$; $\gamma(G) = 1 < \gamma_{\psi_0}(G)$

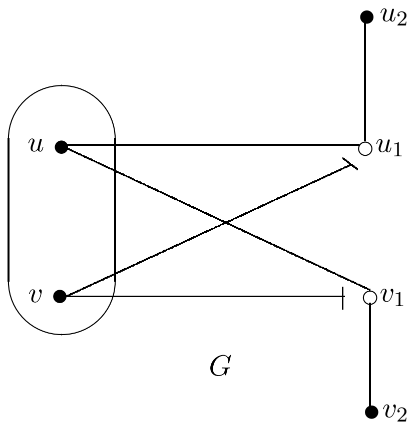

Figure 3(b) $\psi = \{(u_2u_1u), (u_1v), (v_2v_1u), (vv_1)\}$, $\psi \neq E, \gamma_\psi(G) = \gamma(G) = 2$.

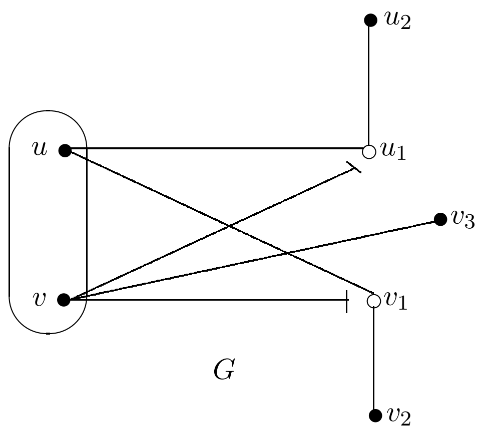

Figure 3(c) $\psi_1 = \{((u_2u_1u), (uv_1v_2), (vu_1), (vv_3), (vv_1)\}$, $\gamma_{\psi_1}(G_1) = 2$, $\gamma(G_1) = 3, \gamma_{\psi_1}(G_1) < \gamma(G_1)$.

Remark 2.6. *Recall from Remark 1.2 that it is possible that a graph may not have any finite domset but may possess a graphoidal cover ψ such that (G,ψ) has a finite ψ-domset.*

This raises the following two new problems, *viz.*

Problem 2.7. *Characterize infinite graphs G, which possess a graphoidal cover ψ such that (G,ψ) has a finite ψ-domset.*

Problem 2.8. *Characterize infinite graphs G, for which the graphoidally covered graph (G,ψ) has a finite ψ-domset for every $\psi \in \mathcal{G}_G$.*

The next question before us is : For which graphoidally covered graphs (G,ψ) every ψ-domset contains a minimal one? Reflecting on this question, we have several possibilities to consider. Firstly, in the infinite split graph $G := \mathcal{S}_{\mathcal{N}}$ with the trivial graphoidal cover mentioned in Remark 1.2, we see that the set of all odd natural numbers constitutes a domset of G which does not contain a minimal one. However, the nontrivial graphoidal cover $\psi \in \mathcal{G}_G$ described therein is such that there exists ψ-domset which has a minimal one. Thus, we are led to the following new open problems.

Problem 2.9. *Characterize infinite graphs G which possess $\psi \in \mathcal{G}_G$, such that every ψ-domset of (G,ψ) contains a minimal one.*

Problem 2.10. *Characterize infinite graphs G such that for every $\psi \in \mathcal{G}_G$, every ψ-domset of (G,ψ) contains a minimal one.*

Problem 2.11. *Characterize infinite graphs G which possess $\psi \in \mathcal{G}_G$ such that (G,ψ) possesses a ψ-domset which contains a minimal one and also a ψ-domset which does not contain a minimal one.*

The graphoidally convered graph (G,ψ) in Remark 1.2 is such an example. The following result gives a class of graphs that provides a partial answer to the above problems and generalizes a well known theorem of Ore [10].

Theorem 2.12. [3] *If G is a locally finite graph then for any $\psi \in \mathcal{G}_G$, every ψ-domset of G has a minimal one.*

However, the hypothesis in the above theorem is not a necessary condition for a graph to possess such a graphoidal cover as the example in Remark 1.2 illustrates.

Next, recall one of our early observations that the set $E := E(G)$ of edges of a graph $G = (V,E)$ is trivially a graphoidal cover of G, whence the E-domination number $\gamma_E(G)$ is written simply as $\gamma(G)$, which is the usual *domination number* of the graph G (Walikar et al. [12]; Haynes et al. [9]).

Observation 2.13. *For any finite graph G, one may have any of the following possibilities.*

(i) *There exists a graphoidal cover* ψ *such that* $\gamma_\psi(G) < \gamma(G)$.

(ii) *There exists a graphoidal cover* ψ' *such that* $\gamma_{\psi'}(G) > \gamma(G)$.

(iii) *There exists a nontrivial graphoidal cover* ψ'' *such that* $\gamma_{\psi''}(G) = \gamma(G)$ *(see* Figure 3*).*

There may exist infnite graphs G for which $\gamma(G)$ is not defined while there exists a nontrivial graphoidal cover ψ of G such that $\gamma_\psi(G)$ is a well-defined number (example, $\mathcal{S}_\mathcal{N}$). Determining graphs which admit one or more of the above types of graphoidal covers seems to be a challenging open problem. It may be noted that a beginning toward this end has already been made in our previous work (*cf.*: Acharya and Purnima Gupta [3]; Theorem 4.20).

3. Cyclomatic Number and Graphoidal Domination

Next, let $\mu := \mu(G)$ denote the *cyclomatic number* of the given finite (p, q)-graph G. Then, it is well known (*cf.*: Harary [8]) that $\mu(G) = q - p + k$ where k is the number of connected components of G. In general, given any finite *hypergraph*, $H = (X, \mathcal{E})$, its *cyclomatic number*, denoted $\mu(H)$, is defined (see Acharya [2]) as

$$\mu(H) = \sum_{E \in \mathcal{E}} |E| - |X(H)| - \varpi(H) \tag{1}$$

where $\varpi(H)$ is the maximum number of edges in a spanning multiforest of the intersection multigraph of H (see Berge [6]).

Observation 3.14. *Given any connected and finite graphoidally covered graph* (G, ψ) *such that any two* ψ*-edges have at most one vertex in common, then since* ψ *is a partition of the edge-set of* G *and may be viewed as a linear hypergraph, we get* $\mu(\psi) = \sum_{P \in \psi} |P| - |V(G)| - (|\psi| - 1) = \sum_{P \in \psi} (\mathcal{L}(P) + 1) - p - (|\psi| - 1) = \sum_{P \in \psi} \mathcal{L}(P) - p + 1 = q - p + 1 = \mu(G)$, *where* $\mathcal{L}(P)$ *denotes the length of* P.

Therefore,

$$\gamma_\psi(G) \geq k = p - q + \mu(\psi) \tag{2}$$

an observation which can be made from a more general consideration (Acharya and Gupta [3]). The following question of immediate interest would then arise : *Precisely for which graphoidally covered graphs* (G, ψ), *equality in (2) would hold?* Determining graphoidally covered graphs (G, ψ) for which $\gamma_\psi(G) = k = \mu + p - q$ is equivalent to finding graphoidally covered graphs (G, ψ) such that for each component G_i of G one has $\gamma_\psi(G_i) = 1$. It would be of interest to determine graphs G which satisfy $\gamma_\psi(G) = k$ and for which

$\mu(G) = 0$; that is, to determine (p, q)-graphs G for which $k = p - q$ and $\gamma_\psi(G) = k$. Trivially, each component of such a graph must be a tree. Ultimately, it reduces to determining trees G and graphoidal covers ψ on G such that $\gamma_\psi(G) = 1$. The following result determines them completely.

Proposition 3.15. *Let G be any tree and $\psi \in \mathcal{G}_G$. Then, $\gamma_\psi(G) = 1$ if and only if $\psi = E(G)$ and G is a* star*, viz., $K_{1,p-1}$.*

Proof. Since G is a tree, according to a known result (cf.: Acharya and Purnima Gupta [3]) for any graphoidal cover ψ of G, $\gamma_\psi(G) \geq (\sup_{P\in\psi} |P|) - 1$ and, therefore, $\gamma_\psi(G) = 1$ if and only if $\sup_{P\in\psi} |P| := \|\psi\| = 2$; that is, $\psi = E$. Thus, G must have a vertex of full degree (that is, a vertex which is adjacent to every other vertex of G) and G being a tree it must be a star. □

However, when a connected graph G satisfies $\mu(G) = 1$ the relation $\gamma_\psi(G) = 1$ may still be valid for some graphoidal cover ψ of G. Since a graph G satisfies $\mu(G) = 1$ if and only if exactly one of the components of G is unicyclic, we shall determine such connected graphs that satisfy $\gamma_\psi(G) = 1$. The following result settles this subproblem and it can also be proved almost on similar lines of argument as in the proof of the above proposition.

Proposition 3.16. *Let G be any connected unicyclic graph of a finite order p and $\psi \in \mathcal{G}_G$. Then, $\gamma_\psi(G) = 1$ if and only if $\psi = E(G)$ and the unique cycle in G is a triangle exactly one vertex of which is a vertex of full degree.*

In fact, if G is a connected graph with a nontrivial graphoidal cover ψ such that $\gamma_\psi(G) = 1$, then $\mu(G)$ can be arbitrary, as seen by the following example: Let $\mu = k \geq 2$, let $G = W_{k+1}$, the k-*wheel* with center v and the *rim* as the cycle $C_k := (u = u_1, u_2, \ldots, u_k, u_1)$ where vu_i, $\forall i$, $1 \leq i \leq k$ are the other edges of G and let $\psi = \{C\} \cup \{\{v, u_i\},\ 1 \leq i \leq k\}$. Then, in (G, ψ), $D = \{v\}$ is a $\gamma_\psi(G)$-*set* (that is, a ψ-domset with $\gamma_\psi(G)$ vertices), and G has a set of exactly μ fundamental cycles, *viz.*, (v, u_i, u_{i+1}) where indices are reduced modulo k when $k \geq 3$. Another similar construction would be to remove the edge u_1u_k from the rim of G and let the remaining path $P_k = (u = u_1, u_2, \ldots, u_k)$, which has length $k - 1$, replace C in ψ, rest being unaltered; let the resulting graph be denoted F_k (called in literature a k-*fan*) and the resulting graphoidal cover ψ'. Then, it is easy to see that F_k has exactly k fundamental cycles too for every $k \geq 2$, but ψ' has no ψ-edge that is a cycle of F_k. An immediate question before us is: Given a finite graph G with arbitrary value of $\mu(G)$, is it possible that there exists a graphoidal cover ψ of G such that $\gamma_\psi = 1$? Notice that we have answered the above question when $\mu(G) \in \{0, 1\}$, in the affirmative and proved that the graphoidal cover in these cases must be trivial. So, it would be of interest to characterize finite as well as infinite graphs G with $\mu(G) \geq 2$ and which possess nontivial graphoidal covers ψ such that $\gamma_\psi(G) = 1$.

4. Acyclic Graphoidal Covers

In general, a graphoidal cover ψ of a graph G is said to be *acyclic* if every ψ-edge is an open path in G. For any $\psi \in \mathcal{G}_G$ we shall denote by ψ^0 the set of open ψ-edges.

Proposition 4.17. *Let G be a graph with a graphoidal cover ψ. Then, ψ can be transformed into an acyclic graphoidal cover ψ' of G.*

Proof. First of all, it is enough to prove the result for connected graphs; so we assume that G is connected. Next, if ψ itself is acyclic, there is nothing to prove. So, let ψ have some cycles of G as members. Since ψ is in fact a partition of the edge set of G satisfying **GC-1** and **GC-2**, we may split each closed ψ-edge Z_i into two nontrivial paths in any arbitrary manner; let such split parts be denoted Z_i^1 and Z_i^2. We then see that the collection $\psi' = \psi^0 \cup \{\{Z_i^1, Z_i^2\} : Z_i \in \psi - \psi^0\}$ is indeed an acyclic graphoidal cover of G. □

Proposition 4.17 suggests some new parameters of a graph as described in the following observation.

Observation 4.18. *Every graph has an acyclic graphoidal cover,* viz., *its edge-set itself (or, the so-called trivial graphoidal cover); we let $\mathcal{A}_G$ denote the set of all acyclic graphoidal covers of G. Hence, for a finite graph G, it is of interest to find the least order of an acyclic graphoidal cover of G; we denote it by $\eta_a(G)$ and call it the acyclic graphoidal covering number of G (*see *Arumugam et al. [5]). One may hence define the* acyclic graphoidal domination number *of G as the integer given by $\gamma_a(G) = min\{\gamma_\psi(G) : \psi \in \mathcal{A}_G\}$. Furthermore, the minimum acyclic graphoidal domination number of G as the integer given by*

$$\gamma_a^0(G) = \min\{\gamma_\psi(G) : \psi \in \mathcal{A}_G^0\}$$

where

$$\mathcal{A}_G^0 = \{\psi \in \mathcal{A}_G : |\psi| = \eta_a(G)\}.$$

Hence, if $\gamma_0(G) = \min\{\gamma_\psi(G) : \psi \in \mathcal{G}_G\}$, then clearly one has

$$\gamma^0(G) \leq \gamma_{\mathcal{A}^0}(G) \leq \gamma_{\mathcal{A}}(G) \leq \gamma(G). \tag{3}$$

References

[1] B. D. Acharya, On the cyclomatic number of a hypergraph, *Discrete Math.*, **27**(2)(1979), 111 - 116.

[2] B.D. Acharya, Polar configurations: I. Foundations, *J. Combin., Info. Sys. Sci.,* **24** (2-4)(1999), 195-218.

[3] B.D. Acharya and Purnima Gupta, Domination in graphoidal covers of a graph, *Discrete Math.,* **206** (1999), 3-33.

[4] B.D. Acharya and E. Sampathkumar, Graphoidal covers and graphoidal covering number of a graph, *Indian J. pure. appl. Math.,* **18**(10)(1987), 882-890.

[5] S. Arumugam, B.D. Acharya and E. Sampathkumar, Graphoidal covers of a graph: A creative review, In: *Graph Theory and its Applications* (*Eds.*: S. Arumugam, B.D. Acharya and E. Sampathkumar), pp.1-28. Tata-Mcgraw Hill Publishing Company, New Delhi, 1996.

[6] C. Berge, *Hypergraphs*, North-Holland Elsevier Publ., Amsterdam, 1989.

[7] G. A. Dirac, On arbitrarily traceable graphs, *Math. Scand.,* **31** (1972), 319-378.

[8] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Massachusetts, 1969.

[9] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[10] O. Ore, Theory of Graphs, *Amer. Math. Soc. Colloq. Publ. No.,***38**, 1962.

[11] E. Sampathkumar, Semigraphs, In: Combinatorial Optimization *Eds.*: N.M. Shikare and B.N. Waphare, pp. 125-138. Narosa Publ. Comp. Pvt. Ltd., New Delhi, 2004.

[12] H.B. Walikar, B.D. Acharya and E. Sampathkumar, Recent Developments in the Theory of Domination in Graphs, MRI Lecture Notes in Mathematics, No.1. The Mehta Research Institute of Mathematics and Mathematical Physics (currently under the new name, 'Harish-Chandra Research Institute'), Allahabad, 1979. pp.252.

[13] D.B. West, *Introduction to Graph Theory*, Prentice Hall Inc., New Jersey, 1996.