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Intersection Graph of Minimal Dominating Sets of a Graph

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Abstract

A subset D of the vertex set $V(G)$ of a graph G is a *dominating set* of G if every vertex of G is either in D or is adjacent to a vertex in D. Almost twenty five years ago, the first author had conjectured that *every graph can be represented as the intersection graph of the minimal dominating sets of some graph.* In this paper, we disprove this conjecture by showing in the first instance that a graph having at least two nontrivial components and in the second that a connected graph having diameter at least four cannot be so represented. We show that, however, every graph can be embedded as an induced subgraph in a graph so representable, thereby showing that they do not have a 'forbidden subgraph' characterization and hence pointing at the need to find other ways to characterize a graph that can be represented as the intersection graph of the minimal dominating sets of some graph.

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1. Introduction

Unless mentioned otherwise, all graphs considered in this article are simple, self-loop-free and not necessarily finite (Ore, [7]). For standard terminology and notations, however, we refer the reader to Harary [4].

A set D of vertices in a graph $G = (V, E)$ is called a *dominating set* of G if every vertex of G is either in D or is adjacent to a vertex in D (*e.g., see* Ore [7], Walikar et al. [8], Haynes et al. [5], Chartrand and Zhang, [2]). The set of all dominating sets of G will be denoted $\mathcal{D}(G)$. A dominating set is *minimal* if it contains no proper dominating set; the set of all minimal dominating sets of G will be denoted $\mathcal{D}^m(G)$.

Given a family $\mathcal{F} = \{F_i\}_{i\in\mathcal{I}}$, $\mathcal{I}$ being an indexing set, of subsets F_i of a given nonempty set X, the well known *intersection graph* $\Omega(\mathcal{F})$ of $\mathcal{F}$ is defined as the graph with vertex set $\mathcal{I}$ and edges defined as precisely those pairs ij of distinct indices $i, j \in \mathcal{I}$ for which $F_i \cap F_j \neq \emptyset$. Further, a graph G is called an *intersection graph* if there exists a family $\mathcal{F} = \{F_i\}_{i\in\mathcal{I}}$ of subsets F_i of some nonempty set X such that $G \cong \Omega(\mathcal{F})$. The *intersection number* $\omega(G)$ of G is then defined as the infimum of the cardinalities of the nonempty sets X such that $G \cong \Omega(\mathcal{F})$ for some family $\mathcal{F} = \{F_i\}_{i\in\mathcal{I}}$ of subsets F_i of X (*see* Harary, [4] for the finite version of this notion). The best known and the most recent fact about the intersection number is the following result.

Theorem 1. [3] *For any graph G with a countable vertex-set, $\omega(G)$ is the minimum number of complete subgraphs of G needed to cover G.*

Reflecting from an open problem in Walikar et al. ([8]; Problem 16, p.229), the first author had surmised that *every graph could be represented as the intersection graph of the minimal dominating sets of some graph.* In the present paper, we disprove this conjecture and pave the way to study the graphs so representable. In the sequel, we shall investigate properties of the intersection graph $\Omega(\mathcal{D}^m(G))$ of the minimal dominating sets of a graph G that is not necessarily finite; for finite graphs, a study of the concept was initiated by Kulli and Janakiram [6].

The following are some fundamental results which will be required for many of our arguments in this paper, where for any vertex x in G, $N(x) = \{y \in V : xy \in E\}$ is the *open neighborhood* of x in G.

Theorem 2. [1] *Every graph possesses an independent dominating set. Furthermore, every independent dominating set of a graph G is a maximal independent set of G and every maximal independent set of G is a minimal dominating set of G.*

Theorem 3. [7] *If G is a graph with no isolates, then the complement $V(G) - D$ of every minimal dominating set D of G is a dominating set of G.*

The above two theorems can be invoked to establish the following general result, which is well-known for finite graphs (Ore, [7]).

Theorem 4. *Every nontrivial graph G without isolates has a dominating set D such that the complement $V(G) - D$ of D is also a dominating set of G.*

We shall also need the following characterization of minimal dominating sets.

Theorem 5. [7] *Let $G = (V, E)$ be any graph and $D \in \mathcal{D}(G)$. Then, D is a minimal dominating set of G if and only if for every vertex $d \in D$ exactly one of the following conditions holds:*

$\overline{\mathbf{T}}$**:** $N(d) \cap D = \emptyset$.

PN: *There exists a vertex* $v \in V - D$ *such that* $N(v) \cap D = \{d\}$.

2. Intersection graph of minimal dominating sets

Suppose a graph G has a vertex v of *full degree*, *i.e.*, a vertex that is adjacent to all the other vertices of G. Clearly, $\{v\} \in \mathcal{D}^m(G)$. This also means that all the other minimal dominating sets of G must be contained in $V(G) - \{v\}$, the latter itself being a dominating set of G in accordance with Theorem 3. Therefore, it follows that $\{v\}$ must be an isolate in the intersection graph $\Omega(\mathcal{D}^m(G))$. Indeed, the following stronger result is seen to hold, generalizing the result for finite graphs obtained by Kulli and Janakiram [6].

Proposition 6. *For any nontrivial graph* $G, \Omega(\mathcal{D}^m(G))$ *is connected if and only if* G *has no vertex of full degree.*

Proof. If G has a vertex of full degree, we have seen above that $\Omega(\mathcal{D}^m(G))$ is disconnected. So, we shall establish the 'sufficiency' part of the proof. Hence, suppose that G has no vertex of full degree. Then for every vertex x in G there is a vertex x' which is not adjacent to x. Let M be any maximal independent set in G. By Theorem 2, $M \in \mathcal{D}^m(G)$. Suppose now that $\Omega(\mathcal{D}^m(G))$ is disconnected. Let $\mathcal{H}$ be the component of $\Omega(\mathcal{D}^m(G))$ that contains M and let $\mathcal{K}$ be any other component of $\Omega(\mathcal{D}^m(G))$. Then, by the definition of $\Omega(\mathcal{D}^m(G))$, for any vertex D in $\mathcal{K}$ we see that the corresponding minimal dominating set D of G does not intersect M, whence $D \subseteq V(G) - M$. Since $V(G) - M$ is a dominating set of G (by virtue of Theorems 2 and 3), for every $x \in D$ one has $N(x) \cap M \neq \emptyset$. Now, $x' \notin M$, for, otherwise, there would be a maximal independent set D' in G that contains both x and x', and (D, D', M) is a 2-path connecting the vertices D and M in $\Omega(\mathcal{D}^m(G))$, a contradiction to our assumption that D and M are in different components of $\Omega(\mathcal{D}^m(G))$. Therefore, $x' \in V(G) - M$. Also, the foregoing argument shows that every vertex of D must be adjacent to every vertex of M. But, then, if any one of D and M has at least two vertices we get a contradiction to the minimality of the other by virtue of Theorem 5. Therefore, we see that both D and M must be singleton sets, whence we see that the vertices in them should both be of full degree in G, which is again a contradiction. Hence $\Omega(\mathcal{D}^m(G))$ is connected □

The following consequence of Proposition 6 disproves the long-standing conjecture of the first author that *every graph can be represented as the intersection graph of the minimal dominating sets of some graph.*

Corollary 7. *A graph having at least two nontrivial components cannot be represented as the intersection graph of the minimal dominating sets of a graph.*

The above result prompts one to ask whether then any graph with at most one nontrivial component could be represented as an intersection graph of the minimal dominating sets of some graph; in particular, whether every connected graph could be so represented. The following result indicates, by and large, negation of this possibility too.

Proposition 8. *If G is a graph having no vertex of full degree then, distance between any two nonadjacent vertices in $\Omega(\mathcal{D}^m(G))$ is at most three.*

Proof. If G has an isolated vertex then it belongs to every dominating set, in particular to every minimal dominating set, whence we see that $\Omega(\mathcal{D}^m(G))$ is a complete graph and our assertion holds.

Hence, let G have no isolates. Let $D_1,\ D_2 \in \mathcal{D}^m(G)$. If $D_1 \cap D_2 \neq \emptyset$ then D_1 and D_2 are adjacent in $\Omega(\mathcal{D}^m(G))$. Since G has no vertex of full degree it follows that $|D_1| \geq 2$ and $|D_2| \geq 2$. Let $x \in D_1$ and $y \in D_2$ be such that x and y are not adjacent in G. Then, there must exist a maximal independent set D_3 containing $\{x, y\}$. Since D_3 is a minimal dominating set, (D_1, D_3, D_2) is a 2-path joining D_1 and D_2 in $\Omega(\mathcal{D}^m(G))$, whence the distance $d_\Omega(D_1, D_2)$ between D_1 and D_2 in $\Omega(\mathcal{D}^m(G))$ is at most two.

Next, suppose that $xy \in E(G)$, for all $x \in D_1$ and for all $y \in D_2$. We consider two cases:

Case 1. There exist vertices $u \in D_1$ and $v \in D_2$ such that every vertex $w \notin D_1 \cup D_2$ is adjacent to either u or v.

In this case the set $\{u, v\}$ is a minimal dominating set of G adjacent to D_1 and D_2 in $\Omega(\mathcal{D}^m(G))$, whence $d_\Omega(D_1, D_2) \leq 2$.

Case 2. For any two vertices $u \in D_1$ and $v \in D_2$ there exists a vertex $w \notin D_1 \cup D_2$ such that w is adjacent to neither u nor v.

Since u and v are adjacent by our assumption we see that there must exist two distinct maximal independent sets D_3 and D_4 containing u and v respectively. It is now easy to see that (D_1, D_3, D_4, D_2) is a path connecting D_1 and D_2 in $\Omega(\mathcal{D}^m(G))$, whence we see that $d_\Omega(D_1, D_2) \leq 3$. □

Corollary 9. *No connected graph having diameter at least four can be represented as an intersection graph of the minimal dominating sets of some graph.*

Next natural question is to ask whether every connected graph having diameter at most three could be represented as an intersection graph of the minimal dominating sets of some graph. Clearly, a graph has diameter one if and only if it is complete and hence the following result settles the problem for these graphs.

Theorem 10. *Every complete graph is the intersection graph of the minimal dominating sets of some graph.*

Proof. Let G be any complete graph. Consider the graph H obtained from G by augmenting a set K of new vertices (that is, $V(G) \cap K = \emptyset$) to it so that each vertex in

K is an isolate in H. Then, $\mathcal{D}^m(H) = \{K \cup \{u\} : u \in V(G)\}$ whence it is easy to see that $\Omega(\mathcal{D}^m(H)) \cong G$. □

Hence, to settle the problem, it remains to address the question only for graphs with diameters two and three. Towards this end, except for a few sporadic infinite classes of such finite graphs (*e.g.*, K_{2^p}, $p \geq 2$; *cf.*, Kulli and Janakiram, [6]), we know little as of now. The following additional facts might be of help for further investigation.

Theorem 11. *Let H be any connected nontrivial graph and $G = H \circ K_1$, the standard* corona *of H obtained by augmenting one new vertex v' for each given vertex v in H and introducing the new edge vv' in each case. Then, $\Omega(\mathcal{D}^m(G)) \cong K_H^*$ where K_H^* is the complete graph of order $2^{|V(H)|}$ minus the edges in a 1-factor of it.*

Proof. Let $V(H) = \{v_1, v_2, \dots, v_p\}$ and $V(G) - V(H) = \{u_1, u_2, \dots, u_p\}$. Then, the union of any r vertices $\{v_{i_1}, v_{i_2}, \dots, v_{i_r}\} \subseteq V(H)$ and any $p - r$ vertices from $\{u_1, u_2, \dots, u_p\} - \{u_{i_1}, u_{i_2}, \dots, u_{i_r}\}$ forms a minimal dominating set in G. Thus, we have 2^p minimal dominating sets of G and every minimal dominating set S intersects all the other minimal dominating sets of G except $V(G) - S$. The conclusion of the theorem is now obvious. □

The following result is an extension of Theorem 10.

Theorem 12. *If G is a graph representable as the intersection graph of the minimal dominating sets of some graph H then so is the case with the graph $G \cup \overline{K}$ where K is a complete graph (possibly, of an infinite order) having no vertex in common with G.*

Proof. By hypothesis, we have $G \cong \Omega(\mathcal{D}^m(H))$. Let $H_1 = H + K$. Then, it is easy to see that $\mathcal{D}^m(H_1) = \mathcal{D}^m(H)\} \cup \{\{v\} : v \in V(K)\}$ and $\Phi : \mathcal{D}^m(H_1) \longmapsto V(G \cup \overline{K})$ given by $\Phi(D) = D$ if $D \in \mathcal{D}^m(H)$ and $\Phi(\{v\}) = v$ if $\{v\} \in \mathcal{D}^m(H_1) - \mathcal{D}^m(H)$ is an isomorphism from $\Omega(\mathcal{D}^m(H_1))$ onto $V(G \cup \overline{K})$. □

3. Embedding in graphs with given property

We shall say that a graph G can be *embedded* in a graph H having a prescribed property provided that there exists an isomorphism Φ that maps G one-to-one onto an induced subgraph of H and then we write $G \prec H$ to indicate this fact; in particular, if Φ maps G onto the whole of H then, we call Φ a *representation* of G when we have in the standard notation $\Phi(G) = H$ or $G \cong H$ whenever Φ is clear from the context. In view of the results presented in the previous section, the following result is a relaxation.

Theorem 13. *Every graph can be embedded as an induced subgraph in the intersection graph of the minimal dominating sets of some graph.*

Proof. Let $G = (V, E)$ be any graph and $H := I(G) = (V \cup E, E^*)$ be its *incidence graph* defined by the adjacency rule:

$$(v, e) \in E^* \Leftrightarrow v \in V,\ e \in E \text{ and } v \in e.$$

Further, if G has a vertex u of degree one and if uv is the pendant edge then in H we introduce the edge uv; let H_1 be the graph obtained from H by introducing all such edges corresponding to the pendant edges in G. Next, if S is the set of isolates in G then, we obtain the graph H_2 from H or H_1 by joining every vertex of S to every vertex of $V - S$ in the latter graph. Now, for each $v \in V(G)$ let $S_v = \{v\} \cup \{e_j \in E(G) : v \in e_j\}$. By the very construction, S_v is a minimal dominating set of H or H_1 or H_2 as the case may be. Next, let $\{S_v : v \in V(G)\}$ and $K = F \cup$ {all the remaining minimal dominating sets of H_1 or H_2}.

Now, consider the intersection graph $\Omega(K)$ of K. Then, $\Omega(F)$ is an induced subgraph of $\Omega(K)$. Define $f : V(G) \to K$ by saying $f(v) = S_v$ for all $v \in V(G)$. It is easy to see that $f(G) = \Omega(F)$ and the proof is complete. □

4. Conclusion and scope

Theorem 13 shows that there is no 'forbidden subgraph' characterization for a graph representable as the intersection graph of the minimal dominating sets of some graph. Thus, in view of Corollary 9 and Theorem 10, obtaining a structural characterization of a graph of diameter two or three and representable as the intersection graph of the minimal dominating sets of some graph has become a challenging open problem.

Next, it is not difficult to see that the following result has the same proof as its finite version, given originally by Kulli and Janakiram [6].

Theorem 14. *A locally finite graph G contains an isolate if and only if $\Omega(\mathcal{D}^m(G)$ is complete.*

In fact, a proof of the above result invokes the well known result of Ore (1962) that *in any locally finite graph, every dominating set contains a minimal one.* Arising from Theorem 14 is hence the natural problem that would require us to find for a given finite graph G a 'host' graph H with least possible order p^* and size q^* such that $G \cong \Omega(\mathcal{D}^m(H))$. This is an open problem as at this stage.

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