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Magic labelings of regular graphs

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Abstract

Let $G(V,E)$ be a graph and λ be a bijection from the set $V \cup E$ to the set of the first $|V|+|E|$ natural numbers. The weight of a vertex is the sum of its label and the labels of all adjacent edges. We say λ is a vertex magic total (VMT) labeling of G if the weight of each vertex is constant. We say λ is an (s,d)-vertex antimagic total (VAT) labeling if the vertex weights form an arithmetic progression starting at s with difference d.

J. MacDougall conjectured that any regular graph with the exception of K_2 and $2K_3$ has a VMT labeling. We give constructions of VAT labelings of any even-regular graphs and VMT labelings of certain regular graphs.

Keywords: Graph labeling, vertex magic total labeling, vertex antimagic total labeling.
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1. Introduction

A *labeling* of a graph $G(V,E)$ is a mapping from the set of vertices, edges, or both vertices and edges to the set of labels. Based on the domain we distinguish vertex labelings, edge labelings and total labelings. In most applications the labels are positive (or nonnegative) integers, though in general real numbers could be used. Various labelings are obtained based on the requirements put on the mapping.

Magic labelings were introduced by Sedláček in 1963 [10]. In general for a magic-type labeling we require the sum of labels related to a vertex (a vertex magic labeling) or to an edge (an edge magic labeling) to be constant all over the graph. In an antimagic labeling we require that all the sums (weights) are different. For many graphs this is not difficult to achieve, but not in general. The conjecture by Hartsfield and Ringel [5] "All graphs except K_2 are antimagic" is still open.

Usually we put additional restrictions on the labeling. We require the weights to form an arithmetic progression. Such labelings were introduced in 2000 [1].

There are many articles published on magic-type graph labelings. For a summary on various labelings see the *Dynamic Survey of Graph Labelings* by Gallian [4]. A monograph *Magic graphs* is due Wallis [11].

We give definitions and describe the problem in Section 2. In Sections 3 and 4 the main results are described in detail and examples are provided.

2. Definitions and known results

We focus on two magic-type labelings: vertex magic total (VMT) labelings and (s,d)-vertex antimagic total (VAT) labelings. In a total labeling both vertices and edges are labeled. By $N(x)$ we denote the set of all vertices adjacent to a vertex x in a given graph. We call the vertices the *neighbors* of x and the set $N(x)$ the *neighborhood* of x. The *weight* of a vertex x in G is the sum of its label and the labels of all adjacent edges:

$$w_\lambda(x) = \lambda(x) + \sum_{y\in N(x)} \lambda(xy). \tag{1}$$

Several classes of graphs allow a VMT and/or VAT labelings. An overview of known results is in [4]. We give the formal definition below.

Definition 2.1 *Let G be a graph with vertex set V and edge set E. We denote $v=|V|$ and $e=|E|$. Let λ be a one-to-one mapping $\lambda : V\cup E \to \{1,2,\dots,v+e\}$.*

The labeling λ is called a vertex magic total (VMT) labeling of *G if there exists a constant h such that for every vertex $x\in V$ is $w_\lambda(x)=h$. The constant h is the* magic constant *for λ.*

The labeling λ is called a vertex antimagic total labeling of *G if the vertex weights* (1) *are pairwise different for all vertices of G. If the weights form an arithmetic progression*

$$s, s+d, \dots, s+(v-1)d,$$

then λ is called an (s,d)-vertex antimagic total (VAT) labeling of *G.*

A graph is a *VMT graph* if it admits a VMT labeling and similarly a *VAT graph* allows a VAT labeling. Examples of VMT and VAT labelings are given in Sections 3 and 4. We use the notation $v=|V|$ and $e=|E|$ through the rest of this section.

For a given graph we can find VMT labelings with different magic constants using the same set of labels. Every vertex label is counted once and every edge label is counted twice, once for each end vertex. Thus, the higher the labels on edges are, the higher the magic constant can be (see Figures 4.2, 4.3, and 4.5). The set of all possible magic constants is the *magic spectrum* of the magic labeling problem. A natural question arises: what is the magic spectrum of a given graph. The full magic spectra is known only for odd complete graphs [9]. Many conjectures are open (see [11]).

For a regular graph we can easily evaluate bounds for the magic constant. If we find a VMT labeling in which the edges are labeled with the first e integers, we get the lowest possible magic constant. On the other hand, if we find a VMT labeling in which the vertices are labeled with the first v integers, we obtain the highest magic constant.

$$\frac{\binom{v+e+1}{2}+\binom{e+1}{2}}{v} \le k \le \frac{2\binom{v+e+1}{2}-\binom{v+1}{2}}{v}.$$

The computation can be found in several papers [11, 2, 6].

An unpublished conjecture by MacDougall says that any regular graph other than K_2 or $2K_3$ is vertex magic total[1]. Because obviously no graph containing K_2 as a component can have a VMT labeling we rephrase the conjecture:

> Any r-regular graph for $r > 1$, with the exception of $2K_3$, has a VMT labeling.

All results in this text support the conjecture.

Often from one magic-type labeling another labeling of the same type or of a different type can be obtained. For a regular graph $G(V,E)$ we can always find the *dual* labeling λ' defined by $\lambda'(z) = v+e+1-\lambda(z)$ for any element $z \in V \cup E$. Deriving new labelings from known ones is one of the basic ideas used throughout the Sections 3 and 4.

3. VAT labelings of even regular graphs

Results in this and the following section are based on the Petersen Theorem.

Proposition 3.1 (Petersen Theorem) *Let G be a $2r$-regular graph. Then there exists a 2-factor in G.*

Notice that after removing edges of the 2-factor guaranteed by the Petersen Theorem we have again an even regular graph. Thus, by induction, an even regular graph has a 2-factorization.

The following theorem allows to find several $(s,1)$-VAT labelings of any even regular graph.

Theorem 3.2 *Let G be a $2r$-regular graph with vertices $x_1, x_2, \ldots, x_n$. Let s be an integer, $s \in \{(rn+1)(r+1)+tn : t = 0,1,\ldots,r\}$. Then there exists an $(s,1)$-VAT labeling λ of G such that $\lambda(x_i) = s+(i-1)$.*

Proof. By induction on r. We show a stronger result. Not only we give an $(s,1)$-VAT labeling of G, but the vertex labels will be consecutive integers. Moreover, we can specify

[1]A note about the conjecture can be found in [8, 6, 7]. The author learned from personal communication with J. MacDougall that the conjecture was not published yet.

which weight $s+(i-1)$ will be assigned to which vertex by ordering the vertices x_i accordingly.

For $r=0$ the statement is trivial. The set of possible values of s is $\{(0n+1)(0+1)+0n\}=\{1\}$. We label the vertices x_i by $1+(i-1)=i$ for $i=1,2,\dots,n$.

In the inductive step we suppose the claim is true for p-regular graphs, $p=0,1,\dots,r$. We show that it is true also for $p=r+1$. Let G be a $(2r+2)$-regular graph with vertices $x_1,x_2,\dots,x_n$. By the Petersen Theorem there exists a 2-regular factor F in G. By G' we denote the the $2r$-regular graph obtained from G by removing the edges of F. By the assumption G' has an $(s,1)$-VAT labeling λ' such that $s\in\{(rn+1)(r+1),(rn+1)(r+1)+n,\dots,(rn+1)(r+1)+rn\}$ and the vertex labels are consecutive integers $\lambda'(x_i)=k+(i-1)$ for some k (k is the smallest vertex label). The factor F is a collection of cycles. We order and orient these cycles arbitrarily. By $out(x_i)$ we denote the end vertex of the outgoing arc from x_i and by $in(x_i)$ we denote the begin vertex of the incoming arc to x_i.

We define the labeling λ_1 of G by

$$\begin{aligned} \lambda_1(x_ix_j) &= \lambda'(x_ix_j) \qquad \forall x_ix_j\in E(G') \quad i,j\in\{1,2,\dots,n\} \\ \lambda_1(x_iout(x_i)) &= \lambda'(x_i) \qquad \forall x_iout(x_i)\in E(F) \\ \lambda_1(x_i) &= n(r+2)+k-\lambda'(in(x_i)) \qquad \forall x_i\in V(F). \end{aligned} \tag{2}$$

Another labeling λ_2 of G is given by

$$\begin{aligned} \lambda_2(x_ix_j) &= \lambda'(x_ix_j) \qquad \forall x_ix_j\in E(G') \quad i,j\in\{1,2,\dots,n\} \\ \lambda_2(x_iout(x_i)) &= n(r+1)-(k-1)+\lambda'(x_i) \qquad \forall x_iout(x_i)\in E(F) \\ \lambda_2(x_i) &= 2k+n-1-\lambda'(in(x_i)) \qquad \forall x_i\in V(F). \end{aligned} \tag{3}$$

It is easy to check that both labelings λ_1 and λ_2 are VAT labelings of G with difference 1. First we verify that each of the labels $1,2,\dots,n(r+2)$ was used exactly once. For the labeling λ_1 the labels $1,2,\dots,n(r+1)$ are used to label the edges of G ($E(G')\cup E(F)$). The labels $n(r+1)+1,n(r+1)+2,\dots,n(r+1)+n$ are used to label the vertices of G. For the labeling λ_2 the labels $1,2,\dots,n(r+1)$ are used to label the edges of $G'=G\setminus E(F)$ and the vertices of G. The labels $n(r+1)+1,n(r+1)+2,\dots,n(r+1)+n$ are used to label the edges of $G\cap E(F)$.

We evaluate the weight of every vertex in the labeling λ_1.

$$\begin{aligned}
w_{\lambda_1}(x_i) &= \lambda_1(x_i) + \sum_{x_j \in N(x_i)} \lambda_1(x_i x_j) \\
&= \lambda_1(x_i) + \sum_{x_j \in N(x_i) \cap E(G')} \lambda_1(x_i x_j) + \sum_{x_j \in N(x_i) \cap E(F)} \lambda_1(x_i x_j) \\
&= n(r+2) + k - \lambda'(in(x_i)) + \sum_{x_j \in N(x_i) \cap E(G')} \lambda'(x_i x_j) + \lambda'(in(x_i)) + \lambda'(x_i) \\
&= n(r+2) + k + \lambda'(x_i) + \sum_{x_j \in N(x_i) \cap E(G')} \lambda'(x_i x_j).
\end{aligned}$$

The labeling λ' is an $(s,1)$-VAT labeling of G', therefore

$$\lambda'(x_i) + \sum_{x_j \in N(x_i) \cap E(G')} \lambda'(x_i x_j) = s + (i-1).$$

The weight of any vertex in labeling λ_1 is $w_{\lambda_1}(x_i) = n(r+2) + k + s + (i-1)$. We know that $s = (rn+1)(r+1) + tn$ for $t \in \{0, 1, \ldots, r\}$. Since k is the smallest vertex label and since we assign always an n-tuple of labels to either edges or vertices of a 2-regular factor, we can write $k = qn + 1 = (q - r + r)n + 1 = (q-r)n + (rn+1)$. The value t specifies that the $(r+1-t)$-th n-tuple was used to label vertices in λ', thus $q \geq r - t$. We have

$$\begin{aligned}
w_{\lambda_1}(x_i) &= n(r+2) + (q-r)n + (rn+1) + (rn+1)(r+1) + tn + (i-1) \\
&= n(r+2) + (rn+1)(r+2) + (t+q-r)n + (i-1) \\
&= ((r+1)n+1)(r+2) + (t+q-r)n + (i-1).
\end{aligned}$$

The labeling λ_1 is an $(s_1,1)$-VAT labeling of the $2(r+1)$-regular graph G where the smallest weight is $s_1 = ((r+1)n+1)(r+2) + (t+q-r)n$.

Similarly we evaluate the weight of every vertex in the labeling λ_2. The weight of any vertex in the labeling λ_2 of the $2(r+1)$-regular graph G is

$$w_{\lambda_2}(x_i) = \lambda_2(x_i) + \sum_{x_j \in N(x_i)} \lambda_2(x_i x_j) = ((r+1)n+1)((r+1)+1) + (t+1)n + (i-1).$$

The labeling λ_2 is an $(s_2,1)$-VAT labeling of the $2(r+1)$-regular graph G where the smallest weight is $s_2 = ((r+1)n+1)(r+2) + (t+1)n$.

Not only both labelings λ_1 and λ_2 assign consecutive integers to the vertices, but also the weight of vertices always increases with the vertex subscript i

$$w_{\lambda_1}(x_i) = s_1 + (i-1) \qquad \wedge \qquad w_{\lambda_2}(x_i) = s_2 + (i-1).$$

This concludes the inductive step and the proof. □

Example 3.3 *We construct several* $(s,1)$ *-VAT labelings of* K_5 *based on the construction given in the proof of Theorem 3.2. For* $r=0$ *is* $G=\overline{K}_5$ *and the construction is trivial, see the Figure 3.1.*

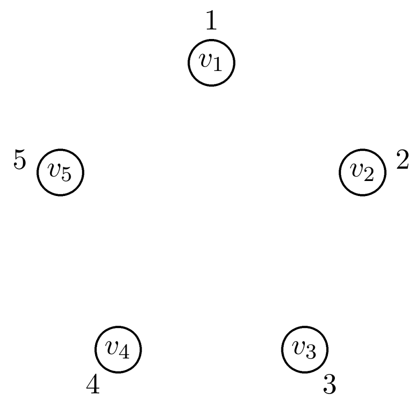


Figure 3.1: $(1,1)$ *-VAT labeling of* $\overline{K}_5$ *.*

Now for $r=1$ *we take the factor* F *to be the cycle* $v_1v_2v_3v_4v_5v_1$ *. We can construct a* $(12,1)$ *-VAT labeling* λ_1 *of* C_5 *or a* $(17,1)$ *-VAT labeling* λ_2 *of* C_5 *(see Figure 3.2). The weights are in bold.*

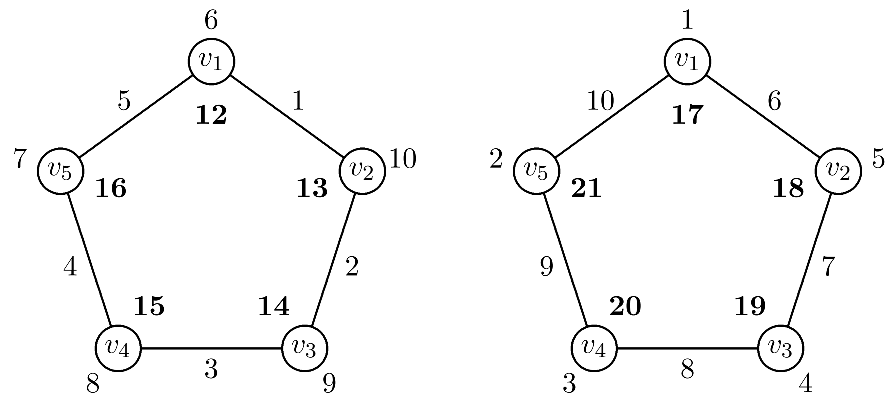


Figure 3.2: *A* $(12,1)$ *-VAT labeling and a* $(17,1)$ *-VAT labeling of* C_5 *.*

For $r=2$ *the factor* F *is the cycle* $v_1v_3v_5v_2v_4v_1$ *. We have two options how to proceed with the construction of an* $(s,1)$ *-VAT labeling of* K_5 *. The first possibility is to extend the* $(12,1)$ *-VAT labeling of* C_5 *. We obtain either a* $(33,1)$ *-VAT labeling* λ_1 *of* K_5 *or a* $(38,1)$ *-VAT labeling* λ_2 *of* K_5 *(see Figure 3.3).*

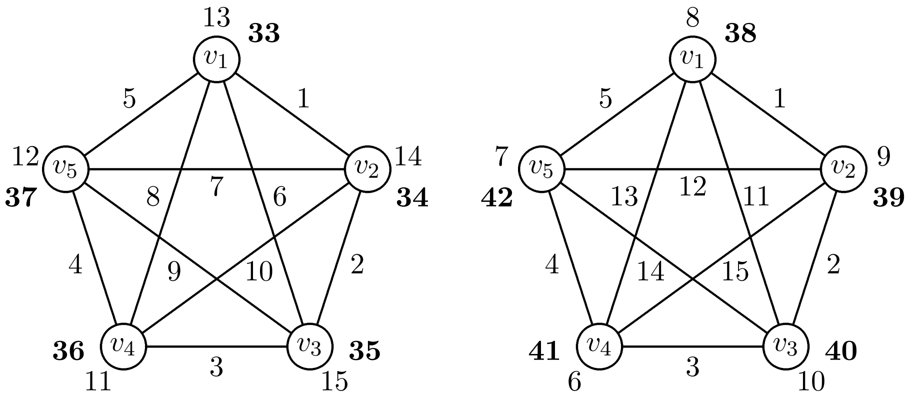


Figure 3.3: *A* $(33,1)$ *-VAT labeling and a* $(38,1)$ *-VAT labeling of* K_5 *.*

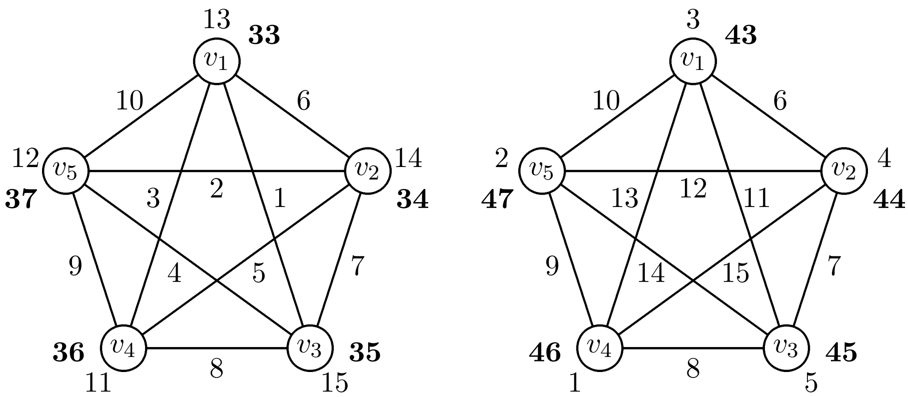


Figure 3.4: *A* $(33,1)$ *-VAT labeling and a* $(43,1)$ *-VAT labeling of* K_5 *.*

The second possibility is to extend the $(17,1)$ *-VAT labeling of* C_5 *. We obtain either another* $(33,1)$ *-VAT labeling* λ_1 *of* K_5 *or a* $(43,1)$ *-VAT labeling* λ_2 *of* K_5 *(see Figure 3.4).*

Both $(33,1)$ *-VAT labelings in this example are different, though this is not necessarily true in general. If* G *would be e.g. a* 4 *-regular graph, which can be decomposed into* C_{2n} *and* $2C_n$ *, then we are sure to obtain two different* $(6n+3,1)$ *-VAT labelings.*

4. VMT labelings of even regular graphs

Further we focus on r-regular graphs on n vertices with VMT labelings in which vertex labels are consecutive integers. First we observe that for such labelings there are restrictions on n and s.

Lemma 4.1 *Let* G *be an* r *-regular graph on* n *vertices. If* G *has a VMT labeling* *such that the vertex labels constitute an arithmetic progression with odd difference, then either r is even and n is odd or r is odd and $n \equiv 0 \pmod 4$.*

Proof. Suppose that there exists a VMT labeling λ of G that assigns labels $k, k+d, \dots, k+(n-1)d$ to the vertices where d is odd. We evaluate the magic h constant of such a graph. There are $\frac{1}{2}rn$ labels assigned to the edges and n labels to the vertices. Each of the $\frac{1}{2}rn+n$ labels is counted twice except those assigned to the vertices. Thus, we have the total sum of weights

$$\begin{aligned}\sum_{v \in V(G)} w_\lambda(v) &= 2 \cdot \frac{1}{2}\left(\frac{1}{2}rn+n\right)\left(1+\frac{1}{2}rn+n\right) - \frac{1}{2}n(2k+(n-1)d) \\ &= n\left(\left(\frac{1}{2}r+1\right)\left(1+\frac{1}{2}rn+n\right) - k - \frac{1}{2}(n-1)d\right).\end{aligned}$$

From the magic property we have also

$$\sum_{v \in V(G)} w_\lambda(v) = nh.$$

Comparing the two results we have

$$h = \left(\frac{1}{2}r+1\right)\left(1+\frac{1}{2}rn+n\right) - k - \frac{1}{2}(n-1)d.$$

Now h is an integer. If r is even then the first product of terms is an integer. Assuming d is odd it immediately follows that n has to be odd.

On the other hand if r is odd then for an odd regular graph n has to be even. We have

$$h = \left(\frac{1}{2}r+1\right)\left(1+\frac{1}{2}rn+n\right) - k - \frac{1}{2}(n-1)d = \frac{1}{2}\left(r+(n-1)d\right) + \frac{1}{4}r^2n + (r+1)n - k.$$

The first term is an integer since the parenthesis contains a sum of two odd numbers. In order for h to be an integer n has to be a multiple of 4. □

The lemma states the limits for the constructions shown below, the constructions require vertex labels to be consecutive integers.

Theorem 4.2 *Let G be a $(2+s)$-regular graph such that it contains an s-regular factor G' which allows a VMT labeling with magic constant h and vertex labels being consecutive integers starting at k. Then G has VMT labelings with magic constants*

$$h = \frac{1}{4}(s+4)(n(s+4)+2) - \frac{1}{2}(n-1) - t,$$

where $t \in \left\{k, \frac{1}{2}n(s+2)+1\right\}$.

Proof. Let $n = |G|$. By the assumption G' has a VMT labeling λ' with the magic constant h and consecutive vertex labels $k, k+1, \ldots, k+(n-1)$. By k we denote the smallest vertex label. Without loss of generality we can denote the vertices of G by x_i so that $\lambda(x_i) = k + (i-1)$ for $i = 1, 2, \ldots, n$.

By F we denote the 2-regular factor obtained from G by removing the edges of G'. The factor F is a collection of cycles. We order and orient these cycles arbitrarily. By $out(x_i)$ we denote the end vertex of the outgoing arc from x_i and by $in(x_i)$ we denote the begin vertex of the incoming arc to x_i.

Let λ_1 be a labeling of G given by

$$\begin{array}{rcll} \lambda_1(x_i x_j) & = & \lambda'(x_i x_j) & \forall x_i x_j \in E(G') \quad i,j \in \{1,2,\ldots,n\} \\ \lambda_1(x_i out(x_i)) & = & \lambda'(x_i) & \forall x_i out(x_i) \in E(F) \\ \lambda_1(x_i) & = & n\left(2 + \frac{s}{2}\right) + k - \lambda'(in(x_i)) & \forall x_i \in V(F). \end{array} \tag{4}$$

Another labeling λ_2 of G is given by

$$\begin{array}{rcll} \lambda_2(x_i x_j) & = & \lambda'(x_i x_j) & \forall x_i x_j \in E(G') \quad i,j \in \{1,2,\ldots,n\} \\ \lambda_2(x_i out(x_i)) & = & n\left(1 + \frac{s}{2}\right) - (k-1) + \lambda'(x_i) & \forall x_i out(x_i) \in E(F) \\ \lambda_2(x_i) & = & 2k + n - 1 - \lambda'(in(x_i)) & \forall x_i \in V(F). \end{array} \tag{5}$$

Again the subscript i is taken modulo n if necessary.

We show that both labelings λ_1 and λ_2 are a VMT labelings of G. First it is easy to verify that we used each of the labels $1, 2, \ldots, n(r+2)$ exactly once. Moreover, both labelings λ_1 and λ_2 have vertices labeled by consecutive integers. The reasoning is identical as in the proof of Theorem 3.2. We evaluate the weight of every vertex in the labeling λ_1.

$$\begin{aligned} w_{\lambda_1}(x_i) &= \lambda_1(x_i) + \sum_{x_j \in N(x_i)} \lambda_1(x_i x_j) \\ &= \lambda_1(x_i) + \sum_{x_j \in N(x_i) \cap E(G')} \lambda_1(x_i x_j) + \sum_{x_j \in N(x_i) \cap E(F)} \lambda_1(x_i x_j) \\ &= n\left(2 + \frac{s}{2}\right) + k - \lambda'(in(x_i)) + \sum_{x_j \in N(x_i) \cap E(G')} \lambda'(x_i x_j) + \lambda'(in(x_i)) + \lambda'(x_i) \\ &= n\left(2 + \frac{s}{2}\right) + k + \lambda'(x_i) + \sum_{x_j \in N(x_i) \cap E(G')} \lambda'(x_i x_j) \end{aligned}$$

and since the weight of every vertex in λ' is h, we have

$$w_{\lambda_1}(x_i) = n\left(2 + \frac{s}{2}\right) + k + h.$$

A VMT labeling of an s-regular graph with consecutive vertex labels starting at k has a magic constant

$$h = \left(1+\frac{s}{2}\right)\left(1+n\left(1+\frac{s}{2}\right)\right) - k - \frac{n-1}{2} \tag{6}$$

(derived in the proof of Lemma 4.1). We obtain

$$\begin{aligned} w_{\lambda_1}(x_i) &= n\left(2+\frac{s}{2}\right) + k + \left(1+\frac{s}{2}\right)\left(1+n\left(1+\frac{s}{2}\right)\right) - k - \frac{n-1}{2} \\ &= \frac{1}{4}(s+4)(n(s+4)+2) - \frac{1}{2}(n-1) - \left(\frac{1}{2}n(s+2)+1\right). \end{aligned}$$

Thus λ_1 is a VMT labeling of the $(2+s)$-regular graph G with the magic constant $h_1 = \frac{1}{4}(s+4)(n(s+4)+2) - \frac{1}{2}(n-1) - t$ for $t = \frac{1}{2}n(s+2)+1$.

Similarly we evaluate the weight of every vertex in the labeling λ_2. We have

$$w_{\lambda_2}(x_i) = \lambda_2(x_i) + \sum_{x_j \in N(x_i)} \lambda_2(x_i x_j) = n\,(3+s) + 1 + h.$$

Using (6) we have

$$\begin{aligned} w_{\lambda_2}(x_i) &= n\,(3+s) + 1 + \left(1+\frac{s}{2}\right)\left(1+n\left(1+\frac{s}{2}\right)\right) - k - \frac{n-1}{2} \\ &= \frac{1}{4}(s+4)(n(s+4)+2) - \frac{1}{2}(n-1) - k. \end{aligned}$$

Also λ_2 is a VMT labeling of the $(2+s)$-regular graph G. The magic constant is $h_2 = \frac{1}{4}(s+4)(n(s+4)+2) - \frac{1}{2}(n-1) - t$ for $t = k$. □

The Theorem 4.2 gives a recursive method for constructing VMT labelings of regular graphs. It can be used repeatedly. Hence the following theorem.

Theorem 4.3 *Let G be a $(2r+s)$-regular graph such that it contains an s-regular factor G' which allows a VMT labeling with magic constant h' and vertex labels being consecutive integers starting at k. Then G has VMT labelings with magic constants*

$$h = \frac{1}{4}(s+2r+2)(n(s+2r+2)+2) - \frac{1}{2}(n-1) - t$$

where $t \in \{k\} \cup \left\{\frac{1}{2}n(s+2i)+1 : i = 1,2,\ldots,r\right\}$.

Proof. The proof goes by induction. For $r=0$ the claim follows immediately from (6) since

$$h = \left(1+\frac{s}{2}\right)\left(1+n\left(1+\frac{s}{2}\right)\right) - k - \frac{n-1}{2} = \frac{1}{4}(s+2)\,(2+n(s+2)) - k - \frac{n-1}{2}.$$

The inductive step follows immediately from Theorem 4.2 and from Petersen Theorem. □

Examples of VMT labelings of even regular graphs with consecutive vertex labels are shown in Example 4.7. The smallest feasible odd degree is $s = 3$ and the smallest feasible n according Lemma 4.1 is $n = 4$. A brute force search reveals that no VMT labeling of K_4 with consecutive vertex labels exists. The graph K_4 is probably too small. The next feasible size is $n = 8$.

Example 4.4 *A 3-regular graph on 8 vertices having a VMT labeling with consecutive vertex labels is in Figure 4.1. Notice that the vertices are denoted in accordance with their labels.*

Using the construction from Theorem 4.2 we can find VMT labelings of several different 5-regular graphs. See Figures 4.2 (factor F is an 8-cycle) and 4.3 (F consists of two 4-cycles). The 2-factor is drawn in thick. The magic constants will be $h_1 = n\left(2 + \frac{s}{2}\right) + k + h = 77$ and $h_2 = n(3 + s) + 1 + h = 85$.

The drawback of the general approach of Theorem 4.3 is that the existence of VMT labelings of regular graphs is based on the existence of regular subgraphs. These do not exist in general (e.g. in some graphs with bridges). Moreover we require the regular subgraphs to have certain VMT labelings. Again these are not guaranteed in general (e.g. for even regular graphs on an even number of vertices such labelings do not exist).

Nonetheless, we can find VMT labelings of a large class of graphs. Through the rest of this section we focus only on even regular graphs on an odd number of vertices which have a VMT labeling with consecutive vertex labels (see Lemma 4.1).

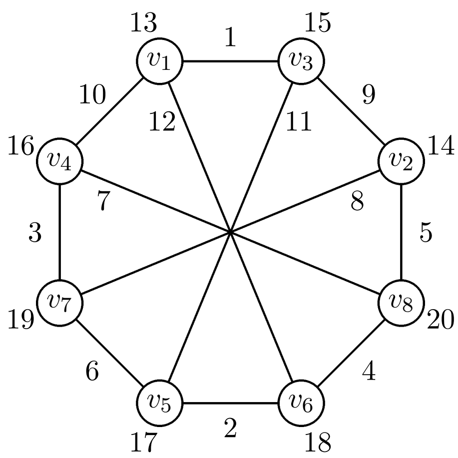


Figure 4.1: *A VMT labeling of a 3-regular graph with the magic constant 36.*

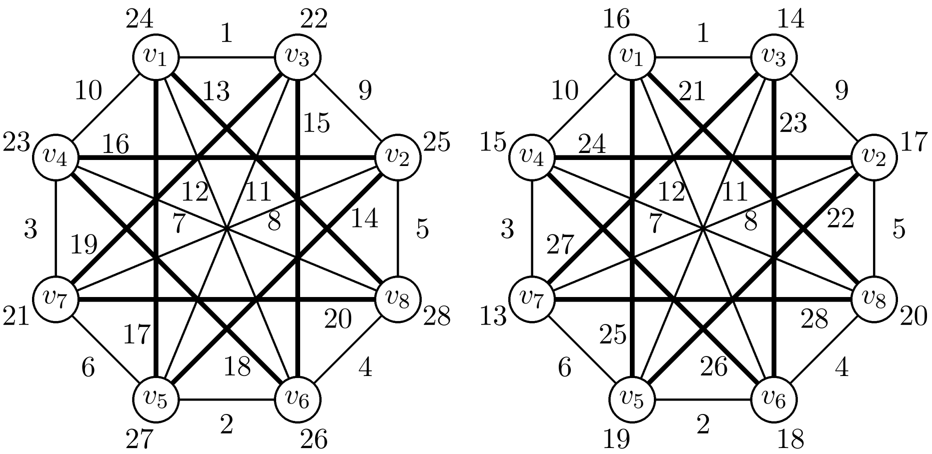


Figure 4.2: *VMT labelings of a 5-regular graph with the magic constants* 77 *and* 85 .

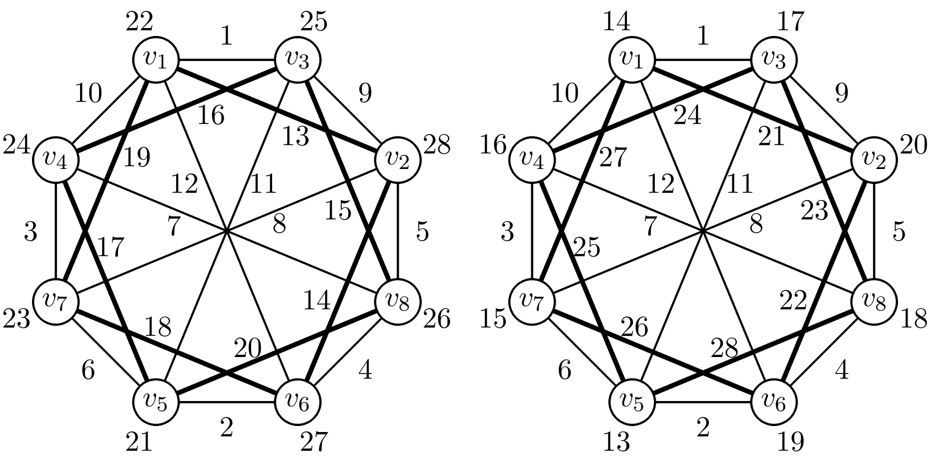


Figure 4.3: *VMT labelings of a 5-regular graph with the magic constants* 77 *and* 85 .

Corollary 4.5 *Let* G *be a* $2r$ *-regular graph on an odd number of vertices which has a Hamiltonian cycle. Then* G *has a VMT labeling with the magic constants*

$$h = \frac{1}{4}\,(2r+2)(n(2r+2)+2) - \frac{1}{2}(n-1) - t$$

where $t \in \{ni+1 : i = 0, 1, \dots, r\}$.

Proof. Obviously a Hamiltonian graph G on an odd number of vertices contains an odd cycle. It was shown in [3] and also in [11] that every odd cycle has an edge magic total labeling where the vertex labels are consecutive integers. An edge magic total (EMT) labeling of the graph $G(V,E)$ has all the first $|V|+|E|$ integers assigned to both edges and vertices so that for every edge the sum of its label and of labels of both endvertices is constant. From an EMT labeling of C_n a VMT labeling of C_n is derived easily by shifting clockwise the vertex labels to edges and edge labels to vertices. The magic constant is $\frac{1}{2}(7n+3)$ and the lowest vertex labels is 1. For $i=1,2,\ldots,r$ the claim follows from Theorem 4.3. For $i=0$ it is enough to observe that $k=1=n\cdot 0+1$. □

An immediate corollary of Dirac's Theorem is that certain dense even regular graphs are VMT.

Corollary 4.6 *Let G be a $2r$-regular graph on an odd number of vertices where $4r>n$. Then G has a VMT labeling.*

Example 4.7 *Based on the construction given in the proof of Theorem 4.3 we construct two VMT labelings of K_5 with the magic constants $h_1=n\left(2+\frac{s}{2}\right)+k+h=35$ and $h_2=n\,(3+s)+1+h=40$.*

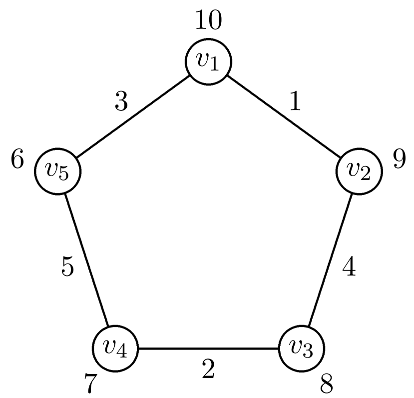
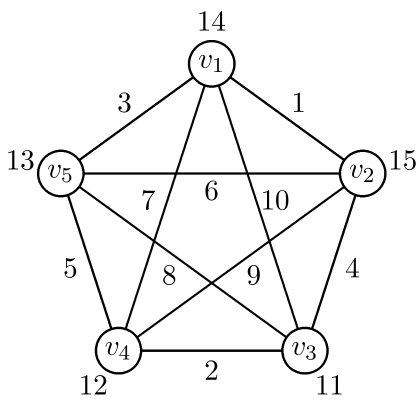


Figure 4.4: *A VMT labeling of C_5 with the magic constant 14.*



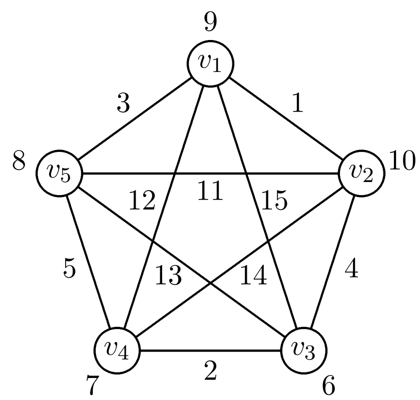


Figure 4.5: *VMT labelings of C_5 and K_5 with magic constants 35 and 40.*

5. Conclusion

As a corollary of Theorem 4.3 many classes of graphs have VMT labelings. Once a VMT labeling with consecutive vertex labels of a given graph exists, any regular graph obtained by adding an arbitrary collection of pairwise edge disjoint 2-factors will have a VMT labeling. Moreover the proof is constructive and shows how to construct labelings with different magic constants. By the nature of the construction not the full magic spectra is covered, only certain values which differ by $n = |G|$.

An exciting observation regarding Theorem 4.3 is that for proving MacDougall's conjecture for even regular graphs on an odd number of vertices it would be enough to find VMT labeling of some 2-regular subgraphs. Unfortunately a brute force search soon reveals that not all 2-regular graphs satisfying Lemma 4.1 have VMT labelings with consecutive vertex labels.

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