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Interpolation theorem on degree-factors with and without specified edges

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Abstract

A spanning subgraph F in a graph G is called an r-factor of G if each vertex has degree r in F. Let G be a graph with two disjoint sets W_1 and W_2 of edges, and let $a < c < b$ be integers, all even or all odd. We shall show that if G has both an a-factor and a b-factor containing W_1 and avoiding W_2, then G also has such a c-factor.

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1. Introduction

In this paper, we consider finite undirected graphs. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G respectively. By $\deg_G(x)$, we denote the degree of $x \in V(G)$ in G. For a function $f:\ V(G) \to \{0, 1, 2, 3 \cdots\}$, a spanning subgraph F of G with $\deg_F(x) = f(x)$ for all $x \in V(G)$ is called an f*-factor* of G. If $f(x) = r$ holds for all $x \in V(G)$, then an f-factor is called an r*-regular factor*, or simply an r*-factor*.

The theory of these factors has been already well-developed in graph theory and various types of theorems on factors have been established in many papers. For example, Katerinis has shown the following theorem in [2]:

Theorem 1. [2] *Let G be a connected graph of even order. Let a, b and c be odd integers such that $1 \leq a < c < b$. If G has both an a-factor and a b-factor, then G has a c-factor.*

In [1], Kano and Yu have extended this theorem with additional conditions on containment of a specified edge.

Theorem 2. [1] *Let G be a connected graph of even order with an edge e, and let a, b and c be odd integers such that $1 \leq a < c < b$. If G has both an a-factor and a b-factor containing e, then G has a c-factor containing e. Similarly, if G has both an a-factor and a b-factor avoiding e, then G has a c-factor avoiding e.*

In this paper, we shall show the following theorem, extending Theorem 2 to a more general form:

Theorem 3. *Let G be a graph with two disjoint sets W_1, $W_2 \subset E(G)$ and let a, b and c be all odd (or all even) integers such that $1 \leq a < c < b$. If G has both an a-factor and a b-factor containing all edges in W_1 and avoiding all edges in W_2, then G has a c-factor containing all edges in W_1 and avoiding all edges in W_2.*

It should be noticed that we cannot omit the condition on the parities of a, b and c in general; they are all odd or all even. We can construct easily an example to show this, as follows.

Let $r \geq 5$ be an odd integer. Prepare two copies of the complete graph K_r with r vertices and join each corresponding pair of vertices in these two with an edge. The edges between two K_r form a perfect matching in the resulting graph G_r, say W_1. Put $a = 1$ and $b = r$, and set $W_2 = \emptyset$. Then G_r has a 1-factor and an r-factor containing W_1 since K_r is $(r-1)$-regular and it must have a c-factor containing W_1 for any odd integer c with $1 = a < c < b = r$, by Theorem 3. However, G_r has no c-factor containing W_1 for any even integer c. For, if G_r has a c-factor F containing W_1 for some even integer c, then $E(F) - W_1$ would induce a $(c-1)$-regular spanning subgraph in each of two K_r's. This is a contradiction since K_r has a odd number of vertices and $c - 1$ is odd.

We shall prove our main theorem, introducing what is called "Tutte's f-Factor Theorem" in Section 2, and show some observations for further studies in Section 3.

2. Proof of Theorem

This section is devoted to prove our main theorem, Theorem 3. However, we shall discuss it under more general situation, as follows.

Let G be a graph and $f : V(G) \to \{0, 1, 2, \cdots\}$ be an integer-valued function. Let S and T be two disjoint subsets of $V(G)$. We define two quantities $h_G(S, T; f)$ and

$\delta_G(S,T;f)$ for S, T and f by:

$$\begin{aligned} h_G(S,T;f) &:= \text{the number of components } C \text{ of } G \setminus (S \cup T) \\ & \quad\ \text{such that } \sum_{x \in V(C)} f(x) + e_G(V(C),T) \equiv 1 \pmod 2 \\ \delta_G(S,T;f) &:= \sum_{x \in S} f(x) + \sum_{x \in T} (\deg_{G \setminus S}(x) - f(x)) - h_G(S,T;f) \end{aligned}$$

where $e_G(X,Y)$ stands for the number of edges between X, $Y \subset V(G)$ in G.

Tutte [4] has characterized those graphs that have f-factors, using these quantities and proved the following theorem, which is often refered as "Tutte's f-Factor Theorem":

Theorem 4. [4] *Let G be a graph and $f : V(G) \to \{0,1,2,\cdots\}$ be an integer-valued function. Then:*

(1) *G has an f-factor if and only if $\delta_G(S,T;f) \geq 0$ for all $S,T \subseteq V(G)$ with $S \cap T = \emptyset$*

(2) *$\delta_G(S,T;f) \equiv \sum_{x \in V(G)} f(x) \pmod 2$ for all $S,T \subseteq V(G)$ with $S \cap T = \emptyset$*

Applying this theorem, we shall prove a kind of an interpolation theorem for f-factors, which is more general, but is slightly technical:

Theorem 5. *Let G be a graph and let f_1, f_2 and $f : V(G) \to \{0,1,2,\cdots\}$ be three functions satisfying one of the following conditions with positive even constants α_1, α_2 for any vertex $x \in V(G)$;*

(i) *$f_1(x) = f(x) = f_2(x)$, or*

(ii) *$f_1(x) + \alpha_1 = f(x) = f_2(x) - \alpha_2$.*

If G has an f_1-factor and an f_2-factor, then G has an f-factor.

Proof of Theorem 5 Let Y be a set of vertices of G satisfying (i) and $X := V(G) \backslash Y$. Then each vertex in X satisfies (ii). Suppose that G has no f-factor. By Theorem 4, the following inequality holds for some two disjoint subsets S and T in $V(G)$:

$$\delta_G(S,T;f) = \sum_{x \in S} f(x) + \sum_{x \in T} (\deg_{G \setminus S}(x) - f(x)) - h_G(S,T;f) < 0$$

Since G has an f_i-factor ($i = 1,2$), we have the following inequality for these S and T:

$$\delta_G(S,T;f_i) = \sum_{x \in S} f_i(x) + \sum_{x \in T} (\deg_{G \setminus S}(x) - f_i(x)) - h_G(S,T;f_i) \geq 0$$

Since $\alpha_i \equiv 0 \pmod 2$, we have the following for any subset $U \subseteq V(G)$:

$$\sum_{x \in U} f_i(x) \equiv \sum_{x \in U} f(x) \pmod 2$$

This implies that $h_G(S,T;f_i) = h_G(S,T;f)$.

From this and the fact that $f_i(x) = f(x)$ holds for $x \in Y$, we can evaluate the difference in $\delta_G(S,T;f_i)$ and $\delta_G(S,T;f)$:

$$\begin{aligned}\delta_G(S,T;f) - \delta_G(S,T;f_1) &= \sum_{x \in S \cap X} (f(x) - f_1(x)) - \sum_{x \in T \cap X} (f(x) - f_1(x)) \\ &= \alpha_1(|S \cap X| - |T \cap X|) < 0\end{aligned}$$

$$\begin{aligned}\delta_G(S,T;f) - \delta_G(S,T;f_2) &= \sum_{x \in S \cap X} (f(x) - f_2(x)) - \sum_{x \in T \cap X} (f(x) - f_2(x)) \\ &= -\alpha_2(|S \cap X| - |T \cap X|) < 0\end{aligned}$$

The first inequality implies $|S \cap X| - |T \cap X| < 0$ (because $\alpha_1 \geq 0$). However, then we have $-\alpha_2(|S \cap X| - |T \cap X|) \geq 0$. This contradicts the second inequality. □

Our proof of Theorem 3 is just an easy application of the above theorem at all:

Proof of Theorem 3 Let G^* be the graph obtained from G by subdividing each edge in $W_1 \cup W_2$ with one vertex. Put $V_i \subset V(G^*)$ be the set of vertices of degree 2 added to edges in W_i for $i = 1, 2$. Define functions $g_\alpha : V(G) \to \{0, 1, 2, \cdots\}$ as follows:

$$g_\alpha(x) = \begin{cases} \alpha & (x \in V(G)) \\ 2 & (x \in V_1) \\ 0 & (x \in V_2) \end{cases}$$

Then we can use g_a, g_b and g_c for G^* as f_1, f_2 and f for G in Theorem 5 with $\alpha_1 = c - a$ and $\alpha_2 = b - c$. Thus, G^* has a g_c-factor, which derives naturally a c-factor containing edges in W_1 and avoiding edges in W_2. □

3. For further studies

In fact, Kano and Yu also have shown the following theorem on factors in regular graphs as an application of Theorem 2:

Theorem 6. [1] *Let G be a connected r-regular graph of even order. If G has a 1-factor containing $e \in E(G)$, then G has a c-factor containing $e \in E(G)$ and another c-factor avoiding e for all integers c such that $1 \leq c \leq r - 1$.*

Corresponding to this, we would like to establish a similar theorem as an application of our theorem. So we ask naturally:

Question. Let G be a connected r-regular graph of even order and W any independent set of edges contained in a 1-factor. Does G have a c-factor containing W and another c-factor avoiding W for all integers c such that $1 \leq c \leq r-1$?

However, the answer to this question is "No" in general. For example, we have already shown a negative example to this question in introduction; that can be obtained from two disjoint copies of K_r by adding a perfect matching between them and the perfect matching W_1 works as W in the question. Furthermore, we can construct such an example to give a negative answer to this question so that the relative size of W is enough small, as follows.

Let $k = 2h+1 \geq 3$ be an odd integer and m a positive integer with $k-1 \leq m \leq 2k-1$. It is easy to construct a connected m-regular graph A with $|V(A)| = 2k$ so that it contains a perfect matching $F = \{a_1b_1, \ldots, a_kb_k\}$; for example, consider the well-known $(2k-1)$-edge-coloring of K_{2k} such that each set of edges with one color forms a perfect matching, and delete the edges with some colors to obtain an m-regular graph. Prepare two copies of this A, say A_1 and A_2, each of which has a perfect matching $F_i = \{a_{i1}b_{i1}, \ldots, a_{ik}b_{ik}\}$ for $i = 1, 2$, and two copies of $K_{2\ell+1}$ with $\ell = m-k+1$, say B_1 and B_2. Put $S_i = \{a_{i1}, \ldots, a_{i\,h+1}, b_{i1}, \ldots, b_{ih}\}$ and $T_i = \{a_{i\,h+2}, \ldots, a_{i\,k}, b_{i\,h+1}, \ldots, b_{ik}\}$ and decompose $V(B_i)$ into two disjoint subsets V_i and U_i with $|V_i| = \ell$ and $|U_i| = \ell+1$. Consider the graph obtained from $A_1 \cup A_2 \cup B_1 \cup B_2$ by adding extra edges $W = \{a_{11}a_{21}, \ldots, a_{1\,h+1}a_{2\,h+1}, b_{11}b_{21}, \ldots, b_{1h}b_{2h}\}$ and by joining all pairs between S_i and V_i and those between T_i and U_i, and denote it by $G_{k,m}$.

Every vertex in S_i is incident to m edges in A_i and to one in W and is adjacent to ℓ vertices in V_i, while every vertex in T_i is incident to m edges in A_i and is adjacent to $\ell+1$ vertices in U_i. Thus, every vertex in $S_i \cup T_i$ has degree $m+\ell+1$ in $G_{k,m}$. On the other hand, every vertex in V_i or in U_i has 2ℓ neighbors in B_i and is adjacent to k vertices in S_i or in T_i, and hence its degree is equal to $2\ell+k = m+\ell+1$, which is an odd integer. Thus, $G_{k,m}$ is $(m+\ell+1)$-regular and is connected. Furthermore, the independent set W of k edges is contained in a 1-factor having edges in F_i which cover $T_i - \{b_{i\,h+1}\}$, a maximum matching in B_i and one edge between $b_{i\,h+1}$ and the one vertex in B_i not covered by the matching for $i = 1, 2$.

Therefore, $G_{k,m}$ and W satisfy the assumption of the above question. However, $G_{k,m}$ does not have any c-factor containing W for any even number c. For, if $G_{k,m}$ did, then each part induced by $A_i \cup B_i$ would contain a subgraph which has an odd number of vertices of odd degree.

To establish a theorem presenting something positive, we should add a stronger assumption to the above. As such an example, we consider the "k-extendability" of graphs and propose the following conjecture. A graph G is said to be k*-extendable* if any independent set of k edges is contained in a perfect matching. (See [3] for detailed.)

Conjecture 1. *Let G be a k-extendable r-regular graph with $|V(G)| \geq 2k+2$. Then* *for any independent set* W *of* k *edges,* G *has a* c*-factor containing* W *and another* c*-factor avoiding* W *for all integers* c *such that* $1 \leq c \leq r-1$.

Note that $G_{k,m}$ does not work as a counterexample to the conjecture since it is not k-extendable. For example, F_1 cannot extend to any perfect matching of $G_{k,m}$; if it could, then B_1 would have a perfect matching since F_1 covers the whole of A_1, but B_1 has an odd number of vertices. On the other hand, the graph G_r given in introduction works as a counterexample to the above with $|V(G)| = 2k$ since k-extendability is trivial in this case. So we need a lower bound for $|V(G)|$ at least.

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