AKCE J. Graphs. Combin., 4, No. 3 (2007), pp. 277-299

Vertex-Distinguishing Colorings of Graphs A Survey of Recent Developments

Henry Escuadro
Mathematics Department
Juniata College
Huntingdon, PA 16652, USA
Futaba Okamoto
Mathematics Department
University of Wisconsin-La Crosse
La Crosse, WI 54601, USA
and
Ping Zhang
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA
e-mail: *ping.zhang@wmich.edu*

Communicated by: S. Arumugam
Received 23 April 2007; accepted 12 December 2007

Dedicated to Gary Chartrand on the occasion of his 70th birthday

Abstract

A research problem in graph theory concerns distinguishing the vertices of a graph by means of graph colorings. We survey various methods, recent results and open questions from this area of research.

Keywords: Vertex-coloring, edge-coloring, color codes.

2000 Mathematics Subject Classification: 05C15.

1. Introduction

A problem in graph theory that has received increased attention during the past 35 years concerns studying methods of distinguishing the vertices of a connected graph from one another.

An earlier method suggested by Sumner [33] and by Entringer and Gassman [13] studied graphs G for which the equality of the open neighborhoods of every two vertices of G

implies that the vertices are the same. In this case, the vertices of G are uniquely determined by their open neighborhoods.

Another idea is to consider the automorphism group $\mathrm{Aut}(G)$ of a connected graph G. If $\mathrm{Aut}(G)$ is the identity group, then all the vertices of G are distinguishable; while if the automorphism group of G is not the identity group, then this group has at least one nontrivial orbit and the vertices that belong to such an orbit are indistinguishable. This means that the vertices of a nontrivial graph cannot be distinguished in all cases by its automorphism group alone. Erwin and Harary [14] introduced the idea of selecting a subset S of the vertex set of a graph G such that the subgroup of $\mathrm{Aut}(G)$ that fixes every vertex of S is the identity group. Another method, introduced by Albertson and Collins [4] and by Harary [23], involves coloring the vertices of G in such a way that the subgroup of color-preserving automorphisms of $\mathrm{Aut}(G)$ is the identity group, thereby distinguishing the vertices of G from one another.

Harary and Melter [24] and Slater [31] introduced the idea of selecting an ordered set $W = \{ w_1, w_2, \cdots, w_k \}$ of vertices in a connected graph G and assigning to each vertex v the ordered k-tuple $c_W(v) = (a_1, a_2, \cdots, a_k)$, called the *distance code* of v (with respect to W), where $a_i = d(v, w_i)$, $1 \le i \le k$. If distinct vertices of G have distinct distance codes, then the vertices of G are distinguishable. That is, the vertices of G are distinguished by their distances to some ordered set of vertices in G.

Harary and Plantholt [25] considered assigning colors to the edges of a graph G in such a way that for every two vertices of G, one of the vertices is incident with an edge assigned a color that the other vertex is not. They referred to the minimum number of colors needed to accomplish this as the *point-distinguishing chromatic index* of G.

It is a well-known result from graph theory that every nontrivial graph contains at least two vertices having the same degree. Thus the vertices of a nontrivial graph cannot be distinguished by their degrees alone. However, it has been observed by Chartrand, Jacobson, Lehel, Oellermann, Ruiz, and Saba that weights can be assigned to the edges of every connected graph of order 3 or more in such a way that the degrees of the vertices (defined as the sum of the weights of its incident edges) of the resulting weighted graph are distinct (see [10]).

For a connected graph F of order 2 or more and a vertex v of a graph G, Chartrand, Holbert, Oellermann and Swart [9] defined the F-*degree* of v in G as the number of copies of F containing v. It has been conjectured that if F is a connected graph of order 3 or more, then there exists some graph G whose vertices can be distinguished by their F-degrees.

In this paper, we describe four more recently introduced methods, which combine many of the features of the methods given above. We survey several results and open questions from this area of research. We refer to the book [12] for graph theory notation and terminology not described in this paper.

2. Detectable Labelings of Graphs

Let G be a connected graph of order $n \geq 3$ and let $c : E(G) \to \{1, 2, \ldots, k\}$ be a coloring (or labeling) of the edges of G for some positive integer k (where adjacent edges may be colored the same). If c uses k colors, then c is a k*-coloring* (or k*-labeling*). The *color code* of a vertex v of G (with respect to the edge coloring c) is the ordered k-tuple

$$\mathrm{code}_c(v) = (a_1, a_2, \cdots, a_k) \text{ (or simply } \mathrm{code}_c(v) = a_1 a_2 \cdots a_k\text{)}, \tag{1}$$

where a_i is the number of edges incident with v that are colored i for $1 \leq i \leq k$. Therefore,

$$\sum_{i=1}^{k} a_i = \deg_G v.$$

If the coloring c is clear, we use $\mathrm{code}(v)$ to denote the color code of a vertex v. The coloring c is called a *detectable labeling* of G if distinct vertices of G have distinct color codes; that is, for every two vertices of G, there exists a color such that the number of incident edges assigned that color is different for these two vertices. The *detection number* $\det(G)$ of G is the minimum positive integer k for which G has a detectable k-labeling. A detectable labeling of a graph G with $\det(G)$ colors is called a *minimum detectable labeling.*

What we have described is an assignment of colors to the edges of a graph (where adjacent edges may be colored the same) that results in a color code assigned to each vertex of the graph. The goal is to use as few colors as possible so that every two distinct vertices are assigned distinct color codes. The detection number of a graph is then defined as the minimum number of colors that can be assigned to the edges of the graph so that the stated goal is achieved. However, it is possible to regard color codes as colors themselves. That is, by assigning colors to the edges, we can produce a color for each vertex, where the color assigned to an edge is an element of the set $\{1, 2, \ldots, k\}$ for some positive integer k and the resulting color assigned to a vertex v is an ordered k-tuple $(a_1, a_2, \cdots, a_k)$, where a_i denotes the number of edges incident with v that are colored i for $1 \leq i \leq k$.

Since every nontrivial graph contains at least two vertices having the same degree, the vertices of a nontrivial connected graph cannot be distinguished by their degrees alone. Therefore, every connected graph of order 3 or more has detection number at least 2. On the other hand, since every coloring that assigns distinct colors to the edges of a connected graph of order $n \geq 3$ is a detectable labeling, the detection number is always defined. This is not the case for $n = 2$ however. Therefore, we only consider connected graphs of order 3 or more.

To illustrate these concepts, consider the graph G shown in Figure 1(a). A labeling of the edges of G is shown in Figure 1(b). For this 3-labeling c, the color codes of its vertices are $\mathrm{code}_c(u) = 110$, $\mathrm{code}_c(v) = 021$, $\mathrm{code}_c(w) = 210$, $\mathrm{code}_c(x) = 201$, $\mathrm{code}_c(y) = 101$, $\mathrm{code}_c(z) = 001$. Since the vertices of G have distinct color codes, c is a detectable labeling.

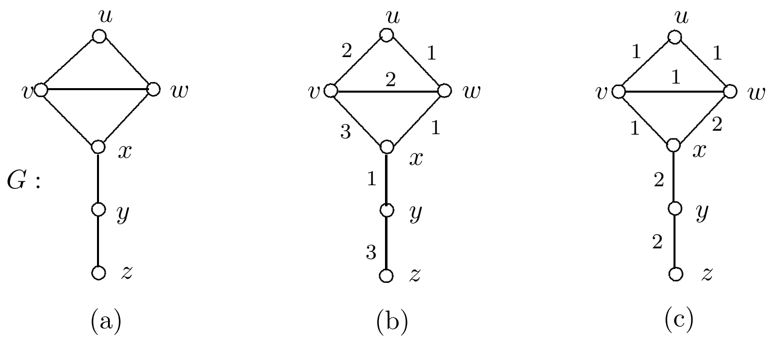

Figure 1: A detectable labeling of a graph

Figure 1(c) shows yet another detectable labeling c' of the graph G of Figure 1(a). For this labeling, $\text{code}_{c'}(u) = 20$, $\text{code}_{c'}(v) = 30$, $\text{code}_{c'}(w) = 21$, $\text{code}_{c'}(x) = 12$, $\text{code}_{c'}(y) = 02$, $\text{code}_{c'}(z) = 01$. The labeling c' uses only two colors. Since G has a detectable 2-labeling, we can immediately conclude that $\det(G) = 2$.

These concepts were introduced and studied by Aigner, Triesch and Tuza in [2, 3] and Burris in [6, 7]. Recently, this topic has been studied further by Chartrand, Escuadro, Okamoto, and Zhang (see [8, 15, 16, 19, 20, 21]). Among the results obtained in these papers are the following.

Theorem 2.1. *If c is a detectable k-labeling of a connected graph G of order at least 3, then G contains at most $\binom{r+k-1}{r}$ vertices of degree r.*

Since vertices with distinct degrees in a connected graph always have distinct color codes, it is most challenging to find minimum detectable labelings of graphs having many vertices of the same degree. The detection numbers of complete graphs, complete bipartite graphs, cycles and paths have been determined and detectable labelings of connected r-regular graphs and trees have been studied as well (see [3, 6, 7, 8, 19]). We state these results as follows.

Theorem 2.2. *The detection number of K_n is 3 for every integer $n \geq 3$.*

Theorem 2.3. *For integers s and t with $1 \leq s \leq t$ and $s + t \geq 3$,*

$$\det(K_{s,t}) = \begin{cases} 3 & \text{if } s = t \geq 2 \\ t & \text{if } 1 = s < t \\ 2 & \text{if } t = s + 1 \\ k & \text{if } 2 \leq s < t - 1 \text{ and } k \text{ is the unique integer} \\ & \text{for which } \binom{s+k-2}{s} < t \leq \binom{s+k-1}{s} \end{cases}$$

Theorem 2.4. *Let $n \geq 3$ be an integer and let $\ell = \left\lceil \sqrt{n/2}\, \right\rceil$. Then*

$$\det(C_n) = \begin{cases} 2\ell & \text{if } 2\ell^2 - \ell + 1 \leq n \leq 2\ell^2 \\ 2\ell - 1 & \text{if } 2(\ell-1)^2 + 1 \leq n \leq 2\ell^2 - \ell. \end{cases}$$

Theorem 2.5. *Let* $n \geq 3$ *and let* $\ell = \left\lceil \dfrac{-3+\sqrt{8n-7}}{4} \right\rceil$. *Then*

$$\det(P_n) = \begin{cases} 2\ell & \text{if } 2\ell^2 - \ell + 2 \leq n \leq 2\ell^2 + 3 \\ 2\ell + 1 & \text{if } 2\ell^2 + 4 \leq n \leq 2\ell^2 + 3\ell + 2. \end{cases}$$

The following result appears in [8].

Theorem 2.6. *If* G *is a connected graph of order* $n \geq 4$, *then*

$$2 \leq \det(G) \leq n - 1.$$

Furthermore, a pair k, n *of positive integers is realizable as the detection number and the order of some nontrivial connected graph if and only if* $k = n = 3$ *or* $2 \leq k \leq n-1$.

An upper bound for the detection number of a connected regular graph was established in terms of its order and its degree of regularity in [6].

Theorem 2.7. *If* G *is a connected* r*-regular graph of order* $n \geq 3$, *then*

$$\det(G) \leq (5e(r+1)!)n)^{\frac{1}{r}},$$

where e *is the natural base.*

An upper bound for the detection number of a tree in terms of the number of its end-vertices and the number of vertices of degree 2 was determined in [7].

Theorem 2.8. *For any tree* T *with at least three vertices,*

$$\det(T) \leq \max\{n_1, 4.62\sqrt{n_2}, 8\} + 1,$$

where n_i *denotes the number of vertices of* T *of degree* i *for* $i = 1, 2$. *The absolute constant* 4.62 *given above may be replaced by a constant* k_T *which depends on* T, *where* $2\sqrt{2} < k_T < 4.62$.

If G is a connected graph of order n and size m, then the number of edges that must be deleted from G to obtain a spanning tree of G is $m - n + 1$. The number $m - n + 1$ is called the *cycle rank* of G. For integers ψ and n, where $\psi \geq 0$ and $n \geq \left\lceil \frac{3+\sqrt{1+8\psi}}{2} \right\rceil$, let $D_\psi(n)$ denote the maximum detection number among all connected graphs of order n with cycle rank ψ and let $d_\psi(n)$ denote the minimum detection number among all connected graphs of order n with cycle rank ψ. Hence, if $\mathcal{G}_{\psi,n}$ denotes the set of all connected graphs of order n with cycle rank ψ, then

$$\begin{aligned} D_\psi(n) &= \max\{\det(G) : G \in \mathcal{G}_{\psi,n}\} \\ d_\psi(n) &= \min\{\det(G) : G \in \mathcal{G}_{\psi,n}\}. \end{aligned}$$

The following results were established in [15, 16, 19, 20, 21].

Theorem 2.9. *Let ψ and n be integers such that $\psi \geq 0$ and $n \geq \left\lceil \frac{3+\sqrt{1+8\psi}}{2} \right\rceil$.*

$$\begin{array}{ll} \textit{If } \psi = 0, & \textit{then } D_0(n) = n-1 \textit{ for } n \geq 3 \textit{ and } d_0(n) = \left\lceil \dfrac{-5+\sqrt{8n+41}}{2} \right\rceil \textit{ for } n \geq 3; \\ \textit{If } \psi = 1, & \textit{then } D_1(n) = n-3 \textit{ for } n \geq 6 \textit{ and } d_1(n) = \left\lceil \dfrac{-5+\sqrt{8n+25}}{2} \right\rceil \textit{ for } n \geq 4; \\ \textit{If } \psi = 2, & \textit{then } D_2(n) = n-4 \textit{ for } n \geq 7 \textit{ and } d_2(n) = \left\lceil \dfrac{-5+\sqrt{8n+9}}{2} \right\rceil \textit{ for } n \geq 6. \end{array}$$

Theorem 2.10. *For integers $\psi \geq 1$, $t \geq 3$, and $n \geq t+3$,*

$$D_\psi(n) \geq n-t \qquad \textit{if } \binom{t-2}{2} + 1 \leq \psi \leq \binom{t-1}{2}.$$

Theorem 2.11. *For integers $\psi \geq 1$ and $n \geq 2+2\psi$,*

$$d_\psi(n) \geq \left\lceil \frac{-5+\sqrt{8n+(41-16\psi)}}{2} \right\rceil.$$

It was conjectured in [20] that the lower bound in each of Theorems 2.10 and 2.11 is, in fact, equality.

3. Detectable Factorizations of Graphs

As described in [8], detectable labelings can be looked at from a different point of view. For a connected graph G of order $n \geq 3$ and a factorization

$$\mathcal{F} = \{F_1, F_2, \cdots, F_k\}$$

of G into k subgraphs F_i $(1 \leq i \leq k)$, the color code of a vertex v of G with respect to $\mathcal{F}$ is the ordered k-tuple

$$\text{code}_{\mathcal{F}}(v) = (a_1, a_2, \cdots, a_k) \ \text{(or simply } \text{code}(v) = a_1 a_2 \cdots a_k\text{)}$$

where $a_i = \deg_{F_i} v$ and so

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} \deg_{F_i} v = \deg_G v.$$

If distinct vertices have distinct color codes, then the factorization $\mathcal{F}$ is called a *detectable factorization* of G. A detectable factorization of G with k factors is called a *detectable k-tuple factorization.* Each ordered detectable k-tuple factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_k\}$

of a graph G gives rise to a detectable k-labeling of the edges of G by assigning color i to the edges of F_i for $1 \leq i \leq k$. On the other hand, let c be a k-coloring of the edges of a connected graph G. For each integer i with $1 \leq i \leq k$, let F_i be the spanning subgraph of G whose edges are labeled i. This produces a k-tuple factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_k\}$ of G. Thus a coloring c of the edges of G is a detectable labeling if and only if for each vertex v of G, there exist two distinct factors F_s and F_t in $\mathcal{F}$ such that $\deg_{F_s} v \neq \deg_{F_t} v$. A factorization of G resulting from some detectable labeling of G is a detectable factorization of G. For example, the detectable 3-tuple factorization $\{F_1, F_2, F_3\}$ that results from the detectable 3-labeling of the graph G of Figure 1 is shown in Figure 2.

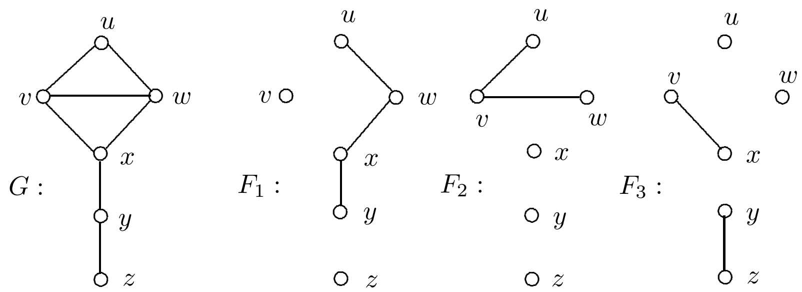

Figure 2: A detectable 3-tuple factorization $\{F_1, F_2, F_2\}$ of a graph

Detectable factorizations of connected regular graphs have been studied in [8, 15, 16], where the following result is stated.

Theorem 3.1. *If G is a regular connected graph of order $n \geq 3$, then* $\det(G) \geq 3$.

The following is a consequence of Theorem 2.1.

Theorem 3.2. *If G is a connected r-regular graph of order n having detection number k, then*

$$n \leq \binom{r+k-1}{r}.$$

By Theorem 3.2 if G is a connected cubic (3-regular) graph of order n with $\det(G) = k$, then $n \leq \binom{k+2}{3}$. In particular, if a cubic graph G contains a detectable 3-labeling, then the order of G is at most 10. The detection number of a connected cubic graph of order at most 10 was determined in [15] (which also provides an answer to a question in [16]).

Theorem 3.3. *Every connected cubic graph whose order is at most* 10 *has detection number* 3.

Thus every connected cubic graph of order at most 10 has a detectable 3-tuple factorization. Such factorizations of cubic graphs were studied in [16]. In particular, it is shown that there are only three possible graphs that can be the factors of a detectable 3-tuple factorization of a connected cubic graph of order 10.

Theorem 3.4. *If G is a connected cubic graph of order* 10 *and $\mathcal{F}$ is a detectable* 3*-tuple factorization of G, then every factor in $\mathcal{F}$ has degree sequence*

$$s : 3, 2, 2, 1, 1, 1, 0, 0, 0, 0.$$

Furthermore, every factor in $\mathcal{F}$ is isomorphic to one of the graphs in Figure 3.

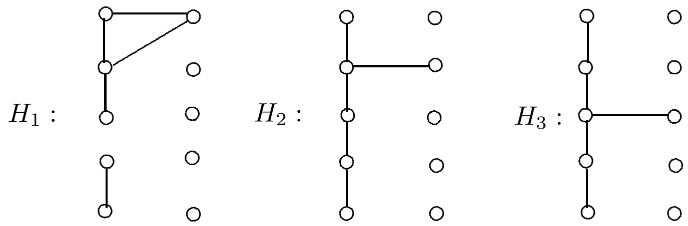

Figure 3: The possible factors in a detectable 3-tuple factorization of connected cubic graph of order 10

Detectable 3-tuple factorizations of a connected cubic graph with additional properties have also been studied in [15, 16]. For graphs F and G, an F*-factorization* of G is a factorization $\mathcal{F}$ of G in which every factor in $\mathcal{F}$ is isomorphic to F. Such a factorization of G is also called an *isomorphic factorization* of G. At the other extreme, if $\mathcal{F}$ is a detectable factorization of a graph G such that no two factors in $\mathcal{F}$ are isomorphic, then $\mathcal{F}$ is an *irregular factorization* of G. The detectable 3-tuple factorizations of the famous Petersen graph have been studied in [16].

Theorem 3.5. *The only graph F for which the Petersen graph has a detectable* 3*-tuple F-factorization is when F is isomorphic to the graph H_2 in Figure* 3.

Since the Petersen graph is triangle-free, the Petersen graph has no irregular factorization by Theorem 3.4. A cubic graph G of order 10 is shown in Figure 4 along with a detectable 3-tuple factorization, which is neither isomorphic nor irregular. In fact, the graph G of Figure 4 has neither a detectable isomorphic 3-tuple factorization nor a detectable irregular 3-tuple factorization (see [16]).

While some cubic graphs of order 10 do not have a detectable isomorphic 3-tuple factorization and some cubic graphs of order 10 do not have a detectable irregular 3-tuple factorization, it is shown in [15] that for the graphs H_1, H_2, and H_3 of Figure 3, the graph of Figure 5 has a detectable 3-tuple factorization containing a factors isomorphic to H_1, b factors isomorphic to H_2, and c factors isomorphic to H_3 for all possible triples (a, b, c) of nonnegative integers a, b and c for which $a+b+c = 3$. That is, (a, b, c) is one of the following: (3,0,0),(0,3,0),(0,0,3),(2,0,1),(2,1,0), (1,2,0),(0,2,1),(0,1,2),(1,0,2),(1,1,1).

We have seen that if G is a connected cubic graph of order n with $\det(G) = k$, then $n \leq \binom{k+2}{3}$. By Theorem 3.1, if G is an r-regular connected graph of order $n \geq 3$, then $\det(G) = k \geq 3$. For $k = 3$, there exists a connected cubic graph G of order

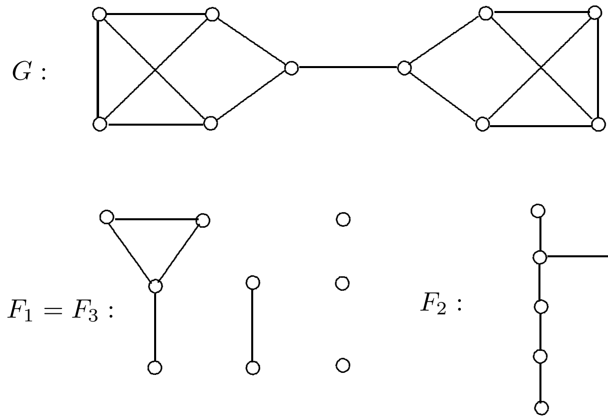

Figure 4: A connected cubic graph G of order 10 and
a detectable 3-tuple factorization G

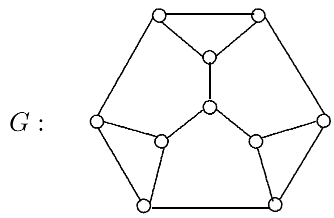

Figure 5: A connected cubic graph G of order 10
having all possible detectable 3-tuple factorizations

$n = \binom{k+2}{3} = \binom{3+2}{3} = 10$ such that $\det(G) = 3$ by Theorem 3.3. On the other hand, it was shown in [16] that for $k \geq 4$, not all connected cubic graphs with detection number k can have order $\binom{k+2}{3}$, however.

Theorem 3.6. *If G is a connected cubic graph of order $\binom{k+2}{3}$ with $\det(G) = k$, then*

$$k \equiv 2 \pmod 4 \ \text{ or } \ k \equiv 3 \pmod 4.$$

If G is a connected cubic graph of order n with detection number 4, then $n \leq 20$. By Theorem 3.6, however, there is no a connected cubic graph of order 20 having detection number 4. Furthermore, the largest order of a connected cubic graph with detection number 5 cannot exceed 34. In fact, the following were shown in [16].

(1) The largest order of a connected cubic graph with detection number 4 is 18.

(2) The largest order of a connected cubic graph with detection number 5 is 32.

The following problem appears in [16]

Problem 3.7. *For each integer $k \geq 6$, what is the largest integer $f(k)$ for which there exists a connected cubic graph of order $f(k)$ with detection number k ?*

For $k = 3$, we have seen that if G is a connected r-regular graph of order $n \geq 3$ having detection number 3, then

$$n \leq \binom{r+2}{r} = \binom{r+2}{2}.$$

It follows from Theorem 3.3 that there exist cubic graphs of order $\binom{3+2}{3} = 10$ whose detection number is 3. Figure 6 shows a connected 4-regular graph of order $\binom{4+2}{2} = 15$ whose detection number is 3. A detectable 3-tuple factorization of this graph is also shown in Figure 6. The next result shows that for some positive integers r, there exists no connected r-regular graph having detection number 3 and order $\binom{r+2}{2.}$

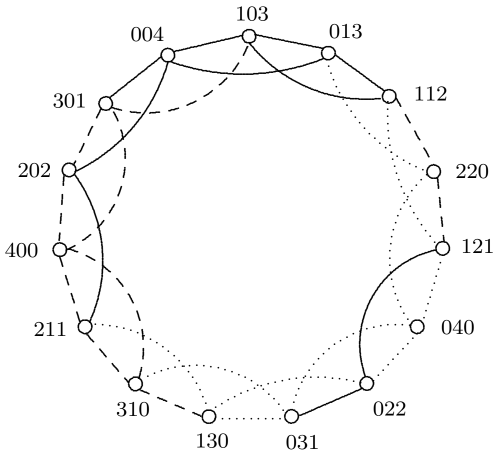

Figure 6: A connected 4-regular graph G of order 15 having detection number 3

Theorem 3.8. *Let G be an r-regular graph of order $n \geq 3$ such that* $\det(G) = 3$. *If* $r \equiv 1 \pmod 4$, *then* $n \leq \binom{r+2}{2} - 1$.

4. Detectable Colorings and a Three-Color Problem

In a proper coloring of a graph G, the object is to assign a color to each vertex of G such that every two adjacent vertices are colored differently but no condition is placed on the colors assigned to nonadjacent vertices. In other words, it is perfectly permissible to assign the same color to two nonadjacent vertices provided that every two adjacent vertices are assigned distinct colors. This suggests the problem of assigning colors from some set $\{1, 2, \ldots, k\}$ to the edges of a graph so that every two *adjacent* vertices are assigned distinct color codes as defined in (1). These color codes are then the colors assigned to the vertices of the graph in this case. We refer to a coloring that accomplishes this as a *detectable coloring* and the smallest number of colors, which when assigned to the edges of a graph results in a detectable coloring, as the *detectable chromatic number* of the graph.

More formally, a coloring c of the edges of a connected graph G is called a *detectable coloring* of G if adjacent vertices of G have distinct color codes. Thus every detectable labeling is a detectable coloring but the converse is not true in general. The graph K_2 is the only connected graph having no detectable coloring. The *detectable chromatic number* $\chi_d(G)$ of G is the minimum positive integer k for which G has a detectable k-coloring. Hence for every connected graph G of order 3 or more, $\chi_d(G) \le \det(G)$. A detectable coloring of a graph G with $\chi_d(G)$ colors is called a *minimum detectable coloring.* Observe that if G is a connected graph of order 3 or more, then $\chi_d(G) = 1$ if and only if no two adjacent vertices of G have the same degree.

Figure 7 shows three graphs G_1, G_2, and G_3, where $\chi_d(G_i) = i$ for $1 \le i \le 3$. A minimum detectable coloring is given in each graph along with the the corresponding color codes of the vertices of G_i for $i = 1, 2$. The corresponding color code for each vertex of G_1 is, in fact, the degree of that vertex.

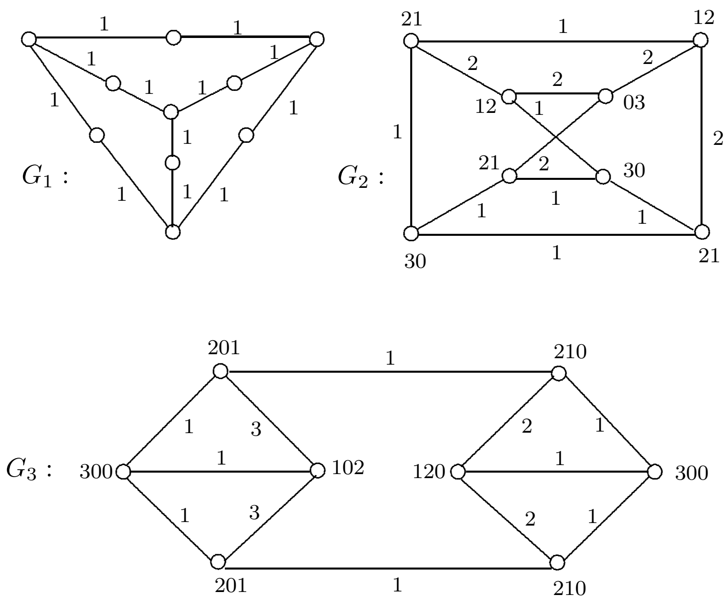

Figure 7: Minimum detectable colorings of graphs

Detectable colorings and detectable chromatic numbers were introduced under different terminology and studied by Karoński, Łuczak, and Thomason [26], who made the following conjecture.

Conjecture 4.1. *For every connected graph G of order 3 or more,*

$$\chi_d(G) \leq 3.$$

In [1] Addario-Berry, Aldred, Dalal, and Reed verified Conjecture 4.1 for all graphs that are either 3-colorable or have minimum degree at least 1000. They also established a general bound for the detectable chromatic number of a graph.

Theorem 4.2. *For every connected graph G of order 3 or more,*

$$\chi_d(G) \leq 4.$$

Whether there is a graph G with $\chi_d(G) = 4$ is not known. However, from the remarks above, any graph G with $\chi_d(G) = 4$ must have chromatic number at least 4 and minimum degree at most 999.

Detectable chromatic numbers were studied further by Chartrand, Escuadro, Okamoto, and Zhang in [17, 18]. The detectable chromatic numbers of several well-known classes of graphs (namely, complete graphs, complete multipartite graphs, cycles, and prisms) were determined.

Theorem 4.3. *If $G = K_{n_1,n_2,\ldots,n_k}$ is a complete k-partite graph of order at least 3, where $k \geq 2$ and $n_1 \leq n_2 \leq \cdots \leq n_k$, then*

$$\chi_d(G) = \begin{cases} 1 & \textit{if } n_1 < n_2 < \cdots < n_k \\ 3 & \textit{if } n_1 = n_2 = \cdots = n_k = 1 \\ 2 & \textit{otherwise.} \end{cases}$$

Theorem 4.4. *For $n \geq 3$,*

$$\chi_d(C_n) = \begin{cases} 2 & \textit{if } n \equiv 0 \pmod 4 \\ 3 & \textit{if } n \not\equiv 0 \pmod 4. \end{cases}$$

Theorem 4.5. *If $n \geq 3$ is an integer, then*

$$\chi_d(C_n \times K_2) = \begin{cases} 3 & \textit{if } n = 3, 5 \\ 2 & \textit{otherwise.} \end{cases}$$

As stated in Theorem 4.4, for $n \geq 3$, $\chi_d(C_n) = 2$ if $n \equiv 0 \pmod 4$ and $\chi_d(C_n) = 3$ if $n \not\equiv 0 \pmod 4$. This result was extended to unicyclic graphs in [17].

Theorem 4.6. *If G a unicyclic graph of order 3 or more, then*

$$\chi_d(G) = \begin{cases} 1 & \textit{if every two adjacent vertices of } G \textit{ have distinct degrees} \\ 3 & \textit{if } G = C_n, \textit{ where } n \not\equiv 0 \pmod 4 \\ 2 & \textit{otherwise.} \end{cases}$$

Theorem 4.6 was extended even further, to several classes of circulants in [18]. For each integer $n \geq 3$ and integers $n_1, n_2, \ldots, n_k$ ($k \geq 1$) such that $1 = n_1 < n_2 < \ldots < n_k \leq \lfloor n/2 \rfloor$, the *circulant* $C_n(n_1, n_2, \ldots, n_k)$ is that graph with n vertices $v_1, v_2, \ldots, v_n$ such that v_i is adjacent to $v_{i \pm n_j (\text{mod } n)}$ for each j with $1 \leq j \leq k$. In particular, if $k = 1$, then $C_n(1) = C_n$. The circulant $C_n(1, 2, \ldots, k)$ is the *k th power* of C_n and is denoted by C_n^k. If $k = \lfloor n/2 \rfloor$, then $C_n^k = K_n$. The circulant $C_n(n_1, n_2, \ldots, n_k)$ is $2k$-regular if $n_k < n/2$ and $(2k-1)$-regular if $n_k = n/2$. Thus circulants are symmetric classes of regular Hamiltonian graphs. The detectable chromatic numbers of all cubic circulants as well as $\chi_d(C_n^k)$ for $2 \leq k \leq 6$ are determined in [18]

Theorem 4.7. *For $n = 2k$, where $k \geq 2$,*

$$\chi_d(C_n(1,k)) = \begin{cases} 3 & \text{if } k = 2 \\ 2 & \text{if } k \geq 3. \end{cases}$$

Theorem 4.8. *For $n \geq 3$,*

$$\begin{aligned} \chi_d(C_n^2) &= \begin{cases} 3 & \text{if } 3 \leq n \leq 5 \\ 2 & \text{if } n \geq 6; \end{cases} \\ \chi_d(C_n^3) &= \begin{cases} 3 & \text{if } 3 \leq n \leq 7 \\ 2 & \text{if } n \geq 8; \end{cases} \\ \chi_d(C_n^4) &= \begin{cases} 3 & \text{if } 3 \leq n \leq 9 \\ 2 & \text{if } n \geq 10; \end{cases} \\ \chi_d(C_n^5) &= \begin{cases} 3 & \text{if } 3 \leq n \leq 11 \\ 2 & \text{if } n \geq 12; \end{cases} \\ \chi_d(C_n^6) &= \begin{cases} 3 & \text{if } 3 \leq n \leq 13 \\ 2 & \text{if } n \geq 14. \end{cases} \end{aligned}$$

Although there is no a complete answer for $\chi_d(C_n^k)$ when $k \geq 7$, we know $\chi_d(C_n^k)$ for all but finitely many integers n for each such integer k.

Theorem 4.9. *Let $k \geq 2$ be an integer. For every sufficiently large integer n,*

$$\chi_d(C_n^k) = 2.$$

It was noted in [18] that

$$\chi_d(C_n^k) = 2 \text{ for } n \geq k(k+1).$$

Consequently, if $\chi_d(C_n^k) > 2$ for some integers n and k, then either

(1) $k = 1$ and $n \not\equiv 0 \pmod{4}$,

(2) $C_n^k = K_n$, or

(3) possibly $2 \le n < k(k+1)$ for some integer $k \ge 7$.

For $n \ge 3$ and $n \not\equiv 0 \pmod 4$, we have seen that the cycle C_n has detectable chromatic number 3. Furthermore, the complete graph K_n also has detectable chromatic number 3. By Theorem 4.8, however, for $2 \le k \le 6$, $\chi_d(C_n^k) = 2$ if and only if C_n^k is neither a cycle nor a complete graph. What characteristics certain circulants possess that result in their detectable chromatic number having the value 3 is not known.

5. Recognizable Colorings of Graphs

In the preceding three sections, we considered edge colorings (or factorizations) of a graph that result in labeling or colorings of the vertices of the graph such that either every two vertices (Sections 2 and 3) or every two adjacent vertices (Section 4) can be distinguished. Another method of uniquely recognizing the vertices of a graph was introduced in [11] by means of coloring the vertices of a graph. Let G be a graph and let $c : V(G) \to \{1, 2, \ldots, k\}$ be a coloring of the vertices of G for some positive integer k (where adjacent vertices may be colored the same). The *color code* of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple

$$\text{code}_c(v) = (a_0, a_1, \ldots, a_k) \;\; (\text{or simply } \text{code}(v) = a_0 a_1 a_2 \cdots a_k\,) \tag{2}$$

where a_0 is the color assigned to v (that is, $c(v) = a_0$) and for $1 \le i \le k$, a_i is the number of vertices adjacent to v that are colored i. Therefore,

$$\sum_{i=1}^{k} a_i = \deg_G v.$$

The coloring c is called *recognizable* if distinct vertices have distinct color codes and the *recognition number* $\text{rn}(G)$ of G is the minimum positive integer k for which G has a recognizable k-coloring. A recognizable coloring of a graph G with $\text{rn}(G)$ colors is called a *minimum recognizable coloring.* These concepts were introduced and studied by Chartrand, Lesniak, VanderJagt, and Zhang in [11].

A 3-coloring of the Petersen graph is given in Figure 8 along with the corresponding color codes of its vertices. Since distinct vertices have distinct color codes, this coloring is recognizable. In fact, it was shown in [11] this 3-coloring is a minimum recognizable coloring of P and so $\text{rn}(P) = 3$.

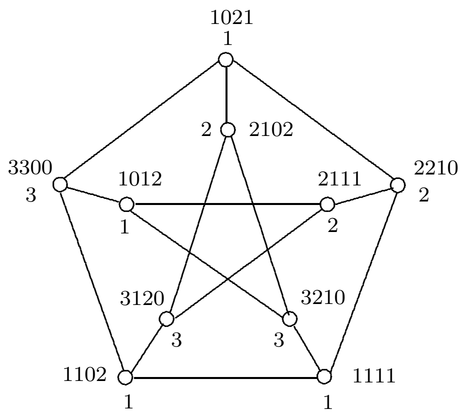

Figure 8: A minimum recognizable coloring of the Petersen graph

Since every coloring that assigns distinct colors to the vertices of a connected graph is recognizable, the recognition number is always defined. On the other hand, every nontrivial graph contains at least two vertices having the same degree. Thus if all vertices of a nontrivial graph are assigned the same color, then any two vertices of the same degree will have the same color code. Therefore, if G is a nontrivial connected graph of order n, then $2 \le \mathrm{rn}(G) \le n$. Among the results established in [11] are the following.

Theorem 5.1. *If c is a recognizable k-coloring of a nontrivial connected graph G, then G contains at most $k\binom{r+k-1}{r}$ vertices of degree r.*

By Theorem 5.1, if G is a connected cubic graph of order n having recognition number 2, then $n \le 8$. It is known that there exists no connected cubic graph of order 8 having recognition number 2, however. Recognition numbers of cycles, paths, and trees in general were investigated.

Theorem 5.2. *Let $k \ge 3$ be an integer. Then $rn(C_n) \ge k$ for all integers n such that*

$$\frac{(k-1)^3+(k-1)^2-2(k-1)+2}{2} \le n \le \frac{k^3+k^2}{2} \qquad \text{if } k \text{ is odd}$$

$$\frac{(k-1)^3+(k-1)^2+2}{2} \le n \le \frac{k^3+k^2-2k}{2} \qquad \text{if } k \text{ is even.}$$

It was conjectured that the lower bound for $\mathrm{rn}(C_n)$ in Theorem 5.2 is, in fact, an equality throughout. With the aid of the proof of Theorem 5.2, a lower bound for $\mathrm{rn}(P_n)$ of a path of order n was established in [11] as well.

Theorem 5.3. *Let $k \ge 3$ be an integer. Then $\mathrm{rn}(P_n) \ge k$ for all integers n such that*

$$\frac{(k-1)^3+(k-1)^2-2(k-1)+10}{2} \le n \le \frac{k^3+k^2+4}{2} \qquad \text{if } k \text{ is odd}$$

$$\frac{(k-1)^3+(k-1)^2+6}{2} \le n \le \frac{k^3+k^2-2k+8}{2} \qquad \text{if } k \text{ is even.}$$

Indeed, it was also conjectured that the lower bound for $\mathrm{rn}(P_n)$ in Theorem 5.3 is an equality. In general, it is known that the maximum recognition number among all trees of order n is $n-1$ for $n \geq 3$. Although there is no known formula for the minimum recognition number among all trees of order $n \geq 3$, we have the following conjecture.

Conjecture 5.4. *For each integer $n \geq 3$, the minimum recognition number among all trees of order n is the unique integer k such that*

$$\frac{(k-1)^3 + 5(k-1)^2 - 2}{2} \leq n \leq \frac{k^3 + 5k^2 - 4}{2}.$$

It is easy to see, however, that the minimum recognition number among all trees of order n is bounded below by the integer k described in Conjecture 5.4.

The recognition number of every complete multipartite graph was also determined in [11]. Let G be a complete k-partite graph for some positive integer k. If every partite set of G has a vertices for some positive integer a, then we write $G = K_{a(k)}$, where then $K_{a(1)} = \overline{K}_a$. If a complete multipartite graph G contains t_i partite sets of cardinality n_i for every integer i with $1 \leq i \leq k$, then we write $G = K_{n_1(t_1), n_2(t_2), \cdots, n_k(t_k)}$.

Theorem 5.5. *Let k and a be positive integers. Then the recognition number of the complete k-partite graph $K_{a(k)}$ is the unique positive integer ℓ for which $\binom{\ell-1}{a} < k \leq \binom{\ell}{a}$. Furthermore, if $G = K_{n_1(t_1), n_2(t_2), \cdots, n_k(t_k)}$, where $n_1, n_2, \ldots, n_k$ are k distinct positive integers, then*

$$\mathrm{rn}(G) = \max\{\mathrm{rn}(K_{n_i(t_i)}) : \ 1 \leq i \leq k\}.$$

Characterizations of those connected graphs of order n having recognition number n or $n-1$ were established in [11].

Theorem 5.6. *If G is a nontrivial connected graph of order n, then*

$$\mathrm{rn}(G) = n \ \textit{if and only if} \ G = K_n.$$

Theorem 5.7. *Let G be a connected graph of order $n \geq 4$. Then*

$$\mathrm{rn}(G) = n-1 \ \textit{if and only if} \ G = K_{1,n-1} \ \textit{or} \ G = C_4.$$

We have seen that if G is a nontrivial connected graph of order n have recognition number k, then $2 \leq k \leq n$. It was shown that every pair k, n of integers with $2 \leq k \leq n$ is realizable as the recognition number and order of some connected graph.

Theorem 5.8. *For each pair k, n of integers with $2 \leq k \leq n$, there exists a connected graph of order n having recognition number k.*

6. Irregular Colorings of Graphs

In the previous section, we described recognizable colorings of a graph where adjacent vertices may be colored the same, that is, a recognizable coloring may not be a proper coloring. In [28], the related parameter concerning proper colorings was developed from recognizable colorings.

Let G be a graph and c a proper k-coloring of the vertices of G for some positive integer k. The *color code* of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple $\text{code}(v) = (a_0, a_1, \ldots, a_k)$ as defined in (2) where a_0 is the color assigned to v and for $1 \leq i \leq k$, a_i is the number of vertices adjacent to v that are colored i. If c is a proper coloring and $\text{code}(v) = (i, a_1, \ldots, a_k)$, then $a_i = 0$. The coloring c is called *irregular* if distinct vertices have distinct color codes and the *irregular chromatic number* $\chi_{ir}(G)$ of G is the minimum positive integer k for which G has an irregular k-coloring. An irregular k-coloring with $\chi_{ir}(G) = k$ is a *minimum irregular coloring.* Since every irregular coloring of a graph G is a proper coloring of G, it follows that

$$\chi(G) \leq \chi_{ir}(G). \tag{3}$$

Irregular colorings and irregular chromatic numbers were introduced and studied by Chartrand, Radcliffe, Okamoto, and Zhang in [28, 29, 30].

A 4-coloring of the Petersen graph is given in Figure 9 along with the corresponding color codes of its vertices. Since distinct vertices have distinct codes, this coloring is irregular and so $\chi_{ir}(P) \leq 4$. Since $\chi(P) = 3$, it follows that $\chi_{ir}(P) = 3$ or $\chi_{ir}(P) = 4$. In fact, it was shown in [28] that $\chi_{ir}(P) = 4$.

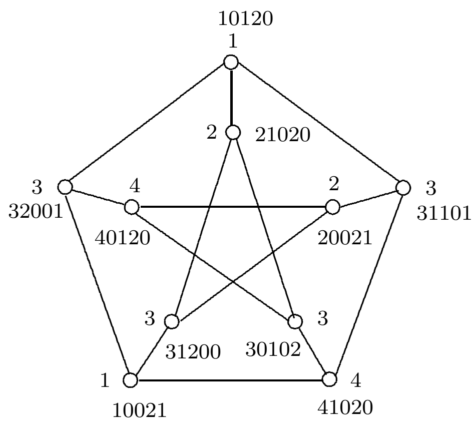

Figure 9: A minimum irregular coloring of the Petersen graph P

It is well-known that if H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$. However, this is not true for the irregular chromatic number of a graph. The following result was presented in [30].

Theorem 6.1. *For every pair* a, b *of integers with* $2 \leq a \leq b$ *there exists a graph* $G_{a,b}$ *containing an induced subgraph* $H_{a,b}$ *such that*

$$\chi_{ir}(G_{a,b}) = a \text{ and } \chi_{ir}(H_{a,b}) = b.$$

Among the results obtained in [28, 29, 30] were the following.

Theorem 6.2. *For every pair* a, b *of integers with* $2 \leq a \leq b$, *there is a connected graph* G *with* $\chi(G) = a$ *and* $\chi_{ir}(G) = b$.

Theorem 6.3. *If a nontrivial connected graph* G *has an irregular* k*-coloring, then* G *contains at most* $k\binom{r+k-2}{r}$ *vertices of degree* r.

Since every nontrivial graph G has at least two vertices of the same degree, every irregular coloring of G must use at least two distinct colors. Furthermore, the coloring of a graph G that assigns distinct colors to distinct vertices of G is irregular and so $\chi_{ir}(G)$ always exists. Therefore, if G is a nontrivial graph of order n, then $2 \leq \chi_{ir}(G) \leq n$. Connected graphs of order $n \geq 2$ having irregular chromatic number 2 or n were characterized in [28], which we state next.

Theorem 6.4. *Let* G *be a nontrivial connected graph of order* n. *Then*

(a) $\chi_{ir}(G) = 2$ *if and only if* n *is even and* $G \cong F_n$, *where* $n = 2k$ *for some positive integer* k *and* F_n *is the bipartite graph with partite sets* $X = \{x_1, x_2, \cdots, x_k\}$ *and* $Y = \{y_1, y_2, \ldots, y_k\}$ *such that* $\deg x_i = \deg y_i = i$ *for* $1 \leq i \leq k$ *;*

(b) $\chi_{ir}(G) = n$ *if and only if* G *is a complete multipartite graph.*

Furthermore, for each pair k, n *of integers with* $2 \leq k \leq n$, *there exists a connected graph of order* n *having irregular chromatic number* k *if and only if* (1) $k = 2$ *and* n *is even or* (2) $k \geq 3$.

Irregular chromatic numbers of disconnected graphs were studied in [29]. Sharp upper and lower bounds for the irregular chromatic number of a disconnected graph were established in terms of the irregular chromatic numbers of its components.

Theorem 6.5. *Let* G_1, G_2, $\ldots$, G_p *be* $p \geq 2$ *graphs and let* $G = G_1 \cup G_2 \cup \cdots \cup G_p$. *Then*

$$\max\{\chi_{ir}(G_i) : 1 \leq i \leq p\} \leq \chi_{ir}(G) \leq \sum_{i=1}^{p} \chi_{ir}(G_i).$$

Both the lower and upper bounds in Theorem 6.5 are sharp. Furthermore, the following result appeared in [29].

Theorem 6.6. *Let G_1, G_2, $\ldots$, G_p be $p \geq 2$ graphs. Then*

$$\chi_{ir}(G_1 \cup G_2 \cup \cdots \cup G_p) = \sum_{i=1}^{p} \chi_{ir}(G_i)$$

if and only if each G_i contains exactly $\chi_{ir}(G_i)$ isolated vertices for all i with $1 \leq i \leq p$.

With the aid of information obtained about the irregular chromatic number of a disconnected graph, all nontrivial graphs G of order n with irregular chromatic number 2 were characterized and irregular chromatic numbers of those disconnected graphs whose components are complete multipartite graphs were determined.

Theorem 6.7. *Let G be a graph of order $n \geq 2$. Then $\chi_{ir}(G) = 2$ if and only if (1) n is even and $G \in \{2K_1, F_n, 2K_1 \cup F_{n-2}\}$, or (2) n is odd and $G = K_1 \cup F_{n-1}$.*

Theorem 6.8. *Let $G = pK_n$ where $p \geq 2$ and $n \geq 2$. Then $\chi_{ir}(G) = k$ if and only if $\binom{k-1}{n} + 1 \leq p \leq \binom{k}{n}$.*

Theorem 6.9. *Let $G = p_1K_{n_1} \cup p_2K_{n_2} \cup \cdots \cup p_tK_{n_t}$, where $p_i \geq 1$ and $n_1 < n_2 < \cdots < n_p$ for $1 \leq i \leq t$. For each integer i with $1 \leq i \leq t$, let k_i be the smallest positive integer such that $\binom{k_i}{n_i}/p_i \geq 1$. Then*

$$\chi_{ir}(G) = \max\{k_i : \ 1 \leq i \leq t\}.$$

Theorem 6.10. *Let $p \geq 2$ and $r \geq 2$ be integers. If s is the smallest integer such that $p \leq \frac{1}{2}\binom{2r}{r}\binom{s}{2}$, then $\chi_{ir}(pK_{r,r}) \leq sr$. Furthermore, if $p = \frac{1}{2}\binom{2r}{r}\binom{s}{2}$, then $\chi_{ir}(pK_{r,r}) = sr$.*

A well-known result involving a graph and its complement provides upper and lower bounds for both the sum and the product of the chromatic numbers of a graph and its complement, which are called the *Nordhaus-Gaddum inequalities* and are due to Nordhaus and Gaddum [27].

Theorem 6.11. *For every graph G of order n,*

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1 \ \text{ and } \ n \leq \chi(G)\chi(\overline{G}) \leq \left(\tfrac{n+1}{2}\right)^2,$$

Nordhaus-Gaddum inequalities for the irregular chromatic number of a graph and its complement were established in [29], which are stated next.

Theorem 6.12. *If G is a graph of order n, then*

$$2\sqrt{n} \leq \chi_{ir}(G) + \chi_{ir}(\overline{G}) \leq 2n \ \text{ and } \ n \leq \chi_{ir}(G)\chi_{ir}(\overline{G}) \leq n^2.$$

Each of the four bounds in Theorem 6.12 is sharp. In fact, more can be said. The following three results appear in [29].

Theorem 6.13. *If G is a graph of order n, then $\chi_{ir}(G) + \chi_{ir}(\overline{G}) = 2n$ (or $\chi_{ir}(G)\chi_{ir}(\overline{G}) = n^2$) if and only if $G = K_n$ or $G = \overline{K}_n$.*

Theorem 6.14. *For every positive even integer n, there exists a connected graph G of order n such that $\chi_{ir}(G)\chi_{ir}(\overline{G}) = n$.*

Theorem 6.15. *For every positive integer p, there exists a graph G of order $n = p^2$ such that $\chi_{ir}(G) + \chi_{ir}(\overline{G}) = 2\sqrt{n}$.*

A point (a, b) in the plane is called a *lattice point* if a and b are integers. If G is a graph of order n such that $\chi_{ir}(G) = a$ and $\chi_{ir}(\overline{G}) = b$, then $2\sqrt{n} \leq a + b \leq 2n$ and $n \leq ab \leq n^2$, which is equivalent to $n \leq ab$ and $a + b \leq 2n$. For this reason, we define a lattice point (a, b) to be *realizable* with respect to an integer n if $n \leq ab$ and $a + b \leq 2n$ and there is a graph G of order n such that $\chi_{ir}(G) = a$ and $\chi_{ir}(\overline{G}) = b$. For the chromatic number, Stewart [32] and and Finck [22] showed that no improvement in Theorem 6.11 is possible (without employing additional conditions).

Theorem 6.16. *Let n be a positive integer. For every two integers a and b such that*

$$2\sqrt{n} \leq a + b \leq n + 1, \text{ and } n \leq ab \leq \left(\tfrac{n+1}{2}\right)^2$$

there is a graph G of order n such that $\chi(G) = a$ and $\chi(\overline{G}) = b$.

There is, however, no corresponding result for irregular chromatic numbers. It was memtioned in [29] that if a, b, and $n \geq 2$ are positive integers such that one of a and b is 2 but the other is none of $\frac{n}{2}$, $\frac{n+1}{2}$, or $\frac{n+2}{2}$, then (a, b) is not realizable with respect to n. The following question appears in [29].

Problem 6.17. *Let $n \geq 2$ be an integer. Which lattice points (a, b) are realizable with respect to n ?*

All realizable lattice points with respect to an integer $n = 9$ have been determined in [29]. Irregular chromatic numbers of cycles and trees were investigated in [28, 30]. In paricular, following result was established in [28].

Theorem 6.18. *Let $k \geq 3$ and $n = k\binom{k}{2}$. If $\chi_{ir}(C_n) = k$, then $\chi_{ir}(C_{n-1}) = k + 1$.*

The irregular chromatic number of every cycle was determined by Anderson, Barrientos, Brigham, Carrington, Kronman, Vitray, and Yellen in [5].

Theorem 6.19. *Let $k \geq 4$. If $(k-1)\binom{k-1}{2} + 1 \leq n \leq k\binom{k}{2}$, then*

$$\chi_{ir}(C_n) = \begin{cases} k & \text{if } n \neq k\binom{k}{2} - 1 \\ k + 1 & \text{if } n = k\binom{k}{2} - 1. \end{cases}$$

As a consequence of Theorem 6.19, the irregular chromatic number of every path was determined in [5].

Corollary 6.20. *Let $n \geq 5$. If k is the unique integer such that*

$$(k-1)\binom{k-1}{2} + 3 \leq n \leq k\binom{k}{2} + 2,$$

then $\chi_{ir}(P_n) = k$.

The irregular chromatic numbers of trees were also studied in [30]. For each integer $n \geq 2$, let $D_T(n)$ be the maximum irregular chromatic number among all trees of order n and let $d_T(n)$ be the minimum irregular chromatic number among all trees of order n. Therefore,

$$2 \leq d_T(n) \leq D_T(n) \leq n$$

for $n \geq 2$. Since $\chi_{ir}(K_{1,n-1}) = n$ for $n \geq 2$, it follows that $D_T(n) = n$ for $n \geq 2$. As for $d_T(n)$, it is easy to see that $d_T(n) = n$ for $n = 2, 3$ and $d_T(4) = 2$, while for $n \geq 5$, there is a lower bound for $d_T(n)$.

Theorem 6.21. *Let $n \geq 5$ be an integer. If k is the unique integer such that*

$$\frac{k^3 - 7k + 4}{2} \leq n \leq \frac{k^3 + 3k^2 - 4k - 4}{2},$$

then $d_T(n) \geq k$.

It was conjectured in [30] that the lower bound in Theorem 6.21 is, in fact, equality. Futhermore, the conjecture has been verified for $5 \leq n \leq 100$ and for many integers $n > 100$.

References

[1] L. Addario-Berry, R. E. L. Aldred, K. Dalal, and B. A. Reed, Vertex colouring edge partitions, *J. Combin. Theory Ser. B,* **94** (2005), 237-244.

[2] M. Aigner and E. Triesch, Irregular assignments and two problems á la Ringel, *Topics in Combinatorics and Graph Theory*, (R. Bodendiek and R. Henn, eds.), Physica, Heidelberg (1990), 29–36.

[3] M. Aigner, E. Triesch and Z. Tuza, Irregular assignments and vertex-distinguishing edge-colorings of graphs, *Combinatorics' 90* Elsevier Science Pub., New York (1992), 1–9.

[4] M. Albertson and K. Collins, Symmetric breaking in graphs, *Electron. J. Combin.,* **3** (1996), R18.

[5] M. Anderson, C. Barrientos, R. C. Brigham, J. R. Carrington, M. Kronman, R. P. Vitray, and J. Yellen, Irregular colorings of some graph classes, *Bull. Inst. Combin. Appl.,* (To appear).

[6] A. C. Burris, On graphs with irregular coloring number 2, *Congr. Numer.,* **100** (1994), 129–140

[7] A. C. Burris, The irregular coloring number of a tree, *Discrete Math.,* **141** (1995), 279–283.

[8] G. Chartrand, H. Escuadro, F. Okamoto, and P. Zhang, Detectable colorings of graphs, *Util. Math.,* **69** (2006), 13-32.

[9] G. Chartrand, K. S. Holbert, O. R. Oellermann, and H. C. Swart, F-Degrees in graphs, *Ars Combin.,* **24** (1987), 133-148.

[10] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz, and F. Saba, Irregular networks, *Congr. Numer.*, **64** (1988), 197-210.

[11] G. Chartrand, L. Lesniak, D. W. VanderJagt, and P. Zhang, Recognizable colorings of graphs, *Discuss. Math. Graph Theory*, (To appear).

[12] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, McGraw-Hill, Boston (2005).

[13] R. C. Entringer and L. D. Gassman, Line-critical point determining and point distinguishing graphs, *Discrete Math.,* **10** (1974), 43-55.

[14] D. Erwin and F. Harary, Destroying authomorphisms by fixing nodes, *Discrete Math.,* **306** (2006), 3244-3252.

[15] H. Escuadro, *Detectable Colorings of Graphs*, Ph.D. Dissertation, Western Michigan University (2006).

[16] H. Escuadro, F. Okamoto, and P. Zhang, On detectable factorizations of cubic graphs, *J. Combin. Math. Combin. Comput.,* **56** (2006), 47-63.

[17] H. Escuadro, F. Okamoto, and P. Zhang, A three-color problem in graph theory, *Bull. Inst. Combin. Appl.,* (To appear).

[18] H. Escuadro, F. Okamoto, and P. Zhang, Circulants and a three-color conjecture, *Congr. Numer.,* **178** (2006), 33-55.

[19] H. Escuadro and P. Zhang, On detectable colorings of graphs, *Math. Bohem.*, **130** (2005), 427-445.

[20] H. Escuadro and P. Zhang, Extremal problems on detectable colorings of connected graphs with cycle rank 2, *AKCE J. Graphs. Combin.*, **2** (2005), 99-117.

[21] H. Escuadro and P. Zhang, Extremal problems on detectable colorings of trees, *Discrete Math.,* (To appear).

[22] H. J. Finck, On the chromatic numbers of a graph and its complement, 1968 *Theory of Graphs*, (Proc. Colloq., Tihany, 1966) 99–113, Academic Press, New York

[23] F. Harary, Methods of destroying the symmetries of a graph, *Bull. Malays. Math. Sci. Soc.,* **24** (2001), 183-191.

[24] F. Harary and R. A. Melter, On the metric dimension of a graph, *Ars Combin.,* **2** (1976), 191-195.

[25] F. Harary and M. Plantholt, The point-distinguishing chromatic index, *Graphs and Applications*, Wiley, New York (1985), 147-162.

[26] M. Karoński, T. Łuczak, and A. Thomason, Edge weights and vertex colours, *J. Combin. Theory Ser. B*, **91** (2004), 151-157.

[27] E. A. Nordhaus and J. W. Gaddum, On complementary graphs, *Amer. Math. Monthly*, **63** (1956) 175-177.

[28] M. Radcliffe and P. Zhang, Irregular colorings of graphs, *Bull. Inst. Combin. Appl.*, **49** (2007), 41-59.

[29] M. Radcliffe and P. Zhang, On irregular colorings of graphs, *AKCE J. Graphs. Combin.,* **3** (2006), 175-191.

[30] M. Radcliffe, F. Okamoto, and P. Zhang, On the irregular chromatic number of a graph, *Congr. Numer.,* **181** (2006), 129-150.

[31] P. J. Slater, Leaves of trees, *Congr. Numer.,* **14** (1975), 549-559.

[32] B. M. Stewart, On a theorem of Nordhaus and Gaddum, *J. Combin. Theory*, **6** (1969), 217-218.

[33] D. P. Sumner, Point determination in graphs, *Discrete Math.*, **5** (1973), 179-187.