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Competition Parameters of a Graph

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Abstract

Competition parameters for a graph are those associated with two players alternately selecting elements to be included in a set S, where S is required to have a certain property. One player attempts to maximize a value, typically the order of the resulting set, while the other player attempts to minimize the value. Several independence/ domination/ enclaveless competitive parameters are discussed, along with a competitive-reachability parameter.

Keywords: Competition parameter, competition-independence parameters, competition-enclaveless parameter, competition-reachability

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1. Introduction

Various graphical games involve two players alternately choosing vertices to be in a vertex set S in a graph G, where the resulting set must have a certain property. Play stops when the addition of any vertex not already in the set destroys the property, and the winner/loser is typically determined by who was last to select a vertex for the set.

As in Finbow and Hartnell [6] and Conway [5], for the Independence Game the resulting set S must be independent, that is, for each edge in $E(G)$ the set S does not contain both endpoints. The independence number of G, the maximum cardinality of an independent vertex set S in $V(G)$, is denoted by $\beta(G)$, and the lower independence number, denoted by $i(G)$, is the minimum cardinality of a maximal independent set. Recall that an independent set is maximally independent if and only if it dominates $V(G)$. For the path P_n with $V(P_n) = \{ v_1, v_2, ..., v_n \}$ and $E(P_n) = \{ v_1v_2, v_2v_3, ..., v_{n-1}v_n \}$ we have

$\beta(P_9) = 5 = |\{v_1, v_3, v_5, v_7, v_9\}|$ and $i(P_9) = 3 = |\{v_2, v_5, v_8\}|$. If the winner is the player who manages to select the last vertex, Player 1 can select v_5 and then play symmetrically. Possible sequences of play include v_5, v_2, v_8 and v_5, v_3, v_7, v_9, v_1, and the resulting set S always has (odd) cardinality three or five. One can easily check that Player 1 could first select v_5 and force the resulting set S to have cardinality four, hence also winning the game when the loser is the one to choose last. Note that Player 1 would lose the "last to play wins" game for path P_4.

Graphical two-player games in which players are not concerned with who played last, but rather with the order of the resulting set, were introduced in Phillips and Slater [9, 10]. One player is the maximizer and the other is the minimizer. The players take turns selecting vertices to go into a set $S \subseteq V(G)$ which is required to have some property such as independence. The maximizer attempts to maximize the order of the resulting set, while the minimizer attempts to minimize its order. The minimizer and maximizer are constructing a single set S, not two separate sets. If both players play optimally, then the cardinality of the resulting set is fixed. Parameters that assign a value to each graph in this manner are called competitive parameters.

For the Competition-Independence Game the resulting set S must be independent. The competition parameters $\beta^+_{com}(G)$ and $\beta^-_{com}(G)$ denote the size of the resulting set S in the Competition-Independence Game when the maximizer and minimizer, respectively, plays first. For path $P_5, \beta^+_{com}(P_5) = 3 = \beta(P_5)$ resulting from Player 1 first choosing v_3, and $\beta^-_{com}(P_5) = 2 = i(P_5)$ resulting from Player 1 first choosing v_2 or v_4. For path P_9 one can verify that $i(P_9) = 3 < \beta^-_{com}(P_9) = 4 = \beta^+_{com}(P_9) < 5 = \beta(P_9)$. Because the resulting set S in this game is a maximally independent (and, hence, dominating) set we have the following theorem.

Theorem 1.1. [10] *For every graph* $G, i(G) \leq \beta^-_{com}(G) \leq \beta(G)$ *and* $i(G) \leq \beta^+_{com}(G) \leq \beta(G)$.

An interesting example is cycle C_8 because $\beta^-_{com}(C_8) > \beta^+_{com}(C_8)$. Because $i(P_n) = \lceil n/3 \rceil$ and $\beta^+_{com}(P_n) = \lceil n/2 \rceil$, we have $(1/3)n \leq \beta^-_{com}(P_n)$, $\beta^+_{com}(P_n) \leq (1/2)(n+1)$. The fraction of vertices in $\beta^+_{com}(P_n)$-sets and $\beta^-_{com}(P_n)$-sets is actually 3/7.

Theorem 1.2. [10] *For path* P_n, $\beta^-_{com}(P_{7k}) = \beta^+_{com}(P_{7k}) = 3k$, $\beta^-_{com}(P_{7k+1}) = \beta^+_{com}(P_{7k+1}) = 3k+1$, $\beta^-_{com}(P_{7k+2}) = \beta^+_{com}(P_{7k+2}) = 3k+1$, $\beta^-_{com}(P_{7k+3}) = 3k+1$, $\beta^+_{com}(P_{7k+3}) = 3k+2$, $\beta^-_{com}(P_{7k+4}) = \beta^+_{com}(P_{7k+4}) = 3k+2$, $\beta^-_{com}(P_{7k+5}) = 3k+2$, *and* $\beta^+_{com}(P_{7k+5}) = 3k+3$, $\beta^-_{com}(P_{7k+6}) = \beta^+_{com}(P_{7k+6}) = 3k+3$.

Note that, if we are interested in the property of domination and we consider alternately choosing vertices for the dominating set S, then, during the process of selecting elements of S, the partially formed set will not have the domination property. This and other problems can be avoided by selecting elements for the complementary set, an enclaveless set. As defined by Alan Goldman and introduced in Slater [17], a vertex $v \in S$ is an enclave of S if all of its neighbors are also in S, that is, the closed neighborhood of

v satisfies $N[v] \subseteq S$. Set S is enclaveless if it does not contain any enclave, that is, for all $v \in S$ we have $N[v] \cap (V(G) - S) \neq \emptyset$. The enclaveless number of G is the maximum cardinality of an enclaveless set S in $V(G)$ and is denoted by $\Psi(G)$, and the lower enclaveless number, denoted by $\psi(G)$, is the smallest cardinality of a maximally enclaveless set. As usual, $\gamma(G)$ and $\Gamma(G)$ denote the domination and upper domination numbers of G. Because a set is enclaveless if and only if its complement is dominating, we have the next complementation theorem.

Theorem 1.3. [17] *For every graph G of order n,* $\gamma(G) + \Psi(G) = n = \Gamma(G) + \psi(G)$.

For the Competition-Enclaveless Game players alternate selecting elements of an enclaveless set S, and $\Psi^+_{com}(G)$ and $\Psi^-_{com}(G)$ denote the size of the resulting enclaveless set S when the maximizer and minimizer, respectively, chooses first.

Because $\psi(P_n) = n - \Gamma(G)$ = n - $\lceil n/2 \rceil$ and $\Psi(G) = n - \gamma(G) = n - \lceil n/3 \rceil$, $\Psi^+_{com}(P_n)$ and $\Psi^-_{com}(P_n)$ will contain between $(1/2)n$ and $(2/3)n$ vertices. The actual fraction is 3/5.

Theorem 1.4. [10] *For path P_n,*

$$\Psi^+_{com}(P_{5k}) = \Psi^+_{com}(P_{5k+1}) = \Psi^-_{com}(P_{5k}) = \Psi^-_{com}(P_{5k+1}) = 3k,$$
$$\Psi^-_{com}(P_{5k+2}) = \Psi^-_{com}(P_{5k+3}) = \Psi^+_{com}(P_{5k+2}) = 3k+1, \textit{and}$$
$$\Psi^+_{com}(P_{5k+3}) = \Psi^+_{com}(P_{5k+4}) = \Psi^-_{com}(P_{5k+4}) = 3k+2.$$

We note that the competition games associated with the formation of acquisition sets are discussed in Slater and Wang [18]. Walsh [19] discusses a competition chromatic number problem. Another competition parameter for another domination related concept is presented in Section 2. In Section 3 we discuss parameters arising when two players alternately make choices as to how to orient the edges of a graph with the goal of maximizing/minimizing a digraph parameter. We introduce a new such parameter, the competition-reachability parameter.

2. Other Competition Parameters

For the competition-independence and competition-enclaveless parameters discussed in Section 1, players alternate selecting elements (in these two cases, vertices) for inclusion in a set S. For competition-acquisition (see [18]), with some specified constraints players alternate forming a sequence of vertices from which "tokens" will be moved with the objective of maximizing/minimizing the number of vertices from which tokens are moved. In contrast to these games in which players alternately try to maximize/minimize the number of "moves" they make, for parameters like the competition-reachability parameter discussed in Section 3 the number of moves is fixed (namely, the number of edges) while players alternate choosing edge orientations in constructing an oriented graph.

In this section we illustrate how to create another competition parameter for a subset parameter.

For the complementary parameters of domination and of being enclaveless, the enclaveless property was better to consider. Similarly, consider total-domination (also called open-domination). Vertex set $S \subseteq V(G)$ is a total-dominating set if each vertex $v \in V(G)$ is adjacent to a vertex in S, that is, $N(v) \cap S = \emptyset$ for every $v \in V(G)$ where open neighborhood $N(v)$ is the set of vertices adjacent to v. For a vertex set $R \subseteq V(G)$ call vertex v an open-enclave of R if $N(v) \subseteq R$. Call $R \subseteq V(G)$ an open-enclaveless set if there are no R-open-enclaves.

Clearly $S \subseteq V(G)$ is a total-dominating set if and only if $V(G) - S$ is open-enclaveless. For total domination number $\gamma_t(G)$, upper total domination number $\Gamma_t(G)$, open-enclaveless number $\Psi_{open}(G)$, and lower open-enclaveless number $\psi_{open}(G)$, we have the following theorem.

Theorem 2.1. *For every graph G of order n without isolated vertices, $\gamma_t(G)$ $+\Psi_{open}(G) = n = \Gamma_t(G) + \psi_{open}(G)$.*

For the Competition-Open-Enclaveless Game players alternate selecting elements of an open-enclaveless set S, and $\Psi^+_{open-com}(G)$ and $\Psi^-_{open-com}(G)$ denote the size of the resulting open-enclaveless set S when the maximizer and minimizer, respectively, chooses first. Note that for path P_n the open neighborhoods are $\{v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_4, v_6\}, ..., \{v_{n-2}, v_n\}$, and $\{v_{n-1}\}$. If $n = 2k$, one can therefore not select two consecutive vertices in v_1, v_3, v_5, . . , v_{n-3} or in v_4, v_6, v_8, . . , v_{n-2}, v_n, two paths with $k - 1 = (n - 2)/2$ vertices. If $n = 2k + 1$, one can not select two consecutive vertices in v_1, v_3, v_5, . . , v_n or in v_4, v_6, . . , v_{2k-2}, paths with $k + 1 = (n + 1)/2$ and $k - 2$ vertices. Thus we have the following theorem which relates $\Psi^+_{open-com}(P_n)$ and $\Psi^-_{open-com}(P_n)$ to competition-independence parameters.

Theorem 2.2. *If $n = 2k$, then*

$$\Psi^+_{open-com}(P_n) = \beta^+_{com}(2P_{k-1}), and \Psi^-_{open-com}(P_n) = \beta^-_{com}(2P_{k-1}).$$

If $n = 2k + 1$, then

$$\Psi^+_{open-com}(P_n) = \beta^+_{com}(P_{k+1} \cup P_{k-2}) and \Psi^-_{open-com}(P_n) = \beta^-_{com}(P_{k+1} \cup P_{k-2}).$$

We conclude this section by noting that there exists a competitive parameter associated with essentially any graphical parameter.

3. Graph Orientations Competition-Reachability

A directed graph D obtained from (undirected) graph G by assigning a direction to each edge in G is called an oriented graph. Various orientation games with different goals for two players have been considered, for example in [1-4]. Bollobas and Szabo [3] and Chartrand, Harary, Schultz and Vanderjagt [4] discuss achievement/avoidance games, namely to create at least one oriented cycle and to have a strong orientation, respectively.

Alon and Tuza [2] consider the game in which the first player successively chooses edges while the second player successively orients each selected edge in the direction he chooses subject to the condition that no directed cycle is created. The first and second players try to minimize and maximize, respectively, the number of edges chosen until there is a unique acyclic orientation that extends the currently oriented edges. Alon, Balogh, Bollobas and Szabo [3] define the game domination number of a graph G for which two players alternately orient an edge of G until all of the edges are oriented, their goals being to minimize and maximize the domination number of the resulting digraph. Here we introduce another game where players alternately orient the edges of graph G until all of the edges are oriented.

The reachability $r(D)$ of a directed graph D is the number of ordered pairs of distinct vertices (x, y) with a directed path from x to y. The problem of orienting an undirected graph G so as to maximize the reachability of the resulting directed graph $D(G)$ is considered, for example, in Hakimi et al [8]. Clearly, $r(D) \leq n(n-1)$ for a digraph D on n vertices, and D is called strongly connected if $r(D) = n(n-1)$. The following result is easy to see.

Theorem 3.1. [11] *A graph G can be oriented so that the resulting oriented graph D is strongly connected if and only if G is 2-edge-connected.*

We can consider a game in which two players alternately choose an orientation for an edge of G until all of the edges have been oriented. The number of moves is clearly fixed, namely $|E(G)|$. For this competitive game the maximizer (respectively, minimizer) will try to maximize (respectively, minimize) the reachability $r(D)$ of the resulting digraph D. The competitive-reachability parameters $r^+_{com}(G)$ and $r^-_{com}(G)$ are the reachability values when the maximizer and minimizer, respectively, moves first. Observe that we always have $r^+_{com}(G) \geq |E(G)|$ and $r^-_{com}(G) \geq |E(G)|$ because for any orientation the $m = |E(G)|$ ordered pairs of vertices determined by the m directed edges provide m directed paths of length one.

Theorem 3.2. *For every graph G of order $n = |V(G)|$ and size $m = |E(G)|$ we have $m \leq r^+_{com}(G) \leq n(n-1)$ and $m \leq r^-_{com}(G) \leq n(n-1)$.*

Note in the next theorem that it is better for each player to be the one to play first when n is even, but it is better to play second for odd paths.

Theorem 3.3. *For the paths P_n on n vertices, $r^-_{com}(P_{4k}) = (6n-8)/4$ and $r^+_{com}(P_{4k}) = (7n-12)/4$; $r^-_{com}(P_{4k+1}) = (7n-7)/4$ and $r^+_{com}(P_{4k+1}) = (6n-10)/4$; $r^-_{com}(P_{4k+2}) = (6n-8)/4$ and $r^+_{com}(P_{4k+2}) = (7n-10)/4$; $r^-_{com}(P_{4k+3}) = (7n-9)/4$ and $r^+_{com}(P_{4k+3}) = (6n-10)/4$.*

Proof. We will verify just some of the equalities. Complete details are presented in Seo and Slater [13].

Claim 1. $r^+_{com}(P_{2k+1}) = 3k - 1 = (6n - 10)/4$. The minimizer can consider the k pairs of consecutive adjacent edges $\{v_1v_2, v_2v_3\}$, $\{v_3v_4, v_4v_5\}$, ..., $\{v_{2k-1}v_{2k},\ v_{2k}v_{2k+1}\}$. Each time the maximizer orients an edge, the minimizer orients the other edge paired with it in the opposite direction. Thus, no three edges in a row are similarly oriented, nor are v_1v_2 and v_2v_3, nor are edges $v_{2k-1}v_{2k}$ and $v_{2k}v_{2k+1}$. Therefore, $r^+_{com}(P_{2k+1}) \leq k - 1 + 2k = 3k - 1$. Also, for each move after the first, the maximizer can choose an edge adjacent to an oriented edge and orient it in the same direction. Therefore, $r^+_{com}(P_{2k+1}) \geq k - 1 + 2k = 3k - 1 = (6n - 10)/4$.

Claim 2. $r^-_{com}(P_{2k}) = 3k-2 = (6n-8)/4$. There are $2k-1$ edges, so the minimizer will make k moves and the maximizer $k-1$. Again, each of the maximizer's $k-1$ moves can make two consecutive edges have the same orientation, so $r^-_{com}(P_{2k}) \geq k - 1 + (2k - 1) = 3k - 2$. The minimizer can orient v_1v_2 and consider the pairs $\{v_2v_3, v_3v_4\}$, $\{v_4v_5, v_5v_6\}$, . . . , $\{v_{2k-2}v_{2k-1},\ v_{2k-1}v_{2k}\}$ and respond to each maximizer's move by orienting the paired edge in the opposite direction. Therefore, $r^-_{com}(P_{2k}) \leq k - 1 + (2k - 1) = 3k - 2$.

Claim 3. $r^-_{com}(P_{4k+1}) = 7k = (7n - 7)/4$. Maximizer can partition the edges into sets of four consecutive edges and always respond to a minimizer's move with a move in the same 4-set of edges. One can think of this as separate play on each of k different P_5's. It can easily be verified that $r^-_{com}(P_5) = 7$, and it follows that $r^-_{com}(P_{4k+1}) \geq 7k$.

Minimizer can adopt the following strategy. Consecutively pair the edges $\{v_1v_2, v_2v_3\}$, $\{v_3v_4, v_4v_5\}$, . . . , $\{v_{2j-1}v_{2j},\ v_{2j}v_{2j+1}\}$, . . . , $\{v_{4k-1}v_{4k},\ v_{4k}v_{4k+1}\}$. To start, orient the first edge from v_2 to v_1. Therefore, whenever maximizer chooses an edge from a new pair, orient the other edge in the pair in the opposite direction. If maximizer orients the edge whose paired edge is already oriented, let v_iv_{i+1} be the first unoriented edge and orient it opposite to the orientation of $v_{i-1}v_i$. Note that in this latter case i must be odd.

Whenever a pair of edges $\{v_{2j-1}v_{2j}, v_{2j}v_{2j+1}\}$ is oriented in the same direction, then the minimizer oriented $v_{2j-1}v_{2j}$ before the maximizer oriented $v_{2j}v_{2j+1}$ and $j \geq 2$ implies that $v_{2j-2}v_{2j-1}$ is oriented oppositely from $v_{2j-1}v_{2j}$. In particular, no four consecutive edges are oriented in the same direction, and if three are then the first two form a pair.

Let $(n_1, n_2, ..., n_t)$ describe the orientation where there are n_1 edges oriented like v_1v_2, then n_2 edges in the opposite direction, then n_3 edges oriented like v_1v_2, etcetera. As noted, each $n_i \leq 3$. Also, $n_i = 3$ implies that $n_{i+1} \leq 2$ because the first of the $n_i + 1$ edges would be the second edge in a pair, and $n_i = 3$ implies that $i \leq t - 1$ because the third of these edges is the left edge of a pair. Further, $n_i = 3$ and $n_{i+j} = 3$ implies that $n_{i+1} + n_{i+2} + ... + n_{i+j-1}$ is odd.

Starting from the beginning of the path we can successively identify collections of edges such that the next remaining edge (if any) is oppositely oriented from its predecessor and the number c of directed paths using edges from each collection is at most seven-fourths the number of edges in the collection. Suppose we have identified such collections for the first $n_1 + n_2 + ... + n_{i-1}$ edges. If $n_i = 1$ or 2, let the next collection consist of precisely these n_i edges. For $n_i = 1$ we have $c = 1$ such directed paths and $c = 1 < (7/4) \cdot 1$, and if $n_i = 2$ then we have two directed paths of length one and one of length two so

$c = 3 < (7/4) \cdot 2$. If $n_i = 3$ and $n_{i+1} = 1$, let the next collection contain these four edges, and we have four directed paths of length one, two of length two, and one of length three. Then $c = 7 \leq (7/4) \cdot 4$. For $n_i = 3$ and $n_{i+1} = 2$ and $n_{i+2} = 1$, let the next collection contain these six edges, and $c = 10 < (7/4) \cdot 6$.

Finally, for $n_i = 3$ and $n_{i+1} = 2$ and $n_{i+2} = 2$, let the next collection contain these seven edges, and $c = 12 < (7/4) \cdot 7$. For the orientation D of P_{4k+1} we therefore have $r(D) \leq (7/4)4k = 7k = (7n-7)/4$. The remaining equalities can likewise be verified, completing the proof. □

Seo and Slater [13, 14] is a much more extensive study of competition- reachability.

4. Addendum

The authors appreciate the opportunity to revise the original form of this paper. Several references have been updated, and some have been added including [15, 16]. In [15] a graph orientation game that considers the number of directed cycles formed is discussed, and [16] presents some infinite families of graphs which are second player optimal for several graph competition-parameters.

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