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Fractional Independence and Fractional Domination Chain in Graphs

S. Arumugam
Core Group Research Facility (CGRF)
National Centre for Advanced Research in Discrete Mathematics (n -CARDMATH)
Kalasalingam University
Anand Nagar, Krishnankoil-626190, India.
E-mail: *s_arumugam_akce@yahoo.com*
and
K. Rejikumar
Department of Mathematics
N.S.S. College, Pandalam-689501, Kerala, India.

Abstract

Let $G = (V, E)$ be a graph. A function $f : V \to [0, 1]$ is called an *independent function* if for every vertex v with $f(v) > 0$, we have $\sum_{u \in N[v]} f(u) = 1$. An independent function f is called a *maximal independent function* (MIF) if for every $v \in V$ with $f(v) = 0$, we have $\sum_{u \in N[v]} f(u) \geq 1$. The fractional independent domination number i_f and the fractional independence number β_{0f} are defined by $i_f = min\{|f| = \sum_{u \in V} f(u) : f$ is an MIF of $G\}$ and $\beta_{0f} = max\{|f| : f$ is an MIF of $G\}$. These parameters along with the fractional irredundance numbers ir_f, IR_f and the fractional domination numbers γ_f, Γ_f lead to the fractional domination chain given by $ir_f \leq \gamma_f \leq i_f \leq \beta_{0f} \leq \Gamma_f \leq IR_f$. In this paper we present several results on the parameters i_f and β_{0f}.

Keywords: Fractional independent function; Fractional independent domination number; Fractional independence number

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1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [4].The six basic parameters relating to domination, independence and irredundance satisfy the chain of inequalities given by $ir \leq \gamma \leq i \leq \beta_0 \leq \Gamma \leq IR$, where ir, IR are the irredundance and upper irredundance numbers, γ, Γ are the domination and upper domination numbers and i, β_0 are the independent domination number and independence number respectively. Hedetniemi et al. [7] introduced the concept of fractional domination, fractional domination number γ_f and the upper fractional domination number Γ_f. Domke et al. [2] introduced the concept of fractional irredundance numbers ir_f and IR_f. In this paper we introduce the concept of fractional independent function and the corresponding fractional parameters i_f and β_{0f}, leading to the fractional analogue of the domination chain

$$ir_f \leq \gamma_f \leq i_f \leq \beta_{0f} \leq \Gamma_f \leq IR_f.$$

In section 2, we present some basic definitions and results on fractional domination. In section 3, we introduce the concept of maximal independent dominating function, the fractional independent domination number i_f and the fractional independence number β_{0f} and present several results on these concepts. In the concluding section we present several interesting open problems.

2. Fractional Domination in Graphs

Let S be a dominating set of $G = (V, E)$. Let $f : V \to \{0, 1\}$ be the characteristic function of S so that

$$f(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \notin S. \end{cases}$$

The dominating property of S is equivalent to the functional inequality

$$\sum_{u \in N[v]} f(u) \geq 1$$

for each $v \in V$. Fractional graph theory deals with the generalisation of integer-valued graph theoretic concepts such that they take on fractional values. One of the standard methods for converting a graph concept from integer version to fractional version is to formulate the concept as an integer program and then to consider the linear programming relaxation. A detailed study of fractional graph theory and fractionalisation of various graph parameters are given in Scheinerman and Ullman[8]. For a review of fractional domination, we refer to chapter 10 of Haynes et al. [5] and chapters 2, 3 and 5 of Haynes et al. [6]. We first present some basic definitions and results on fractional domination.

Definition 2.1. [7] *Let $G = (V, E)$ be a graph. A function $f : V \to [0, 1]$ is called a dominating function (DF) if*

$$f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1$$

for every $v \in V$. A dominating function f is called a minimal dominating function (MDF) if there does not exist a dominating function $g \neq f$ for which $g(v) \leq f(v)$ for all $v \in V$. Equivalently, f is an MDF if for every vertex v with $f(v) > 0$, there exists a vertex $w \in N[v]$ such that

$$\sum_{u \in N[w]} f(u) = 1.$$

The fractional domination number γ_f *and the upper fractional domination number* Γ_f *are defined by* $\gamma_f = min\{|f| : f$ *is an MDF of* $G\}$ *and* $\Gamma_f = max\ \{|f| : f$ *is an MDF of* $G\}$, *where* $|f| = \sum_{v \in V} f(v)$.

Definition 2.2. [2] *A function* $f : V \to [0, 1]$ *is called an irredundant function if for every* $v \in V$ *with* $f(v) > 0$, *there exists a vertex* $w \in N[v]$ *such that*

$$\sum_{u \in N[w]} f(u) = 1.$$

An irredundant function g *is maximal if for all functions* $f : V \to [0, 1]$ *with* $f > g$, f *is not an irredundant function. The fractional irredundance number* ir_f *and the upper fractional irredundance number* IR_f *are defined by*

$$ir_f = min\{|f| : f \text{ is a maximal irredundant function}\}$$

and

$$IR_f = max\{|f| : f \text{ is a maximal irredundant function}\}.$$

Since every MDF is an irredundant function, we have

$$ir_f \le \gamma_f \le \Gamma_f \le IR_f.$$

Definition 2.3. [1] *For a DF* f *of* G, *the boundary set* B_f *and the positive set* P_f *are defined by* $B_f = \{v \in V : \sum_{u \in N[v]} f(u) = 1\}$ *and* $P_f = \{v \in V : f(v) > 0\}$.

Definition 2.4. [1] *Let* $G = (V, E)$ *be a graph and let* $A, B \subseteq V$. A *is said to dominate* B *if each* $v \in B - A$ *is adjacent to a vertex in* A. *If* A *dominates* B, *we write* $A \to B$.

Theorem 2.5. [1] *A DF* f *of* G *is an MDF if and only if* $B_f \to P_f$.

Though the convex combination of two DFs is again a DF, the convex combination of two MDF's f and g need not be an MDF. The following theorem shows that either all convex combinations of f and g are MDF's or none of them is an MDF.

Theorem 2.6. [1] *Let* f *and* g *be MDF's of* G. *Let* $h_\lambda = \lambda f + (1 - \lambda)g$, *where* $0 < \lambda < 1$. *Then* h_λ *is an MDF of* G *if and only if* $B_f \cap B_g \to P_f \cup P_g$.

Theorem 2.7. [3] $\gamma_f(K_{r,s}) = \frac{r(s-1)}{rs-1} + \frac{s(r-1)}{rs-1}$.

3. Fractional Independence in Graphs

In chapter 3 of Haynes et al. ([6],p85), Domke et al. raised the following question:

Can we define the concept of fractional independent function as a function

$f : V \to [0, 1]$ in such a way that

(i) the characteristic function of every independent set of vertices is an independent function,

(ii) every maximal independent set of vertices corresponds to a maximal independent function, and

(iii) every maximal independent function is a minimal dominating function?

We now proceed to answer this question.

Let $G = (V, E)$ be a graph and let $S \subseteq V$ be an independent set. Let $f : V \to [0, 1]$ be the characteristic function of S. If $v \in S$, then $f(v) = 1$ and since no neighbour of v is in S, we have $\sum_{u \in N[v]} f(u) = 1$. Further if S is a maximal independent set, then for each $v \notin S$, at least one neighbour of v is in S and hence $\sum_{u \in N[v]} f(u) \geq 1$.

These observations motivate the following definition.

Definition 3.1. *Let $G = (V, E)$ be a graph. A function $f : V \to [0, 1]$ is called an independent function if for every vertex v with $f(v) > 0$, we have $\sum_{u \in N[v]} f(u) = 1$.*

An independent function f is called a maximal independent function (MIF) if for every $v \in V$ with $f(v) = 0$, we have $\sum_{u \in N[v]} f(u) \geq 1$.

Clearly if S is an independent set in G, then $f = \chi_s$ is an independent function. If further S is maximal, then χ_s is an MIF.

Remark 3.2. *If a function $f : V \to [0, 1]$ is an independent function, then $P_f \subseteq B_f$.*

Theorem 3.3. *Every MIF is an MDF.*

Proof. Let f be an MIF of G. It follows from the definition that $\sum_{u \in N[v]} f(u) \geq 1$ for all $v \in V$. Hence f is a dominating function. Further $P_f \subseteq B_f$ so that $B_f \to P_f$. Hence it follows from Theorem 2.5 that f is an MDF. □ □

Definition 3.4. *The fractional independence number β_{0_f} and the fractional independent domination number i_f are defined by*

$$\beta_{0_f}(G) = max\{|f| : f \text{ is an MIF of } G\}$$

and

$$i_f(G) = min\{|f| : f \text{ is an MIF of } G\}.$$

Since every MIF is an MDF, we have $\gamma_f(G) \leq i_f(G) \leq \beta_{0_f}(G) \leq \Gamma_f(G)$. Hence we obtain the following fractional domination chain:

$$ir_f(G) \leq \gamma_f(G) \leq i_f(G) \leq \beta_{0_f}(G) \leq \Gamma_f(G) \leq IR_f(G).$$

Remark 3.5. *The convex combination of two independent functions need not be an independent function. Consider the path* $P_3 = (v_1, v_2, v_3)$. *Define*

$$f_1(v_1) = f_1(v_3) = 0, f_1(v_2) = 1\,;$$
$$f_2(v_1) = f_2(v_3) = 1, f_2(v_2) = 0\,;$$
$$f_3(v_1) = f_3(v_2) = f_3(v_3) = 1/2.$$

Clearly f_1 *and* f_2 *are independent functions,* $f_3 = \frac{1}{2}f_1 + \frac{1}{2}f_2$ *and* f_3 *is not an independent function. Further* f_1 *and* f_2 *are both MIF's and hence a convex combination of two MIF's need not be an independent function.*

Remark 3.6. *Let* f *and* g *be two independent functions. Let* $h_\lambda = \lambda f + (1-\lambda)g$, *where* $0 < \lambda < 1$. *Then* h_λ *is an independent function if and only if* $P_f \cup P_g \subseteq B_f \cap B_g$. *Hence either all convex combinations of* f *and* g *are independent functions or none of them is an independent function. A similar result is true for MIF's. Further if* f *and* g *are MIF's and if* h_λ *is an independent function, then* h_λ *is an MIF.*

Lemma 3.7. *Let* f *and* g *be two MIF's. Then either all convex combinations of* f *and* g *are MIF's or none of them is an MIF's.*

Proof. Let $h_\lambda = \lambda f + (1-\lambda)g$ where $0 < \lambda < 1$. Suppose that h_{λ_1} is an MIF and let $\lambda \neq \lambda_1$. We claim that h_λ is an MIF. Let $v \in V$. Suppose $h_\lambda(v) = 0$. Then $f(v) = g(v) = 0$. Since f and g are MIF's, we have $f(N[v]) \geq 1$ and $g(N[v] \geq 1$. Hence it follows that $h_\lambda(N[v]) \geq 1$.

Now suppose $h_\lambda(v) > 0$. Then either $f(v) > 0$ or $g(v) > 0$. Hence $h_{\lambda_1}(v) > 0$. Since h_{λ_1} is an MIF, $h_{\lambda_1}(N[v]) = 1$, so that $f(N[v]) = g(N[v]) = 1$. Hence $h_\lambda(N[v]) = 1$. Thus $P_{h_\lambda} \subseteq B_{h_\lambda}$ and $h_\lambda(N[v]) \geq 1$ for all $v \in V$. Hence h_λ is an MIF. □

Lemma 3.8. *Any MDF* f *of a graph* G *with* $B_f = V$ *is an MIF of* G.

Proof. Since $B_f = V$, we have $P_f \subseteq B_f$ and hence f is an independent function. Also since f is an MDF, $f(N[v]) \geq 1$ for all $v \in V$ and hence f is an MIF. □

Corollary 3.9. *If* G *is a graph with* $B_f = V$ *for every MDF* f *of* G, *then* $\gamma_f = i_f$ *and* $\beta_{0_f} = \Gamma_f$.

Theorem 3.10. $i_f(G+H) \leq min\{i_f(G), i_f(H)\}$ *and* $\beta_{0_f}(G+H) \geq \beta_{0_f}(G) + \beta_{0_f}(H)$.

Proof. Let f and g be MIF's of G and H respectively such that $|f| = i_f(G)$ and $|g| = i_f(H)$. Define $f_1 : V(G) \cup V(H) \to [0,1]$ by

$$f_1(v) = \begin{cases} f(v) & \text{if } v \in V(G) \\ 0 & \text{if } v \in V(H). \end{cases}$$

Clearly, $P_{f_1} = P_f$ and $B_{f_1} = B_f$. Since f is an MIF of G, it follows that $P_f \subset B_f$ and hence $P_{f_1} \subset B_{f_1}$. Thus f_1 is an independent function of $G+H$. Since f is an MIF of G it follows that f_1 is an MIF of $G+H$. Hence $i_f(G+H) \leq |f_1| = |f| = i_f(G)$. Similarly $i_f(G+H) \leq i_f(H)$. Hence $i_f(G+H) \leq min\{i_f(G), i_f(H)\}$.

Similarly, we can prove that $\beta_{0_f}(G+H) \geq \beta_{0_f}(G) + \beta_{0_f}(H)$. □

The following lemma is useful in computing the value of i_f and β_{0_f} for complete multipartite graphs.

Lemma 3.11. *Let f be an MIF of the complete multipartite graph $G = K_{r_1,r_2,\ldots,r_n}$ and let $V_1, V_2, \ldots, V_n$ be the partition of $V(G)$ into independent sets with $|V_i| = r_i$ $(1 \leq i \leq n)$. Then $f|V_i$ is a constant function for all i.*

Proof. Let $v_1 \in V_i$.

Case i. $f(v_1) = 0$.

Suppose there exists a vertex $v_2 \in V_i$ such that $f(v_2) > 0$. Then v_2 belongs to $P_f \subset B_f$. Hence $\sum_{x \in N[v_2]} f(x) = f(v_2) + \sum_{x \in N(v_2)} f(x) = 1$. Since $f(v_2) > 0$, it follows that $\sum_{x \in N(v_2)} f(x) < 1$. However $N(v_1) = N(v_2)$ and hence $\sum_{x \in N[v_1]} f(x) = f(v_1) + \sum_{x \in N(v_1)} f(x) = \sum_{x \in N(v_2)} f(x) < 1$, which is a contradiction. Hence $f(v) = 0$ for all $v \in V_i$.

Case ii. $f(v_1) > 0$.

It follows from case(i) that $f(v) > 0$ for all $v \in V_i$. Hence $f(N[v]) = 1$ for all $v \in V_i$. So $f(v_1) + \sum_{x \in N(v_1)} f(x) = f(v) + \sum_{x \in N(v)} f(x) = 1$ for all $v \in V_i$.

Since $N(v_1) = N(v)$, it follows that $f(v_1) = f(v)$ for all $v \in V_i$. Hence the result follows. □

Example 3.12. *Consider the graph $G = K_{2,3,4}$. Let V_1, V_2, V_3 be a partition of $V(G)$ with $|V_1| = 2, |V_2| = 3$ and $|V_3| = 4$. Let $V_1 = \{v_1, v_2\}, V_2 = \{u_1, u_2, u_3\}$ and $V_3 = \{w_1, w_2, w_3, w_4\}$. Let f be an arbitrary MIF of G. Then by Lemma 3.11, $f(v_1) = f(v_2) = x, f(u_1) = f(u_2) = f(u_3) = y$ and $f(w_1) = f(w_2) = f(w_3) = f(w_4) = z$.*

We have the following cases.

Case i. $x \neq 0, y \neq 0$ *and* $z \neq 0$.

Then $f(N[v]) = 1$ for all $v \in V$. Corresponding system of equations is given by

$$x + 3y + 4z = 1$$

$$2x + y + 4z = 1$$

$$2x + 3y + z = 1.$$

Solving for x, y and z, we get $x = 6/23$, $y = 3/23$ and $z = 2/23$.

Case ii. $x = 0, y \neq 0$ *and* $z \neq 0$.

Then

$$y + 4z = 1$$
$$3y + z = 1.$$

Solving these equations, we get $x = 0, y = 3/11$ and $z = 2/11$.

Case iii $y = 0, x \neq 0$ *and* $z \neq 0$.

Then

$$x + 4z = 1$$
$$2x + z = 1.$$

Solving these equations, we get $x = 3/7, y = 0$ and $z = 1/7$.

Case iv $z = 0, x \neq 0$ *and* $y \neq 0$.

Then

$$x + 3y = 1$$
$$2x + y = 1.$$

Solving these equations, we get $x = 2/5, y = 1/5$ and $z = 0$.

Case v. $x = y = 0$.

Then $z = 1$.

Case vi. $x = z = 0$.

Then $y = 1$.

Case vii. $y = z = 0$.

Then $x = 1$.

Thus there are exactly seven MIF's which are given below

$f_1(u_i) = 6/23$	$f_1(v_i) = 3/23$	$f_1(w_i) = 2/23$	with	$\lvert f_1\rvert = 29/23$;
$f_2(u_i) = 0$	$f_2(v_i) = 3/11$	$f_2(w_i) = 2/11$	with	$\lvert f_2\rvert = 17/11$;
$f_3(u_i) = 3/7$	$f_3(v_i) = 0$	$f_3(w_i) = 1/7$	with	$\lvert f_3\rvert = 10/7$;
$f_4(u_i) = 2/5$	$f_4(v_i) = 1/5$	$f_4(w_i) = 0$	with	$\lvert f_4\rvert = 7/5$;
$f_5(u_i) = 0$	$f_5(v_i) = 0$	$f_5(w_i) = 1$	with	$\lvert f_5\rvert = 4$;
$f_6(u_i) = 0$	$f_6(v_i) = 1$	$f_6(w_i) = 0$	with	$\lvert f_6\rvert = 3$;
$f_7(u_i) = 1$	$f_7(v_i) = 0$	$f_7(w_i) = 0$	with	$\lvert f_7\rvert = 2$.

Hence $i_f(G) = 29/23$ and $\beta_{0f}(G) = 4$.

Theorem 3.13. *If* $r, s \geq 2$, *then*

$$i_f(K_{r,s}) = \frac{r(s-1)}{rs-1} + \frac{s(r-1)}{rs-1}$$

and

$$\beta_{0_f}(K_{r,s}) = max\,\{r, s\}\,.$$

Proof.

(i) By Theorem 2.7, $\gamma_f(K_{r,s}) = \frac{r(s-1)}{rs-1} + \frac{s(r-1)}{rs-1}$.

Hence it is enough to prove that $i_f(K_{r,s}) = \gamma_f(K_{r,s})$. Obviously $\gamma_f(K_{r,s}) \leq i_f(K_{r,s})$. Further if $A = \{a_1, a_2, \ldots, a_r\}$ and $B = \{b_1, b_2, \ldots, b_s\}$ is a bipartition of $K_{r,s}$, then $g : V \to [0,1]$ defined by,

$$g(v) = \begin{cases} \frac{(s-1)}{rs-1} & \text{if } v \in A \\ \frac{(r-1)}{rs-1} & \text{if } v \in B. \end{cases}$$

is an MIF.

Also $\sum_{v \in A \cup B} g(v) = \frac{r(s-1)}{rs-1} + \frac{s(r-1)}{rs-1} = \gamma_f(K_{r,s})$. Hence $i_f(K_{r,s}) \leq \gamma_f(K_{r,s})$.
Thus $i_f(K_{r,s}) = \gamma_f(K_{r,s})$.

(ii) Let f be any MIF of $K_{r,s}$. It follows from Lemma 3.11 that $f|A$ and $f|B$ are both constant functions. If $f(v) = 0$ for all $v \in A$, then $f(v) = 1$ for all $v \in B$. Also if $f(v) = 0$ for all $v \in B$, then $f(v) = 1$ for all $v \in A$. Now suppose $f(x) > 0$ for all $x \in A \cup B$ and let $f(v) = a$ for all $v \in A$ and $f(v) = b$ for all $v \in B$. Since $\sum_{x \in N[v]} f(x) = 1$ for all $x \in A \cup B$, it follows that $a + sb = 1$ and $ra + b = 1$.

Hence $a = \frac{s-1}{rs-1}$ and $b = \frac{r-1}{rs-1}$.
Thus there are exactly three MIF's $f_1, f_2,$ and f_3 of $K_{r,s}$ and they are given by

$$f_1(v) = \begin{cases} 0 & \text{if } v \in A \\ 1 & \text{if } v \in B; \end{cases}$$

$$f_2(v) = \begin{cases} 1 & \text{if } v \in A \\ 0 & \text{if } v \in B; \end{cases}$$

and

$$f_3(v) = \begin{cases} \frac{s-1}{rs-1} & \text{if } v \in A \\ \frac{r-1}{rs-1} & \text{if } v \in B. \end{cases}$$

Also $|f_1| = s, |f_2| = r$ and $|f_3| < 2$.
Hence it follows that $\beta_{0f}(K_{r,s}) = max\{r, s\}$. □

4. Conclusion and Scope

We have introduced the concept of independent dominating functions, fractional independent domination number and fractional independence number. The following are some interesting problems for further investigation.

(i) Characterize the class of graphs for which

  (a) $\gamma_f = i_f$.
  (b) $\Gamma_f = \beta_{0_f}$.
  (c) $\gamma_f = i_f$ and $\Gamma_f = \beta_{0_f}$.

(ii) Determine i_f and β_{0_f} for standard families of graphs such as paths, cycles, regular graphs etc.

(iii) Another interesting area is to define the edge analogue of independent functions and develop its theory.

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