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Independence Saturation and Extended Domination Chain in Graphs

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Abstract

The six basic parameters relating to domination, independence and irredundance satisfy a chain of inequalities given by $ir \leq \gamma \leq i \leq \beta_0 \leq \Gamma \leq IR$ where ir, IR are the irredundance and upper irredundance numbers, γ, Γ are the domination and upper domination numbers and i, β_0 are the independent domination number and independence number respectively. In this paper, we introduce the concept of independence saturation number IS of a graph which extends the above domination chain. We also consider the edge-analogue of this extended domination chain.

Keywords: Irredundance; Independence; Domination; Independence saturation; Edge independence saturation.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [5].

One of the fastest growing areas in graph theory is the theory of domination which is closely related to subset problems such as independence, irredundance, covering and matching. An excellent treatment of fundamentals of domination in graphs and several advanced topics in domination are given in Haynes et al. [6, 7].

The six parameters of domination, independence and irredundance are connected by a chain of inequalities and is called the *domination chain* of a graph. In chapter 14 of Haynes et al. [7] Cockayne presents several results on this topic. In this paper, we introduce the concept of *independence saturation number* and its edge-analogue. In section 2, basic definitions and results leading to the domination chain and a few results, which we need subsequently, are presented. In section 3, we introduce the concept of independence saturation number of a graph which extends the domination chain and in section 4, the corresponding edge analogue is presented. A list of interesting problems for further investigation is presented in section 5.

2. Domination Chain - A brief review

Definition 2.1. *Let $G = (V, E)$ be a graph. A subset S of V is said to be a dominating set in G if every vertex in $V - S$ is adjacent to some vertex in S. A dominating set S is called a minimal dominating set if no proper subset of S is a dominating set of G. The domination number γ of G is the minimum cardinality taken over all minimal dominating sets in G. The upper domination number Γ of G is the maximum cardinality taken over all minimal dominating sets in G.*

Definition 2.2. *Let S be a subset of vertices of a graph G and let $u \in S$. A vertex v is called a private neighbour of u with respect to S if $N[v] \cap S = \{u\}$. The private neighbour set of u with respect to S is defined as $pn[u, S] = \{v/N[v] \cap S = \{u\}\}$. The set S is called an irredundant set if for every $u \in S$, $pn[u, S] \neq \phi$, that is, every vertex $u \in S$ has at least one private neighbour. An irredundant set S is called a maximal irredundant set if no proper superset of S is irredundant. The minimum cardinality of a maximal irredundant set in G is called the irredundance number of G and is denoted by $ir(G)$. The maximum cardinality of an irredundant set in G is called the upper irredundance number of G and is denoted by $IR(G)$.*

Theorem 2.3. *A dominating set S is a minimal dominating set if and only if it is dominating and irredundant.*

Definition 2.4. *A subset S of V in a graph G is said to be independent if no two vertices in S are adjacent. An independent set S is called a maximal independent set if no proper superset of S is independent.*

Theorem 2.5. [6] *An independent set S in a graph G is maximal independent if and only if S is minimal as a dominating set.*

Definition 2.6. *The minimum cardinality of a maximal independent set is called the independent domination number of G and is denoted by $i(G)$. The maximum cardinality of an independent set in G is called the independence number of G and is denoted by $\beta_0(G)$.*

Definition 2.7. *A subset S of V is called a vertex cover of G if every edge of G has at least one end in S. The number of vertices in a minimum vertex cover of G is called the vertex covering number of G and is denoted by α_0 .*

Theorem 2.8. [6] *For any graph G,*

$$\frac{\gamma(G)}{2} < ir(G) \leq \gamma(G) \leq 2ir(G) - 1.$$

Theorem 2.9. [3] *For any graph G,*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G).$$

Since its first publication in 1978, this inequality chain has been the focus of more than 100 research papers.

Cockayne and Mynhardt [4] answered the following question which naturally arises.

Given an integer sequence (a, b, c, d, e, f) does there exist a graph G such that $ir(G) = a$, $\gamma(G) = b$, $i(G) = c$, $\beta_0(G) = d$, $\Gamma(G) = e$ and $IR(G) = f$. If such a graph exists then (a, b, c, d, e, f) is called a dii-sequence.

Theorem 2.10. [4] *A sequence (a, b, c, d, e, f) of positive integers is a dii-sequence if and only if*

(*i*) $a \le b \le c \le d \le e \le f$.

(*ii*) $a = 1$ *implies that* $c = 1$.

(*iii*) $d = 1$ *implies that* $f = 1$.

(*iv*) $b \le 2a - 1$.

3. Independence saturation and extended domination chain

Acharya[1], introduced the concept of *domsaturation number* $ds(G)$ of a graph, which is defined to be the least positive integer k such that every vertex of G lies in a dominating set of cardinality k. Arumugam and Kala [2] observed that for any graph G, $ds(G) = \gamma(G)$ or $\gamma(G) + 1$ and obtained several results on $ds(G)$. Motivated by this concept Subramanian [10] introduced the concept of *independence saturation number* of a graph.

Definition 3.1. *Let $G = (V, E)$ be a graph and let $v \in V$. Let $IS(v)$ denote the maximum cardinality of an independent set in G which contains v. Then $IS(G) = min\,\{IS(v) : v \in V\}$ is called the independence saturation number of G. Thus $IS(G)$ is the largest positive integer k such that every vertex of G lies in an independent set of cardinality k. Let $v \in V$ be such that $IS(v) = IS(G)$. Then any independent set of cardinality $IS(G)$ containing v is called an IS-set.*

Example 3.2.

(*i*) *For the cycle C_n of length n, $IS(v) = \lfloor n/2 \rfloor$ for every vertex v and hence $IS(C_n) = \lfloor n/2 \rfloor$.*

(*ii*) $IS(K_n) = 1$ *and* $IS(K_{m,n}) = min\,\{m, n\}$.

Remark 3.3. *Any IS-set is a maximal independent set and hence is a dominating set. Thus*

$$i(G) \le IS(G) \le \beta_0(G). \tag{1}$$

These inequalities can be strict for the tree T given in Figure 1, we have IS =3, $\beta_0 = 6$ and $i = 2$.

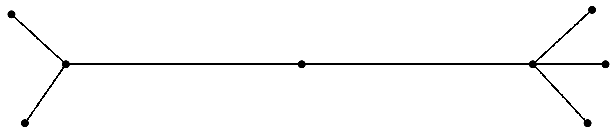

Figure 1

Further (1) extends the domination chain and we have

$$ir \leq \gamma \leq i \leq IS \leq \beta_0 \leq \Gamma \leq IR.$$

We observe that ir, γ *and* i *are minimum parameters,* β_0, Γ *and* IR *are maximum parameters and* IS *is a min-max parameter.*

The parameters i, IS and β_0 can be arbitrary as shown in the following theorem.

Theorem 3.4. *Let* a, b *and* c *be three positive integers with* $2 \leq a \leq b \leq c$. *Then there exists a graph* G *with* $i(G) = a, IS(G) = b$ *and* $\beta_0(G) = c$.

Proof. **Case i.** $a = 2$.

Let

$$k = \begin{cases} 0 & \text{if } c \leq 2b - 1 \\ c - 2b + 1 & \text{if } c > 2b - 1 \end{cases}$$

and let

$$\alpha = \begin{cases} 2b - 1 - c & \text{if } c \leq 2b - 1 \\ 0 & \text{if } c > 2b - 1. \end{cases}$$

Let $P_3 = (v_1, v_2, v_3)$ be a path on three vertices. Attach $b - 1$ pendant vertices $u_1, u_2, \ldots, u_{b-1}$ to v_1 and $b - 1 + k$ pendant vertices $w_1, w_2, \ldots, w_{b-1+k}$ to v_3. Add the edges $u_1w_1, u_2w_2, \ldots, u_\alpha w_\alpha$. For the resulting graph G, we have $i(G) = 2 = a$, $IS(G) = b$ and $\beta_0(G) = c$.

Case ii. $a > 2$.
Let

$$k = \begin{cases} 0 & \text{if } c \leq 2b - a \\ c - 2b + a & \text{if } c > 2b - a \end{cases}$$

and let

$$\alpha = \begin{cases} 2b - a - c & \text{if } c \leq 2b - a \\ 0 & \text{if } c > 2b - a. \end{cases}$$

Let $P = (v_1, v_2, \ldots, v_a)$ be a path on a vertices. Attach $b - (a - 1)$ pendant vertices $u_1, u_2, \ldots, u_{b-(a-1)}$ to v_1, attach $b - (a - 1) + k$ pendant vertices $w_1, w_2, \ldots, w_{b-(a-1)+k}$ to v_a and attach a pendant vertex x_i to each v_i, $2 \leq i \leq a - 1$. If $c \leq 2b - a$, we add the edges $u_1w_1, u_2w_2, \ldots, u_\alpha w_\alpha$. For the resulting graph G, we have $i(G) = a$, $IS(G) = b$ and $\beta_0(G) = c$. $\square$

Theorem 2.1 and Theorem 3.4 naturally lead to the following.

Problem 3.5. *Given seven positive integers $a \leq b \leq c \leq d \leq e \leq f \leq g$, find a necessary and sufficient condition for the existence of a graph G such that $ir(G) = a, \gamma(G) = b, i(G) = c, IS(G) = d, \beta_0(G) = e, \Gamma(G) = f$ and $IR(G) = g$.*

Theorem 3.6. *The problem of determining whether $IS(G) \geq k$ for any graph G is NP-complete.*

Proof. Let G_1 be the graph consisting of disjoint union of $G + K_1$ and G. Then $\beta_0(G) \geq k$ if and only if $IS(G_1) \geq k + 1$. Since the problem of determining whether $\beta_0(G) \geq k$ is NP-complete, the results follows. □

We now proceed to obtain bounds for $IS(G)$.

Theorem 3.7. *If G is an r-regular graph with $r > 0$, then $IS(G) \leq p/2$. Further equality holds if and only if G is bipartite.*

Proof. Let A be an independent set of cardinality IS. Then $r.IS \leq |E(G)| = \frac{pr}{2}$ and hence $IS \leq \frac{p}{2}$. Now, if $IS = \frac{p}{2}$, then $r.IS = \frac{pr}{2} = |E(G)|$ and hence A is a minimum vertex cover of G so that $IS = \alpha_0 = \frac{p}{2}$. Further $\alpha_0 + \beta_0 = p$ and hence $\beta_0 = \frac{p}{2}$ and $V - A$ is independent. Hence G is bipartite.

Conversely, let G be a bipartite graph with bipartition (X, Y). Since G is regular, $|X| = |Y| = \frac{p}{2}$ and G has a perfect matching. Hence it follows that $IS = \frac{p}{2}$. □

Theorem 3.8. *For any graph G, $IS \leq p - \Delta$. Further for a tree $T, IS = p - \Delta$ if and only if $V - N(v)$ is an independent set for every vertex v of degree Δ and $p_u \leq p_v$ for every $u \in N(v)$, where p_x is the number of pendant vertices adjacent to x.*

Proof. Let $v \in V$. Clearly, the maximum cardinality of an independent set containing v is at most $p - \Delta$ and hence $IS \leq p - \Delta$.

Now, let G be a tree with $IS = p - \Delta$. Let v be a vertex of degree Δ in G. Since v lies in an independent set of cardinality $p - \Delta$, it follows that $V - N(v)$ is an independent set. Now let $u \in N(v)$. Then the maximum cardinality of an independent set in G containing u is $p - \Delta - p_u + p_v$ and hence $p_u \leq p_v$.

Conversely, Let G be a tree satisfying the conditions of the theorem. If v is a vertex of degree Δ, then $V - N(v)$ is an independent set of cardinality $p - \Delta$. Also any vertex $u \in N(v)$ lies in an independent set of cardinality $p - \Delta - p_u + p_v \geq p - \Delta$. Hence it follows that $IS = p - \Delta$. □

Theorem 3.9. *Let G be any graph with at least three vertices. Then*

(*i*) $3 \leq IS + \overline{IS} \leq p + 1 - (\Delta - \delta)$ *and* $2 \leq IS.\overline{IS} \leq (p - \Delta)(\delta + 1)$.

(ii) *The following are equivalent.*

(a) $IS + \overline{IS} = 3.$

(b) $IS.\overline{IS} = 2.$

(c) G *or* $\overline{G}$ *has the property that it has a unique vertex of degree* $p-1$ *and has at least one pendant vertex.*

(iii) $IS + \overline{IS} = p+1$ *if and only if* G *is either* K_p *or* $\overline{K_p}$*.*

Proof.

(i) Since $IS \geq 1$ and $\overline{IS} \geq 2$ when $IS = 1,$ it follows that $IS+\overline{IS} \geq 3$ and $IS.\overline{IS} \geq 2.$ By Theorem 3.8, $IS \leq p - \Delta$ and hence it follows that $IS + \overline{IS} \leq p+1-(\Delta-\delta)$ and $IS.\overline{IS} \leq (p-\Delta)(\delta+1).$

(ii) Obviously (a) and (b) are equivalent. To prove (a) is equivalent to (c), we assume without loss of generality that, $IS = 1$ and $\overline{IS} = 2.$ Since $IS = 1,$ we have $\Delta = p-1.$ Since $\overline{IS} = 2,$ it follows that G has exactly one vertex of degree Δ and has a pendant vertex. Thus (a) implies (c). If (c) holds then either $IS = 1$ and $\overline{IS} = 2$ or $IS = 2$ and $\overline{IS} = 1$ hence (c) implies(a).

(iii) Suppose $IS + \overline{IS} = p+1.$ Then it follows from (i) that G is regular. If G is r-regular and $0 < r < p-1,$ then it follows from Theorem 3.7 that $IS + \overline{IS} \leq p.$ Hence $r = 0$ or $p-1$ so that G is either K_p or $\overline{K_p}.$ The converse is obvious.

□

Lampert and Slater [8] introduced the concept of *neighbourhood knockout numbers* $nk(G)$ and $NK(G)$ of a graph $G.$

Definition 3.10. *A knockout sequence is a sequence* $(S_0, S_1, \ldots, S_t)$ *of subsets of* V *such that* $S_0 = V$ *and for every* $i, 1 \leq i \leq t,$ *the set* S_i *is obtained from* S_{i-1} *by deleting a vertex* $v \in S_{i-1}$ *that is adjacent to another vertex* $u \in S_{i-1}$*. We say that the vertex* u *knocks out* v *in step* i *of the knockout process. Further, it is required that a knockout sequence terminates with a set* S_t *only when no further knockouts are possible.*

Thus a knockout sequence terminates with an independent set.

A knockout sequence need not terminate with a maximal independent set. To have knockout sequences that produce maximal independent sets upon termination some additional constraint is required. One such method is knockout with replacement.

Definition 3.11. *In knockout-with-replacement each knockout of* v *from a set* S_i *results in a set* $S_{i+1} = (S_i - \{v\}) \cup \{w : w \in V(G)$ *and* $N[w] \cap (S_i - \{v\} = \phi\}$*. Any vertex* w *with* $N[w] \cap S_i - \{v\} = \phi$ *is called a replacement vertex.*

Theorem 3.12. [6] *Each knockout-with-replacement sequence on a graph of order* n *and size* m *terminates in at most* m *steps with a maximal independent set.*

However, with no additional constraints a knockout-with-replacement sequence can produce any maximal independent set and hence does not give rise to any new parameter interior to $i(G) \leq \beta_0(G)$. An additional constraint produces such parameters.

Definition 3.13. [6] *In a neighbourhood knockout-with-replacement sequence, a vertex* u *knocks out a vertex* v *in step* $i+1$ *going from* S_i *to* S_{i+1} *provided the order of the neighbourhood of* u *in* $\langle S_i \rangle$ *is at least as large as the order of the neighbourhood of* v *in* $\langle S_i \rangle$. *The neighbourhood knockout numbers* $nk(G)$ *and* $NK(G)$ *denote the minimum and maximum cardinalities of a terminating set* S_t *in a maximal neighbourhood knockout sequence.*

Since every terminating set in a neighbourhood knockout-with-replacement sequence is a maximal independent set, we have

$$i(G) \leq nk(G) \leq NK(G) \leq \beta_0(G).$$

For the graph G given in Figure 2, we have $i(G) = 8, nk(G) = 9, NK(G) = 14$ and $\beta_0(G) = 15$.

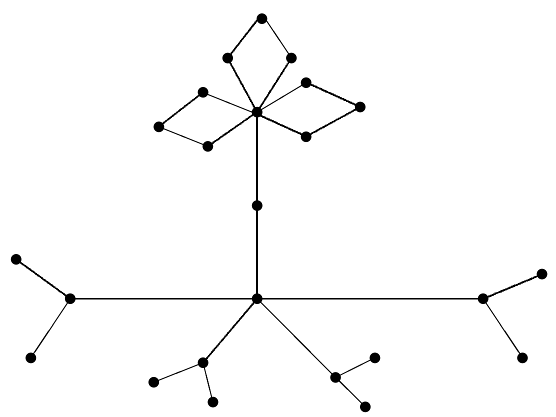

Figure 2

We observe that for this graph, $IS(G) = 13$ and hence $IS(G)$ is distinct from the other parameters.

Problem 3.14. *Is there any relation between* $IS(G)$ *and neighbourhood knockout numbers?*

Slater [9] pointed out that we can have $IS(G) > NK(G)$. For example if H is the graph consisting of three copies of G, where G is the graph given in Figure 2, we have $IS(H) = 43$ and $NK(H) = 42$. This leads to the following natural questions.

Problem 3.15. *Does there exist a graph G for which $IS(G) < nk(G)$?*

Problem 3.16. *Characterise the class of graphs G for which $nk(G) \leq IS(G) \leq NK(G)$.*

For example the graph given in Figure 2 satisfies this inequality.

4. Edge analogue of domination chain

Let $G = (V, E)$ be a graph. Two edges of G are said to be adjacent if they have a vertex in common. A set of edges $F \subseteq E$ is an *edge dominating set* if every edge in $E - F$ is adjacent to an edge in F. Let $\gamma'(G)$ and $\Gamma'(G)$ denote the minimum and maximum number of edges in a minimal edge dominating set of G. A set of edges $F \subseteq E$ is called an *independent set of edges* or a *matching* if no two edges in F are adjacent. Let $\beta_1(G)$ and $i'(G)$ denote respectively the maximum and minimum number of edges in a maximal independent set of edges in G. A set of edges $F \subseteq E$ is called an *irredundant edge set* if for every edge uv in F, there exists an edge wx which is adjacent to uv but is not adjacent to any edge in $F - \{uv\}$. The edge wx is called a *private neighbour* of uv. Let $ir'(G)$ and $IR'(G)$ denote respectively the minimum and maximum number of edges in a maximal irredundant set of edges in G. These six edge parameters satisfy the following inequality chain:

$$ir'(G) \leq \gamma'(G) \leq i'(G) \leq \beta_1(G) \leq \Gamma'(G) \leq IR'(G).$$

This chain can be rewritten in the form

$$ir(L(G)) \leq \gamma(L(G)) \leq i(L(G)) \leq \beta_0(L(G)) \leq \Gamma(L(G)) \leq IR(L(G)).$$

Since $L(G)$ is $K_{1,3}$-free, $\gamma(L(G)) = i(L(G))$, so that two parameters in this chain are always equal.

We now proceed to introduce the edge analogue of independence saturation.

Definition 4.1. *Let $G = (V, E)$ be a graph. Let $e \in E$. Let $EIS(e)$ denote the maximum cardinality of a matching in G which contains the edge e. Then $EIS(G) = min\,\{EIS(e) : e \in E\}$ is called the edge independence saturation number of G. Thus $EIS(G)$ is the largest positive integer k such that every edge of G lies in a matching of cardinality k. Let $e \in E$ be such that $EIS(e) = EIS(G)$. Then any matching of cardinality $EIS(G)$ containing e is called an EIS-set of G.*

Example 4.2. $EIS(P_n) = \left\lfloor \frac{n-1}{2} \right\rfloor, EIS(K_n) = EIS(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$ *and* $EIS(K_{m,n}) = min\,\{m, n\}$.

Remark 4.3. *For any graph G, we have $1 \leq EIS(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$ and these bounds are sharp.*

Remark 4.4. *Since any EIS-set of G is a maximal independent set of edges, we have $i'(G) \leq EIS(G) \leq \beta_1(G)$ and these bounds are sharp.*

In the following theorems we prove the existence of relations connecting these parameters.

Theorem 4.5. *For any graph G, we have $\beta_1 = EIS$ or $EIS + 1$.*

Proof. Let e be an edge of G with $EIS(e) = EIS(G)$. Let A be a set of independent edges such that $e \in A$ and $|A| = EIS(e) = EIS(G)$. Let B be a maximum matching in G so that $|B| = \beta_1$. If $e \in B$, then $EIS = \beta_1$. Suppose $e \notin B$. Let $H = G[A\Delta B]$ be the edge induced subgraph of G induced by $A\Delta B$, where $A\Delta B$ denotes the symmetric difference of A and B. Then each component of H is either an even cycle or a path with edges alternately in A and B. Further any path component of H which does not contain e is an even path. Hence the only possible odd component of H is a path component C_1 of H which contains e. If C_1 contains k edges from A and $k+1$ edges from B, then $|B| = |A| + 1$, so that $\beta_1 = EIS + 1$. Hence $\beta_1 = EIS$ or $EIS + 1$. □

Problem 4.6. *Characterise the class of graphs for which $\beta_1 = EIS$.*

Theorem 4.7. *For any graph G, we have $EIS \leq 2i' - 1$.*

Proof. Let A be an edge dominating set of G with $|A| = i'$. Since every edge of G is adjacent to an edge of A and there can be at most two independent edges adjacent with any edge of A, it follows that $EIS \leq 2i' - 1$. □

Problem 4.8. *Characterise the class of graphs for which $EIS = 2i' - 1$.*

Theorem 4.9. *Given three positive integers a, b and c with $2 \leq a \leq b \leq c$, there exists a graph G with $i'(G) = a, EIS(G) = b$ and $\beta_1(G) = c$ if and only if $b \leq 2a - 1$ and $c = b$ or $b + 1$.*

Proof. Necessity follows from Theorem 4.5 and Theorem 4.7. Conversely, let $b \leq 2a-1$ and $c = b$ or $b+1$. Let $b = a + k$, where $0 \leq k \leq a - 1$.
Case i. $b = c$.

We construct a graph G as follows:

Let $\{u_1v_1, u_2v_2, \ldots, u_av_a\}$ be a set of a independent edges. Add vertices $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$; Join x_i with u_i, u_{i+1} ; Join y_i with v_i, v_{i+1} for all $i, 1 \leq i \leq k$. Add a vertex z and join z with u_{k+j} and v_{k+j} for all j, $2 \leq j \leq a-k$. Also join u_{k+1}, u_{k+2} and v_{k+1}, v_{k+2}. For the resulting graph G, we have $i'(G) = a$ and $EIS(G) = \beta_1(G) = b$.
Case ii. $b = c - 1$.

Construct a graph G as follows:

Let $\{u_1v_1, u_2v_2, \ldots, u_av_a\}$ be a set of a independent edges. Add vertices $x_1, x_2, \ldots, x_{k+1}, y_1, y_2, \ldots, y_{k+1}$; Join x_i with u_i and join y_i with v_i for all $i, 1 \leq i \leq k+1$. Also add a vertex z and join z with u_i and v_i for all i with $k+2 \leq i \leq a$. Further add the edges $u_{k+1}u_{k+2}$ and $v_{k+1}v_{k+2}$. For the resulting graph G, we have $i'(G) = a, EIS(G) = b$ and $\beta_1(G) = c$. □

Problem 4.10. *Let a, b, c, d, e, f and g be positive integers with $a \leq b \leq c \leq d \leq e \leq f \leq g$. Obtain a necessary and sufficient condition for the existence of a graph G with $ir'(G) = a, \gamma'(G) = b, i'(G) = c, EIS(G) = d, \beta_1(G) = e,\ \Gamma'(G) = f$ and $IR'(G) = g$.*

5. Conclusion and Scope

In this paper, we have introduced the concept of independece saturation number and edge independence saturation number which extend the domination chain and its edge analogue. Slater [9] suggested the following generalization of independence saturation and several questions arising out of this generalisation.

Let $G = (V, E)$ be a graph. Let k be an integer with $0 \leq k \leq \beta_0$. Let $R \subseteq V$ be an independent set in G with $|R| = k$. Define

$$IS_k(R) = Max\{|S| : S \textit{ is an independent set in } G \textit{ and } R \subseteq S\}$$
$$IS_k(G) = Min\{IS_k(R) : R \subseteq V \textit{ is independent and } |R| = k\}$$
$$is_k(R) = Min\{|S| : S \textit{ is a maximal independent set and } R \subseteq S\}$$

and

$$is_k(G) = \text{Max } \{is_k(R) : R \subseteq V \text{ is independent and } |R| = k\}.$$

We observe that $IS_1(G) = IS(G)$, $IS_0(G) = IS_{\beta_0(G)}(G) = is_{\beta_0(G)}(G) = \beta_0(G)$ and $is_0(G) = IS_{i(G)}(G) = i(G)$. Further $i(G) \leq is_k(G) \leq \beta_0(G)$ and $i(G) \leq IS_k(G) \leq \beta_0(G)$

Problem 5.1. *Consider the sequence $\beta_0(G) = IS_0(G), IS_1(G), IS_2(G), \ldots, IS_{\beta_0(G)}(G) = \beta_0(G)$. Is this sequence unimodal?.*

In other words, is it true that if $0 \leq k \leq i(G)$, then $IS_k(G) \geq IS_{k+1}(G)$ and if $i(G) \leq k \leq \beta_0(G)$, then $IS_k(G) \leq IS_{k+1}(G)$?

Problem 5.2. *Study the sequence $i(G) = is_0(G), is_1(G), \ldots, is_{\beta_0(G)}(G) = \beta_0(G)$.*

Problem 5.3. *Find $min\,\{k : IS_k(G) = i(G)\}$.*

Problem 5.4. *Find $min\,\{k : is_k(G) = \beta_0(G)\}$.*

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