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Hamilton Cycles in Cubic Graphs

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Abstract

A graph is *cubic* if each of its vertex is of degree 3 and it is *hamiltonian* if it contains a cycle passing through all its vertices. It is known that if a cubic graph is hamiltonian, then it has at least three Hamilton cycles. This paper is about those works done concerning the number of Hamilton cycles in cubic graphs and related problems.

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A graph is called *hamiltonian* if it contains a cycle passing through all its vertices. Such a cycle is called a *Hamilton cycle*. A graph is *cubic* if each of its vertex is of degree 3. Cubic graphs have been much studied in graph theory and they seem to be among the most desirable regular graphs.

Historically, cubic hamiltonian graphs have been associated with the Four-Color Theorem. In 1880, Tait [14] has shown that one can prove the Four-Color Theorem by showing that every simple 2-edge connected cubic planar graph is 3-edge colorable. If a cubic graph G has a Hamilton cycle C, then C has an even number of vertices so that its edges can be colored alternatively with two colors. The remaining edges of G can then be colored with a different color which means that G is 3-edge colorable. Thus if it is true that all bridgeless cubic planar graphs are hamiltonian, then the Four-Color Theorem is established. This has led Tait to conjecture that every cubic 3-connected planar graph is hamiltonian.

Although Tait's conjecture turned out to be false (as was pointed out by Tutte [16] in 1946), it has nevertheless stirred up people's interest on the hamiltonicity of cubic graphs. A more detailed account on cubic hamiltonian graphs will be presented in a forthcoming paper [7]. In this paper, we confine our attention on the problem concerning the number of Hamilton cycles in cubic graphs and also those problems related to it.

The earliest result on the number of Hamilton cyces seems to be the following one which appeared in [16].

Theorem 1. [16] *Every edge of a cubic graph lies on an even number of Hamilton cycles. Consequently a cubic hamiltonian graph has at least three Hamilton cycles.*

Some questions naturally arise. (Q1) Does there exist cubic graph having only three Hamilton cycles? (Q2) If so, can such cubic graphs be characterized?

Let G be a graph with chromatic index k. Then G is said to be *uniquely k-edge colorable* if there is only one way of partitioning its edge-set into color-classes.

It is easy to see that if a cubic graph G is uniquely 3-edge colorable, then the union of any two color classes forms a Hamilton cycle. This means that G has precisely three Hamilton cycles.

Theorem 2. [10] *Let G be a uniquely 3-edge-colorable cubic graph. Then G has precisely three Hamilton cycles.*

The converse of Theorem 2 was conjectured to be true by Greenwell and Kronk [10]. However, in [15], Thomason constructed an infinite family of cubic graphs which have only three Hamilton cycles but they are not uniquely 3-edge-colorable. The general form of this family of graphs is given below.

Let n and k be two integers such that $1 \leq k \leq n-1$. The *generalized Petersen graph $GP(n,k)$* is defined to have vertex-set $\{u_i, v_i : i = 0, 1, \ldots, n-1\}$ and edge-set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 0, 1, \ldots, n-1 \text{ with subscripts reduced modulo } n\}$.

In [15], Thomason showed that $GP(n,2)$ has only three Hamilton cycles for each $n \equiv 3 \pmod 6$ and that it is uniquely 3-edge-colorable if and only if $n = 3$ or $n = 9$.

More generally, Schwenk [13] enumerated the number of Hamilton cycles in $GP(n,2)$. Let F_m denote the m-th Fibonacci number.

Theorem 3. [13]

For $n \geq 3$, the number of Hamilton cycles in $GP(n,2)$ is

(i)	$2(F_{\frac{n}{2}+2} - F_{\frac{n}{2}-2} - 1)$	*if* $n \equiv 0,\ 2 \pmod 6$,
(ii)	n	*if* $n \equiv 1 \pmod 6$,
(iii)	3	*if* $n \equiv 3 \pmod 6$,
(iv)	$n + 2(F_{\frac{n}{2}+2} - F_{\frac{n}{2}-2} - 1)$	*if* $n \equiv 4 \pmod 6$,
(v)	0	*if* $n \equiv 5 \pmod 6$.

In [13], Schwenk asked the following question.

Problem 1. [13]

(i) *Can we determine the number of Hamilton cycles in* $GP(n,k)$ *for* $k \geq 3$ *?*

(ii) *Which other cubic graphs are amenable to have their Hamilton cycles enumerated?*

Responding to these, some works have been done in [8] and [4].

While unable to determine the number of Hamilton cycles in $GP(n,3)$, the set of all Hamilton cycles in $GP(n,3)$ have been determined in [8]. Moreover, a lower bound was obtained for the general case $GP(n,k)$.

Theorem 4. [8] *Let* $G = GP(n,k)$ *be a hamiltonian generalized Petersen graph with* $k \geq 3$. *Assume that* G *is neither* $GP(12,4)$ *nor* $GP(2k+2,k)$ *where* $k \geq 3$ *is odd. Then* G *has at least* n *Hamilton cycles.*

Let n and k be two integers such that $k \geq 3$ is odd and $n \geq k$. Let $H(n,k)$ denote the graph with vertex set $\{u_0, u_1, \ldots, u_{2n-1}\}$ and edge set $\{u_i u_{i+1}, u_{2j} u_{2j+k} : i = 0,1,\ldots,2n-1, j = 0,1,\ldots,n-1\}$ with the operations on the subscripts reduced modulo $2n$. Incidentally, note that the graph $H(n,k)$ is isomorphic to the Cayley graph $\Gamma(D_n, S)$ on the dihedral group $D_n =< a,b : a^n = b^2 = 1, bab = a^{-1} >$ with respect to the generating set $S = \{b, a^{\frac{k-1}{2}}b, a^{n-1}b\}$. More details about Cayley graphs may be found in [2]. Note that $H(7,5)$ is the Heawood graph.

The sets of all Hamilton cycles in the graphs $H(n,3)$ and $H(n,5)$ and also for the graphs $H(k,k)$ and $H(k+1,k)$, $k \geq 3$ have been determined in [4]. In particular, the authors determined the numbers of Hamilton cycles in the graphs $H(n,3)$ and $H(k,k)$.

Theorem 5. [4]

(i) *For each* $n \geq 3$, *the graph* $H(n,3)$ *has* $n+2$ *Hamilton cycles if* n *is even, and has* $n+3$ *Hamilton cycles if* n *is odd.*

(ii) *For each* $k \geq 3$, *the graph* $H(k,k)$ *has precisely* $k+3$ *Hamilton cycles.*

We now consider the problem of characterizing cubic graphs with given number of Hamilton cycles.

Let G be a connected graph. Then G is called *(2,3)-regular* if the degree of each vertex is either 2 or 3. Let G be a $(2,3)$-regular graph. For any two vertices x and y of degree 2 in G, let $p(x,y)$ denote the number of Hamilton paths in G with x and y as end vertices. The sequence of all such numbers of paths is called the *Hamilton path sequence* of G or just *path sequence* of G for brevity.

Let X and Y be two (2,3)-regular graphs whose set of vertices of degree 2 are $A = \{x_1, x_2, \ldots, x_k\}$ and $B = \{y_1, y_2, \ldots, y_k\}$ respectively. By a *merger* of X and Y, we mean a graph obtained by joining the vertices x_j and y_j, $j = 1,2,\ldots,k$. We shall extend the definition to the case when X is a (2,3)-regular graph with only three vertices of degree 2, and Y is an isolated vertex. In this case, the *merger* of X and Y is the graph obtained by joining Y to all the three vertices of degree 2 in X. In any case, a merger will result in a cubic graph.

Note that if G is a cubic graph, then deleting any vertex x from G together with all edges incident to it will yield a (2,3)-regular graph with three vertices of degree 2.

Suppose that $k = 3$. Assume that $p(x_1, x_2) = m_3$, $p(x_2, x_3) = m_1$ and $p(x_3, x_1) = m_2$, and $p(y_1, y_2) = n_3$, $p(y_2, y_3) = n_1$ and $p(y_3, y_1) = n_2$. If G is a merger of X and Y, then the number of Hamilton cycles in G is given by

$$m_1n_1 + m_2n_2 + m_3n_3$$

In the event that Y is an isolated vertex, we may take $n_1 = 1 = n_2 = n_3$ so that the number of Hamilton cycles in G is $m_1 + m_2 + m_3$.

A graph is called a *tup* if it is either an isolated vertex or else a (2,3)-regular graph with path sequence $\{1, 1, 1\}$. An isolated vertex is called *trivial tup*. We can easily obtain a tup from a cubic graph with precisely three Hamilton cycles simply by deleting a vertex together with all edges incident to it.

Theorem 6. [8] *Let G be a cubic hamiltonian graph. If G is of edge-connectivity 2, then G has $4n$ Hamilton cycles and hence has at least four Hamilton cycles. Equality holds if and only if G is a merger of two $(2, 3)$-regular graphs each with path sequence $\{2\}$.*

Figure 1 shows a cubic graph with precisely 4 Hamilton cycles. It is the merger two $(2, 3)$-regular graphs (each is obtained by deleting an edge from the complete graph on 4 vertices).

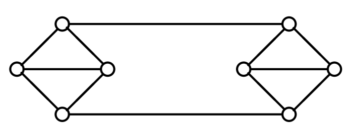


Figure 1: Cubic graph with 4 Hamilton cycles

Theorem 7. [8] *Let G be cubic hamiltonian graph. Then G has precisely three Hamilton cycles if and only if G is a merger of two tups. Moreover, if G has more than two vertices, then G is uniquely 3-edge-colorable if and only if G is either the merger of a trivial tup and a non-trivial tup that is uniquely 3-edge-colorable, or else a merger of two non-trivial tups that are uniquely 3-edge-colorable.*

We can use Theorem 7 to construct infinitely many non-planar uniquely 3-edge-colorable cubic graphs. To see this, take two non-planar uniquely 3-edge-colorable cubic graphs (for example, two copies of the generalized Petersen graph $G(9,2)$) and delete a vertex from each graph (together with all edges incident to it). Each resulting graph is a $(2,3)$-regular graph which is uniquely 3-edge-colorable (by Lemma 3.1 of [8]). Take a merger of these graphs. The result is a non-planar cubic graph which is uniquely 3-edge-colorable. The construction can be repeated using $GP(9,2)$ and the resulting graph. This shows that Conjecture B of [9] (which states that the only non-planar uniquely 3-edge-colorable cubic graph is the graph $GP(9,2)$) is not true.

Problem 2. [8] *When is a sequence $\{m_1, m_2, m_3\}$ the path sequence of a $(2,3)$-regular graph? Is there a construction for $(2,3)$-regular graph with a given path sequence?*

The following result is relevant to the above question.

Theorem 8. [8] *Let H be a connected $(2,3)$-regular graph with path sequence $\{m_1, m_2, m_3\}$. Then all the m_i 's are of the same parity.*

Some constructions for $(2,3)$-regular graph with a given path sequence have been discussed in [8] but not a great deal has been done.

Before we turn our attention to cubic bipartite graphs, let's take note of the following conjecture. Although the converse of Theorem 2 is not true, the following conjecture of Cantoni (mentioned in [17]) remains unsettled (since the generalized Petersen graph $GP(n,2)$ is non-planar if $n \geq 5$ is odd).

Conjecture 9. *Let G be a planar cubic graph. If G has precisely three Hamilton cycles, then G is uniquely 3-edge-colorable.*

The most general result concerning the number of Hamilton cycles in cubic bipartite graphs is the following result of Bosák [3].

Theorem 10. [3] *Every cubic bipartite graph has an even number of Hamilton cycles.*

Theorems 1 and 10 imply that every cubic bipartite graph have at least 4 Hamilton cycles. It seems to be case that there is no cubic bipartite graph having precisely 4 Hamilton cycles. This could even be the case when restricted only to planar cubic bipartite graphs. However, we are unable to prove this. More generally, the following problem was considered.

Problem 3. [6] *For a given integer n, determine all cubic bipartite planar graphs with precisely n Hamilton cycles.*

The *cyclic connectivity* of a graph G is the least number of edges in G whose removal results in a disconnected graph with two components each containing a cycle. In the event that G is a cubic bipartite planar graph, then the cyclic connectivity of G is no more than 4.

Figure 2 shows a cubic graph (which is called the *cube*) with 6 Hamilton cycles while Figure 3 shows a cubic graph with 12 Hamilton cycles. Note that the graph in Figure 3 has cyclic connectivity 3.

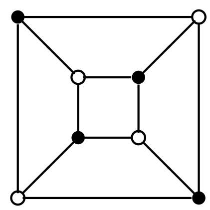


Figure 2: A cubic graph with precisely 6 Hamilton cycles

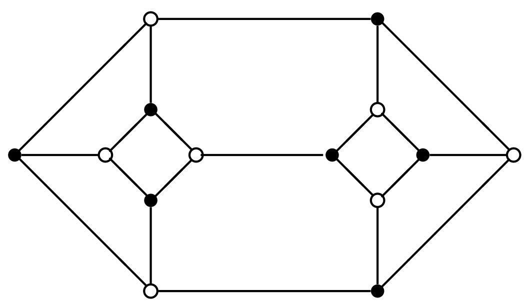


Figure 3: A cubic graph graph with precisely 12 Hamilton cycles

Theorem 11. [6] *Suppose every 3-connected cubic bipartite planar graph is hamiltonian. Let G be a cubic bipartite planar graph.*

(i) *Then G has precisely six Hamilton cycles if and only if G is the cube.*

(ii) *Suppose G has cyclic connectivity 3. Then G has precisely twelve Hamilton cycles if and only if G is the graph in Figure 3.*

Problem 4. *Is it the case that Theorem 11 remains true even if we remove the assumption that every 3-connected cubic bipartite planar graph is hamiltonian?*

Remark 1. *In [1], Barnette conjectured that all 3-connected cubic bipartite planar graphs are hamiltonian (Conjecture 5, Page 343). This conjecture (which is now known* *as Barnette's conjecture) has been shown to be true by Holton, Manvel and McKay [11] for all such graphs having vertices up to and including* 64.

Let C_n, P_n and K_n denote a cycle, a path and a complete graph each on n vertices. Let H_n, denote the graph obtained from the cartesian product $C_n \times K_2$ by deleting an edge not lying on the subgraph C_n. The graph H_4 is depicted in Figure 5. Note that H_n has path sequence $\{4\}$ if n is even and $\{2\}$ otherwise.

Let L_k denote the graph obtained from the graph $P_{k+2} \times K_2$ by deleting the two edges whose two end vertices are of degree 2. Now if $k = 0$, then L_k is the graph $2K_2$ which consists of two copies of K_2. The graph L_k is depicted in Figure 4.

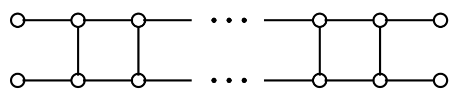
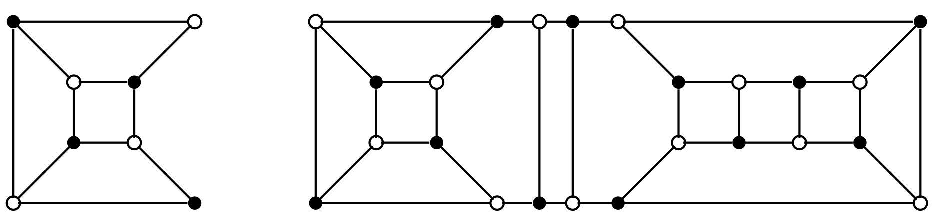


Figure 4: The graph L_k

Let $G_{m,n,k}$ denote the graph obtained by taking the merger of H_m and H_n and then replacing the two new edges (which join up H_m and H_n) by the graph L_k. Then, for any $r, s \geq 2$ and $k \geq 0$, $G_{2r,2s,k}$ has precisely 16 Hamilton cycles. The graph $G_{4,6,2}$ is depicted in Figure 5. Note that if m or n is odd, then $G_{m,n,k}$, is not bipartite.



H_4 $G_{4,6,2}$

Figure 5: The graphs H_4 and $G_{4,6,2}$

Problem 5. [5] *Let G be a cubic bipartite planar graph with connectivity 2. Is it true that G has precisely 16 Hamilton cycles if and only if G is the graph $G_{2r,2s,k}$?*

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