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A note on a longest cycle which is vertex dominating

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Abstract

A cycle C is *vertex dominating* in G if $N_G(u) \cap V(C) \neq \emptyset$ holds for every $u \in V(G-C)$. In 1987, Bondy and Fan considered the degree condition for the existence of vertex dominating cycles. But they did not refer to the length of a vertex dominating cycle. In this paper, we prove two results on the degree condition which ensure the existence of a longest cycle which is vertex dominating.

Keywords: Vertex dominating cycle, minimum degree

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1. Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. We denote the degree of a vertex x by $d_G(x)$. Let $\alpha(G)$, $\kappa(G)$ and $\delta(G)$ be the independence number, the connectivity and the minimum degree of a graph G, respectively.

A subgraph H of a graph G is called *edge dominating* if $E(G-H)=\emptyset$, and *vertex dominating* if each $u \in V(G-H)$ has a neighbor in H. The concept "domination" in graphs has been studied in many directions.

For example, Bondy investigated the degree condition which ensure the existence of a longest cycle which is edge dominating, and proved the following theorem. Let

$$\sigma_k(G) := \begin{cases} \min\left\{\sum_{x\in S} d(x) : S \text{ is an independent set and } |S|=k\right\} & \text{if } \alpha(G)\geq k \\ +\infty & \text{if } \alpha(G)<k. \end{cases}$$

Theorem 1. [1] *Let G be a 2-connected graph on n vertices. If $\sigma_3(G) \geq n+2$, then every longest cycle in G is edge dominating.*

On the other hand, compared with an edge dominating cycle, there are a few results on a vertex dominating cycle. Among them, we introduce two results. The first one was proved by Bondy and Fan. A vertex set S is called r-*stable* if $dist(x,y) \geq r$ for every $x,y \in S$, where $dist(x,y) := \min\{|E(P)| : P \text{ is an } xy\text{-path}\}$.

Theorem 2. [2] *Let G be a k-connected graph on n vertices. Suppose that for any 3-stable set S of order $k+1$, we have $\sum_{x\in S} d_G(x) \geq n-2k$. Then there exists a vertex dominating cycle.*

In [2], it is shown that the degree condition in Theorem 2 is best possible. Let H_i be a graph obtained by joining a vertex u_i and all vertices of K_ℓ. G_1 is constructed from $K_k \cup \bigcup_{i=1}^{k+1} H_i$ by joining each vertex of $K_\ell \subset H_i$ and each vertex of K_k. (See Figure 1.) Then $|V(G_1)| = n = k + (k+1)(\ell+1) = (k+1)\ell + 2k + 1$. Any cycle C in G can not pass through H_i for some H_i. Hence there exists a vertex u_i whose neighbors are not contained in C. Therefore, G has no vertex dominating cycles. On the other hand, $\{u_1, u_2, \ldots, u_{k+1}\}$ is 3-stable set and $\sum_{i=1}^{k+1} d_G(u_i) = (k+1)\ell = n-2k-1$.

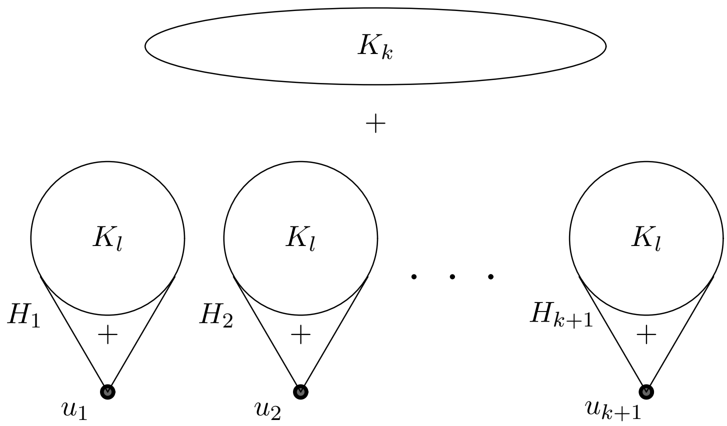

Figure 1: The graph G_1

Next, we will show the result by Saito and Yamashita. Let $\omega(G)$ be a number of components of G. A graph G is called t-*tough* if $|S| \geq t \cdot \omega(G-S)$ for every cut set $S \subset V(G)$.

Theorem 3. [6] *Let G be a k-connected graph on n vertices with $k \geq 2$ and let $t > \frac{k}{k+1}$. Suppose for any 4-stable set S of order $k+1$, we have $\sum_{x\in S} d_G(x) \geq n-2k-2$. If G is t-tough, then there exists a vertex dominating cycle.*

In [2] and [6], the length of a vertex dominating cycle is not taken into account. In this paper, we consider the existence of a longest cycle which is vertex dominating as it is shown in Theorem 1. On a $\sigma_{k+1}(G)$ condition, we prove the following theorem.

Theorem 4. *Let $2 \le k \le 4$ and G be a k-connected graph on n vertices. If $\sigma_{k+1}(G) \ge n + k(k-2)$, then every longest cycle in G is vertex dominating.*

This degree condition is best possible. For k graphs $G_1, G_2, \ldots, G_k$, the graph G_1+G_2 is defined the join of G_1 and G_2 and the sequential join $G_1 + G_2 + \cdots + G_k$ is defined the union of $k-1$ joins $G_i + G_{i+1}$ for $1 \le i \le k-1$. Let k and ℓ be integers with $\ell \ge k+2$ and H'_1 and H'_2 be graphs isomorphic to K_k. We consider the graph $G_2 = K_1 + H'_1 + H'_2 + kK_\ell$. (See Figure 2.) Let u be the unique vertex in K_1. Since every longest cycle C passes through exactly H'_2 and kK_ℓ, all neighbors of u, that is, H'_1 must not be contained in C. Thus, no longest cycle in G_2 is vertex dominating. On the other hand, $|V(G_2)| = n = k\ell + 2k + 1$ and $\sigma_{k+1}(G_2) = k + k(k+\ell-1) = k\ell + k^2 = n + k(k-2) - 1$.

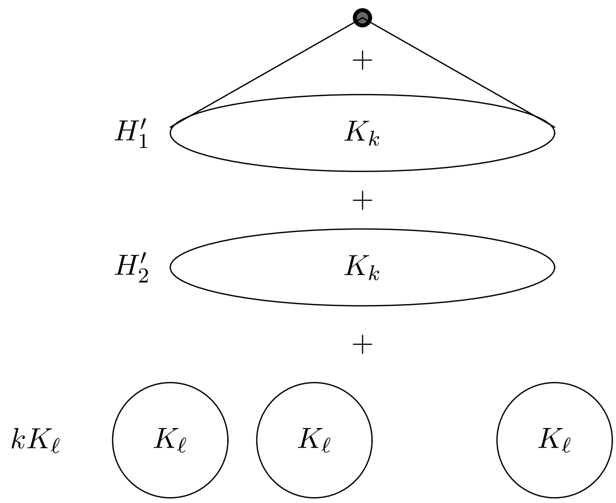

Figure 2: The graph G_2

We conjecture that Theorem 4 holds also for $k \ge 5$.

Conjecture 5. *Let $k \ge 2$ and G be a k-connected graph on n vertices. If $\sigma_{k+1}(G) \ge n + k(k-2)$, then every longest cycle in G is vertex dominating.*

The graph G_2 also shows that the degree condition is sharp if this conjecture is true.

From Theorem 1, we have the following corollary.

Corollary 6. *Let G be a 2-connected graph on n vertices. If $\delta(G) \ge \frac{1}{3}(n+2)$, then every longest cycle in G is edge dominating.*

If a cycle C is edge dominating, then it is also vertex dominating. So it is expected that we can reduce the minimum degree condition if we replace 'edge dominating' to 'vertex dominating'. On this topic, we prove the following.

Theorem 7. *Let G be a 2-connected graph on n vertices with $n \geq 65$. If $\delta(G) \geq \frac{1}{3}(n-4)$, then every longest cycle in G is vertex dominating.*

Note that the degree conditions in Corollary 6 and Theorem 7 is differ only by 2. The graph G_1, which is constructed by Bondy and Fan, shows that the degree condition of Theorem 7 is best possible again.

To get the conclusion "every longest cycle in G is vertex dominating", $\delta(G) \geq \frac{|V(G)|}{3}$ is necessary if we apply Theorem 4. Therefore, Theorem 7 with $k=2$ says that we can decrease the bound on $\delta(G)$ by considering a graph G of sufficiently large order.

For graph-theoretic terminology not explained in this paper, we refer the reader to [7]. Let $u \in V(G)$. A neighborhood of u is denoted by $N_G(u)$. For $U \subset V(G)$, we define $N_G(U)$ by $N_G(U) := \bigcup_{u \in U} N_G(u)$. For a subgraph H of G, we write $N_H(u)$, $N_H(U)$ and $d_H(u)$ instead of $N_G(u) \cap V(H)$, $N_G(U) \cap V(H)$ and $|N_H(u)|$, respectively. We write $N_G(H)$ instead of $N_G(V(H))$.

A path joining u and v is called a *uv-path.* For a subgraph H, a uv-path P is called an *H-path* if $V(P) \cap V(H) = \{u, v\}$ and $E(P) \cap E(H) = \emptyset$.

Let C be a cycle in G. We give an orientation to C and write the oriented cycle C by $\overrightarrow{C}$. For $x, y \in V(C)$, we denote an xy-path along $\overrightarrow{C}$ by $x\overrightarrow{C}y$, and write the reverse sequence of $x\overrightarrow{C}y$ by $y\overleftarrow{C}x$. For $x \in V(C)$, we denote the h-th successor and the h-th predecessor of x on $\overrightarrow{C}$ by x^{+h} and x^{-h}, respectively. For $X \subset V(C)$, we define $X^{+h} := \{x^{+h} : x \in X\}$ and $X^{-h} := \{x^{-h} : x \in X\}$. We often write x^+, x^-, X^+ and X^- for x^{+1}, x^{-1}, X^{+1} and X^{-1}, respectively.

For a path P, we sometimes give an orientation to P and at that time define the same terminology as we did for a cycle C.

2. Proof of Theorem 4

Let C be a longest cycle in G. If $N_C(x) \neq \emptyset$ for every $x \in V(G-C)$, then there is nothing to prove. Therefore, we may assume that there exists $x_0 \in V(G-C)$ such that $N_C(x_0) = \emptyset$. Let H be a component of $G-C$ with $x_0 \in V(H)$.

Since G is k-connected, there exist k disjoint paths $P_1, P_2, \ldots, P_k$ joining x_0 and C such that $|V(P_i) \cap V(C)| = 1$. Let v_i be the end vertex of P_i which is distinct from x_0 and let $x_i = v_i^+$ for $1 \leq i \leq k$. Then, the following claim holds.

Claim 2.1. *For every $1 \leq i \neq j \leq k$,*

(i) $N_H(x_i) = \emptyset$.

(ii) *There exists no C-path joining x_i and x_j.*

(iii) *If there exists $w \in V(x_i^{+l}\overrightarrow{C}v_j)$ such that $wx_i \in E(G)$, then $w^{-l}x_j \notin E(G)$ for any $1 \leq l \leq 3$.*

Proof. It is easy to prove (i) and (ii). Hence we shall only show (iii). Suppose that there exists $w \in V(x_i^{+l} \overrightarrow{C} v_j)$ such that $x_i w \in E(G)$ and $x_j w^{-l} \in E(G)$ for some $1 \le l \le 3$ and let $C' := v_i P_i x_0 P_j v_j \overleftarrow{C} w x_i \overrightarrow{C} w^{-l} x_j \overrightarrow{C} v_i$. Note that $|V(P_i)| \ge 3$ and $|V(P_j)| \ge 3$ since $N_C(x_0) = \emptyset$. Furthermore, $|V(C - C')| = |V(w^{-l+1} \overrightarrow{C} w^-)| \le 2$ and $|V(C' - C)| = |V((P_i \cup P_j) \setminus \{v_i, v_j\})| \ge 3$ since $l \le 3$. But this implies $|V(C')| > |V(C)|$, a contradiction. □

Let $C_i := x_i \overrightarrow{C} v_{i+1}$ for every $1 \le i \le k$.

Claim 2.2. $\sum_{j=1}^k d_{C_i}(x_j) \le |V(C_i)| + k - 2$ *for every* $1 \le i \le k$.

Proof. By the symmetry, we prove the case $i = 1$.

If $k = 4$, then we have $N_{C_1}(x_1)^{-3} \cup N_{C_1}(x_2) \cup N_{C_1}(x_3)^- \cup N_{C_1}(x_4)^{-2} \subset V(C_1) \cup \{v_1, v_1^-\}$ and by Claim 2.1 (iii), $N_{C_1}(x_1)^{-3}$, $N_{C_1}(x_2)$, $N_{C_1}(x_3)^-$ and $N_{C_1}(x_4)^{-2}$ are pairwise disjoint. Hence we obtain $\sum_{j=1}^k d_{C_1}(x_j) \le |V(C_1)| + 2 = |V(C_1)| + k - 2$.

For the case $k = 3$ and $k = 2$, we can show the inequality similarly. □

By Claim 2.1 (i) and (ii), we have $d_H(x_i) = 0$ and $N_{G-C-H}(x_i) \cap N_{G-C-H}(x_j) = \emptyset$. Hence $\sum_{j=1}^k d_{G-C-H}(x_j) \le |V(G - C - H)|$. Then by Claim 2.2,

$$\begin{aligned} \sum_{j=0}^{k} d_G(x_j) &\le \sum_{i=1}^{k} (|V(C_i)| + k - 2) + |V(G - C - H)| + |V(H)| - 1 \\ &= |V(C)| + k(k-2) + |V(G - C)| - 1 \\ &= n + k(k-2) - 1. \end{aligned}$$

This contradicts $\sigma_{k+1}(G) \ge n + k(k-2)$. □

3. Preliminary lemma

In this section, we show a lemma for the proof of Theorem 7. Hereafter, we use the term “dominating” instead of “vertex dominating” for convenience.

The following theorems are used in our proof. A graph G is called *hamilton-connected* if for any two distinct vertices, G has a hamilton path connecting them.

Theorem 8. [3] *Let G be a 2-connected graph on n vertices with $n \ge 4$ and $u, v, w \in V(G)$. If $d_G(x) \ge d$ for every $x \in V(G) - \{u, v, w\}$, then there exists a uv-path P such that $|V(P)| \ge d + 1$.*

Theorem 9. [5] *Let G be a graph on n vertices. If $\sigma_2(G) \ge n + 1$, then G is hamilton-connected.*

Especially, we will use a minimum degree version.

Theorem 10. *Let G be a graph on n vertices. If $\delta(G) \geq \frac{1}{2}(n+1)$, then G is hamilton-connected.*

We define G_3 to be a graph as follows. Let w, u, v be three vertices of a triangle. We join $\{w, u\}$ and $K_m \cup K_{m-1}$. Moreover, we join a new vertex x and K_{m-1}. (See Figure 3.)

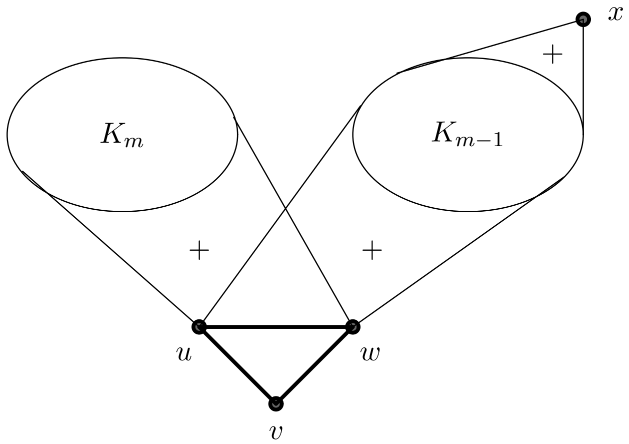

Figure 3: The graph G_3

We show the following lemma.

Lemma 11. *Let G be a 2-connected graph on n vertices with $n \geq 27$ and let $u, v \in V(G)$ such that $G - \{u, v\}$ is connected. If $d_G(x) \geq \frac{1}{2}(n-5)$ for every $x \in V(G) - \{u, v\}$, then every longest uv-path is dominating or G is a spanning subgraph of G_3.*

Proof of Lemma 11. Suppose that there exists a longest uv-path P which is not dominating. Then, there exists $x \in V(G-P)$ such that $N_P(x) = \emptyset$. We give an orientation to P from u to v and let H be a component of $G-P$ with $x \in V(H)$.

By Theorem 8, we have

$$|V(P)| \geq \frac{1}{2}(n-5) + 1 = \frac{1}{2}(n-3). \tag{1}$$

Since $N_G(x) \subset V(H) - \{x\}$,

$$|V(H)| \geq d_G(x) + 1 \geq \frac{1}{2}(n-3).$$

Therefore,

$$|V(P)| \leq n - |V(H)| \leq n - \frac{1}{2}(n-3) = \frac{1}{2}(n+3). \tag{2}$$

Let $|V(P)| = \frac{1}{2}n + t$. By (1) and (2), we have $-\frac{3}{2} \leq t \leq \frac{3}{2}$.

Claim 3.1. $G - P - H = \emptyset$.

Proof. Suppose that there exists $x' \in V(G-P-H)$. Note that $|V(G-P-H)| \leq n - (\frac{1}{2}n + t) - \frac{1}{2}(n-3) = \frac{3}{2} - t$. Then,

$$\begin{aligned} d_P(x') &= d_G(x') - d_{G-P-H}(x') \\ &\geq \frac{1}{2}(n-5) - \left(\frac{3}{2} - t - 1\right) = \frac{1}{2}n - 3 + t. \end{aligned}$$

On the other hand, since $N_P(x') \cap N_P(x')^+ = \emptyset$, we obtain

$$d_P(x') \leq \frac{1}{2}(|V(P)|+1) = \frac{1}{4}n + \frac{1}{2} + \frac{1}{2}t.$$

Then $n \leq -2t + 14 \leq 17$, contradicting $n \geq 27$. □

By Claim 3.1, we have $|V(H)| = n - |V(P)| = \frac{1}{2}n - t$.

Claim 3.2. $|N_P(H)| \leq t + 3$.

Proof. Suppose that $|N_P(H)| \geq t+4$. Let $y_1, y_2, \cdots, y_{t+4} \in N_P(H)$. We may assume that $y_1, y_2, \cdots, y_{t+4}$ appear in this order along $\overrightarrow{P}$. Since P is a longest uv-path, $N_G(y_1^+) \cap (V(H) \cup \{y_2^+, \cdots, y_{t+3}^+\}) = \emptyset$. Then, $d_G(y_1^+) \leq |V(P)| - (t+2) - 1 = \frac{1}{2}n - 3$, but this is a contradiction. □

Claim 3.3. *H is hamilton-connected.*

Proof. By Claim 3.2, we have

$$\begin{aligned} \delta(H) &\geq \delta(G) - |N_P(H)| \\ &\geq \frac{1}{2}(n-5) - (t+3) = \frac{n}{2} - t - \frac{11}{2} \\ &\geq \frac{1}{2}\left(\frac{1}{2}n - t + 1\right) = \frac{1}{2}(|V(H)|+1), \end{aligned}$$

since $n \geq 27$ and $t \leq \frac{3}{2}$. Then by Theorem 10, H is hamilton-connected. □

Since G is 2-connected, there exist $y_1, y_2 \in N_P(H)$ such that $|N_H(\{y_1, y_2\})| \geq 2$. We choose y_1, y_2 so that $|V(y_1 \overrightarrow{P} y_2)|$ is as small as possible. By Claim 3.3, $|V(y_1^+ \overrightarrow{P} y_2^-)| \geq |V(H)| = \frac{1}{2}n - t$. Hence we have

$$|V(P - y_1^+ \overrightarrow{P} y_2^-)| \leq \left(\frac{1}{2}n + t\right) - \left(\frac{1}{2}n - t\right) = 2t \leq 3. \tag{3}$$

If $|N_P(H)| \geq 3$, or if $y_1 \neq u$ and $y_2 \neq v$, then we can easily find four or more vertices in $P - y_1^+ \overrightarrow{P} y_2^-$, a contradiction. Thus, $|N_P(H)| = 2$ and $y_1 = u$ or $y_2 = v$. By the symmetry, we may assume that $y_1 = u$.

Suppose that $y_2 = v$. Since $|N_P(H)| = 2$, $G - \{u, v\}$ has two components, that is, one is H and the other is $P - \{u, v\}$, contradicting the assumption. Therefore, we have $y_2 = v^-$ by (3).

Suppose that $N_P(v) - \{u, y_2\} \neq \emptyset$. Let $a \in N_P(v) - \{u, y_2\}$ and let Q be a hamilton uy_2-path in $G[V(H) \cup \{u, y_2\}]$. Then $P' := u \overrightarrow{Q} y_2 \overleftarrow{P} av$ is a uv-path and $|V(P')| = |V(Q)| + |V(y_2 \overleftarrow{P} a) - \{y_2\}| + |\{v\}| \geq \frac{1}{2}n - t + 4 > \frac{1}{2}n + t = |V(P)|$, a contradiction. Therefore, $N_G(v) = \{u, v^-\}$ and G is a spanning subgraph of G_3 because w, $V(K_m)$ and $V(K_{m-1})$ on G_3 correspond with v^-, $V(P) - \{u, v^-, w\}$ and $V(H) - \{x\}$, respectively. □

4. Proof of Theorem 7

The following theorem is used in our proof of Theorem 7.

Theorem 12. [4] *Suppose that G is a 2-connected graph. If $\delta(G) \geq d$, then there exists a cycle with length at least* $\min\{2d, |V(G)|\}$.

Proof of Theorem 7. Suppose that G is 3-connected. Since $n \geq 65$, we have

$$\sigma_4(G) \geq 4\delta(G) \geq \frac{4}{3}(n-4) > n+3.$$

Then by Theorem 4, every longest cycle is dominating.

Thus, we may assume $\kappa(G) = 2$. Let $S := \{u, v\} \subset V(G)$ be a 2-cut set of G.

Claim 4.1. $\omega(G - S) = 2$.

Proof. Suppose that $\omega(G-S) \geq 3$ and let $D_1, D_2, \cdots, D_m$ $(m \geq 3)$ be components of $G - S$ such that $|V(D_1)| \leq |V(D_2)| \leq \cdots \leq |V(D_m)|$. For each $1 \leq i \leq m$ and $x_i \in V(D_i)$, we have $N_G(x_i) \subset \big(V(D_i) - \{x_i\}\big) \cup S$ and hence $|V(D_i)| \geq d_G(x_i) - |S| + 1 \geq \frac{1}{3}(n-7)$. Suppose that $m \geq 4$. Then by the choice of D_1, $n = |V(G)| \geq m|V(D_1)| + |S| \geq \frac{4}{3}(n-7) + 2 \geq \frac{4n}{3} - \frac{22}{3}$. This implies that $n \leq 22$, a contradiction. Thus, we have $m = 3$. Then for $1 \leq i \leq 3$, $|V(D_i)| = n - |S| - \big(\sum_{j \neq i} |V(D_j)|\big) \leq \frac{1}{3}(n+8)$ and

$$\delta(D_i) \geq \delta(G) - |S| \geq \frac{1}{3}(n-10) > \frac{1}{6}(n+11) \geq \frac{1}{2}(|V(D_i)| + 1).$$

Hence by Theorem 10, each D_i is hamilton-connected for $1 \leq i \leq 3$.

If $|V(D_1)| \geq \frac{1}{3}(n-1)$, then $n \geq 2 + \sum_{i=1}^{3} |V(D_i)| \geq n+1$, a contradiction. Moreover, if there exists $x_1 \in V(D_1)$ such that $N_S(x_1) = \emptyset$, then $|V(D_1)| \geq d_G(x_1) + 1 \geq \frac{1}{3}(n-1)$.

Hence $|V(D_1)| < \frac{1}{3}(n-1)$ and $N_S(x) \neq \emptyset$ for every $x \in V(D_1)$. Let D_i be a minimum component of $G-S$. Then $|V(D_i)| = |V(D_1)|$. By the same way as above, we obtain that for every $x \in V(D_i)$, $N_S(x) \neq \emptyset$.

Since D_i is hamilton-connected for every $1 \leq i \leq 3$, each longest cycle passes all vertices in G except for the minimum component of $G-S$. Moreover any vertex in such minimum component has a neighbor in S. Hence every longest cycle is dominating. □

By Claim 4.1, we have $\omega(G-S) = 2$ for every 2-cut set S. Let D_1 and D_2 be components of $G-S$ with $|V(D_1)| \leq |V(D_2)|$. By the assumption,

$$\frac{1}{3}(n-7) \leq |V(D_1)| \leq \frac{1}{2}(n-|S|) = \frac{1}{2}(n-2). \tag{4}$$

Then,

$$\delta(D_1) \geq \delta(G) - |S| \geq \frac{1}{3}(n-10) \geq \frac{n}{4} \geq \frac{1}{2}(|V(D_1)|+1).$$

Hence by Theorem 10, D_1 is hamilton-connected.

Let C be a longest cycle in G. By Theorem 12, we may assume that $|V(C)| \geq 2\delta(G) \geq \frac{2}{3}(n-4)$. Therefore, we have $V(C) \cap V(D_2) \neq \emptyset$. We consider the following three cases.

Case 1. $V(C) \cap S = \emptyset$.

Since G is 2-connected, there are two disjoint paths such that one joins u and w_1 and the other joins v and w_2, where $w_1, w_2 \in V(C)$ and no vertices are contained in C except for w_1 and w_2. By (4), $|V(w_1^+ \overrightarrow{C}\, w_2^-)|$, $|V(w_2^+ \overrightarrow{C}\, w_1^-)| \geq |V(D_1) \cup S| \geq \frac{1}{3}(n-7)+2 = \frac{1}{3}(n-1)$. Hence $n \geq |V(D_1)| + |S| + |V(w_1^+ \overrightarrow{C}\, w_2^-)| + |V(w_2^+ \overrightarrow{C}\, w_1^-)| + |\{w_1, w_2\}| \geq \frac{1}{3}(n-7) + 2 + 2 \cdot \frac{1}{3}(n-1) + 2 = n+1$, a contradiction.

Case 2. $|V(C) \cap S| = 1$.

Without loss of generality, we may assume that $u \in V(C)$ and $v \notin V(C)$. In this case, note that $V(C) \subset V(D_2) \cup \{u\}$. Therefore $|V(D_2 - C)| \leq \frac{2}{3}(n+1) - \frac{2}{3}(n-4) + 1 \leq \frac{13}{3}$. Since $\delta(G) \geq \frac{1}{3}(n-4)$ and $n \geq 65$, for every $x \in V(D_2 - C)$, $|N_C(x)| \geq |N_G(x)| - |V(D_2 - C)| - |\{v\}| \geq \frac{1}{3}(n-4) - \frac{13}{3} - 1 \geq 15$, that is, $N_C(x) \neq \emptyset$. Hence C dominates $V(D_2)$. Suppose that $N_C(v) = \emptyset$. Since G is 2-connected, there is a vw-path P with $u \notin V(P)$, where $w \in V(C)$. Since $N_C(v) = \emptyset$, $|V(P)| \geq 3$. Hence $|V(u^+ \overrightarrow{C}\, w^-)|$, $|V(w^+ \overrightarrow{C}\, u^-)| \geq |V(D_1) \cup V(P) - \{w\}| \geq \frac{1}{3}(n-1)$. Then $n \geq |V(D_1)| + |V(P)| + |V(u^+ \overrightarrow{C}\, w^-)| + |V(w^+ \overrightarrow{C}\, u^-)| + |\{u\}| \geq \frac{1}{3}(n-7) + 3 + 2 \cdot \frac{1}{3}(n-1) + 1 = n+1$, a contradiction. Therefore, $N_C(v) \neq \emptyset$ and let $w' \in N_C(v)$.

If $xu \in E(G)$ hold for every $x \in V(D_1)$, then C is a dominating cycle. So we may assume that there exists $x_1 \in V(D_1)$ such that $x_1 u \notin E(G)$. By the assumption,

$|V(D_1)| \geq d_G(x_1) - 1 + 1 \geq \frac{1}{3}(n-4)$. Since C is longest, $|V(u^+ \overrightarrow{C} w'^-)|$, $|V(w'^+ \overrightarrow{C} u^-)| \geq |V(D_1) \cup \{v\}| \geq \frac{1}{3}(n-1)$. This implies that $n = |V(D_1)| + |\{v\}| + |V(u^+ \overrightarrow{C} w'^-)| + |V(w'^+ \overrightarrow{C} u^-)| + |\{u, w'\}| \geq \frac{1}{3}(n-4) + 1 + 2 \cdot \frac{1}{3}(n-1) + 2 = n+1$, a contradiction.

Case 3. $u, v \in V(C)$.

Suppose that $V(C) \cap V(D_1) = \emptyset$. If there exists $x_2 \in V(D_2)$ such that $N_C(x_2) = \emptyset$, then $|V(D_2)| \geq |V(C) - S| + |N_G(x_2)| + |\{x_2\}| \geq \frac{2}{3}(n-4) - 2 + \frac{1}{3}(n-4) + 1 = n-5$, and hence $n = |S| + |V(D_1)| + |V(D_2)| \geq 2 + \frac{1}{3}(n-7) + n - 5 = \frac{4}{3}(n-16)$, contradicting $n \geq 65$. Thus, C is a dominating cycle in $G[V(D_2) \cup S]$. Moreover, if $N_S(x) \neq \emptyset$ for every $x \in V(D_1)$, then C is a dominating cycle. Therefore, we may assume that there exists $x_1 \in V(D_1)$ such that $N_S(x_1) = \emptyset$. By the degree condition, $|V(D_1)| \geq d_G(x_1) + 1 \geq \frac{1}{3}(n-1)$ and hence $n \geq |V(D_1)| + |S| + |V(u^+ \overrightarrow{C} v^-)| + |V(v^+ \overrightarrow{C} u^-)| \geq \frac{1}{3}(n-1) + 2 + 2 \cdot \frac{1}{3}(n-1) = n+1$, a contradiction.

Then we may assume that $V(C) \cap V(D_1) \neq \emptyset$. Since D_1 is hamilton-connected, $V(D_1) \subset V(C)$. Let $H := G[V(D_2) \cup S]$. Clearly $H - S$ is connected. On the other hand, by (4), $|V(H)| = n - |V(D_1)| \leq \frac{1}{3}(2n+7)$. For every $w \in V(D_2)$, $d_H(w) = d_G(w) \geq \frac{1}{3}(n-4) = \frac{1}{2}\{\frac{1}{3}(2n+7) - 5\} \geq \frac{1}{2}(|V(H)| - 5)$. Moreover, by (4) and $n \geq 65$, $|V(H)| \geq n - \frac{1}{2}(n-2) = \frac{1}{2}n + 1 \geq 27$. Therefore, by Lemma 11, every longest uv-path in H is dominating or H is isomorphic to a spanning subgraph of G_3. But if H is isomorphic to a spanning subgraph of G_3, then G has a 2-cut set S with $\omega(G-S) = 3$, contradicting Claim 4.1. Thus, $C \cap H$ is a dominating path and then, C is a dominating cycle. □

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