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# Signed Degree Sequences in Signed Bipartite Graphs

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**Abstract**

A signed bipartite graph is a bipartite graph in which each edge is assigned a positive or a negative sign. Let $G(U,V)$ be a signed bipartite graph with $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$. Then signed degree of $u_i$ is $sdeg(u_i) = d_i = d_i^+ - d_i^-$, where $1 \leq i \leq p$ and $d_i^+$ ( $d_i^-$ ) is the number of positive(negative) edges incident with $u_i$, and signed degree of $v_j$ is $sdeg(v_j) = e_j = e_j^+ - e_j^-$, where $1 \leq j \leq q$ and $e_j^+$ ( $e_j^-$ ) is the number of positive(negative) edges incident with $v_j$. Clearly, $|d_i| \leq q$ and $|e_j| \leq p$. So the sequences $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ are called the signed degree sequences of $G(U,V)$. In this paper, we give characterizations of signed degree sequences in signed bipartite graphs.

**Keywords:** Signed graph, signed bipartite graph, signed degree, signed degree sequence.

**2000 Mathematics Subject Classification:** 05C22.

## 1. Introduction

A signed graph is a graph in which each edge is assigned a positive or a negative sign. These were first introduced by Harary [3]. Let $G$ be a signed graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Then signed degree of $v_i$ is $sdeg(v_i) = d_i = d_i^+ - d_i^-$, where $1 \leq i \leq n$ and $d_i^+$ ( $d_i^-$ ) is the number of positive(negative) edges incident with $v_i$. So the sequence $\sigma = [d_1, d_2, \cdots, d_n]$ in non-increasing order is called signed degree sequence of $G$. An integral sequence is $s$-graphical if it is the signed degree sequence of a signed graph. Also, a non-zero sequence $\sigma = [d_1, d_2, \cdots, d_n]$ is a standard sequence if $\sigma$ is non-increasing, $\sum_{i=1}^{n} d_i$ is even, $d_1 > 0$, each $|d_i| < n$, and $|d_1| \geq |d_n|$.

The following result, due to Chartrand et al. [1], gives a necessary and sufficient condition for an integral sequence to be s-graphical, which is similar to Hakimi's result for degree sequences in graphs [2].

**Theorem 1.1.** *A standard integral sequence* $\sigma = [d_1, d_2, \cdots, d_n]$ *is* $s$*-graphical if and only if* $\sigma' = [d_2 - 1, d_3 - 1, \cdots, d_{d_1+s+1} - 1, d_{d_1+s+2}, \cdots, d_{n-s}, d_{n-s+1} + 1, \cdots, d_n + 1]$, *is* $s$*-graphical for some* $s$, $0 \leq s \leq \frac{n-1-d_1}{2}$.

The next characterization for signed degree sequences in signed graphs is given by Yan et al. [5].

**Theorem 1.2.** *A standard integral sequence* $\sigma = [d_1, d_2, \cdots, d_n]$ *is* $s$*-graphical if and only if* $\sigma'_m = [d_2 - 1, d_3 - 1, \cdots, d_{d_1+m+1} - 1, d_{d_1+m+2}, \cdots, d_{n-m}, d_{n-m+1} + 1, \cdots, d_n + 1]$, *is* $s$*-graphical, where* $m$ *is the maximum non-negative integer such that* $d_{d_1+m+1} > d_{n-m+1}$.

The set of distinct signed degrees of the vertices in a signed graph is called its signed degree set. Pirzada et al. [4] proved that every non-empty set of positive(negative) integers is the signed degree set of some connected signed graph and determined the smallest possible order for such a signed graph.

## 2. Characterizations of signed degree sequences in signed bipartite graphs

Two sequences $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ of integers are $s$-graphical if $\alpha$ and $\beta$ are the signed degree sequences of some signed bipartite graph. We denote a positive edge $xy$ by $xy^+$ and a negative edge $xy$ by $xy^-$.

**Remark 2.1.** *We note that Theorems 1.1 and 1.2 are valid for signed bipartite graphs as every signed bipartite graph is a signed graph. Now we proceed to give the characterizations of signed degree sequences in signed bipartite graphs based on the property that a signed bipartite graph has a pair of signed degree sequences. Our results need not hold for signed graphs in general as every signed graph need not be a signed bipartite graph.*

First we have the following result, which is analogous to Theorem 4 [1].

**Theorem 2.2.** *Let* $G(U, V)$ *be a signed bipartite graph with* $m$ *edges. Then,*

$$g = \sum_{u \in U} sdeg(u) = \sum_{v \in V} sdeg(v) \equiv m (\bmod 2),$$

*and the number of positive edges and negative edges of* $G(U, V)$ *are respectively* $\frac{m+g}{2}$ *and* $\frac{m-g}{2}$.

*Proof.* Taking $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$, let $u_i$ $(1 \leq i \leq p)$ be incident with $d_i^+$ positive edges and $d_i^-$ negative edges, while $v_j$ $(1 \leq j \leq q)$ be incident with $e_j^+$ positive edges and $e_j^-$ negative edges. Then, $sdeg(u_i) = d_i^+ - d_i^-$ while $deg(u_i) = d_i^+ + d_i^-$ for $1 \leq i \leq p$, and $sdeg(v_j) = e_j^+ - e_j^-$ while $deg(v_j) = e_j^+ + e_j^-$ for $1 \leq j \leq q$. Clearly,

$$\sum_{i=1}^{p} \deg(u_i) = \sum_{j=1}^{q} \deg(v_j) = m\,.$$

Suppose that $G(U,V)$ have $x$ positive edges and $y$ negative edges. Then, $m = x + y$, $\sum_{i=1}^{p} d_i^+ = \sum_{j=1}^{q} e_j^+ = x$ and $\sum_{i=1}^{p} d_i^- = \sum_{j=1}^{q} e_j^- = y$. Also, $\sum_{i=1}^{p} sdeg(u_i) = \sum_{i=1}^{p} \left(d_i^+ - d_i^-\right) = \sum_{i=1}^{p} d_i^+ - \sum_{i=1}^{p} d_i^- = x - y$. Similarly, $\sum_{j=1}^{q} sdeg(v_j) = x - y$. Thus, $g = \sum_{i=1}^{p} sdeg(u_i) = \sum_{j=1}^{q} sdeg(v_j) = x - y = m - y - y = m - 2y$, so that $g \equiv m (\mathrm{mod}\, 2)$. From $x + y = m$ and $x - y = g$, we have $x = \frac{m+g}{2}$ and $y = \frac{m-g}{2}$. □

**Remark 2.3.** *A necessary condition for the two sequences* $\alpha = [d_1, d_2, \cdots, d_p]$ *and* $\beta = [e_1, e_2, \cdots, e_q]$ *of integers to be* $s$*-graphical is that* $\sum_{i=1}^{p} d_i = \sum_{j=1}^{q} e_j$.

A zero sequence is a finite sequence each term of which is 0. Obviously, every two finite zero sequences are the signed degree sequences of a signed bipartite graph. If $\gamma = [a_1, a_2, \cdots, a_n]$ is a sequence of integers, then the negative of $\gamma$ is the sequence $-\gamma = [-a_1, -a_2, \cdots, -a_n]$. From these definitions, we have the following observation.

**Remark 2.4.** *Let* $\alpha = [d_1, d_2, \cdots, d_p]$ *and* $\beta = [e_1, e_2, \cdots, e_q]$ *be the signed degree sequences of a signed bipartite graph* $G(U,V)$*, then* $-\alpha = [-d_1, -d_2, \cdots, -d_p]$ *and* $-\beta = [-e_1, -e_2, \cdots, -e_q]$ *are the signed degree sequences of the signed bipartite graph* $G'(U,V)$ *obtained from* $G(U,V)$ *by interchanging positive edges with the negative edges.*

Two sequences of integers $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ are said to be standard sequences if

(A) (i) $\alpha$ is non-zero,

(ii) $\alpha$ is non-increasing and $|d_1| \geq |d_p|$, for we may always replace $\alpha$ and $\beta$ by $-\alpha$ and $-\beta$ if necessary,

(iii) $\sum_{i=1}^{p} d_i = \sum_{j=1}^{q} e_j$,

(iv) $d_1 > 0$,

(v) each $|d_i| \leq q$, each $|e_j| \leq p$, and each $|e_j| \leq |d_1|$.

or

(B) (i) $\beta$ is non-zero,

(ii) $\beta$ is non-increasing and $|e_1| \geq |e_q|$, for we may always replace $\alpha$ and $\beta$ by $-\alpha$ and $-\beta$ if necessary,

(iii) $\sum_{i=1}^{p} d_i = \sum_{j=1}^{q} e_j$,

(iv) $e_1 > 0$,

(v) each $|d_i| \leq q$, each $|e_j| \leq p$, and each $|d_i| \leq |e_1|$.

For the next results, we assume (A).

A complete signed bipartite graph is a complete bipartite graph in which each edge is assigned a positive or a negative sign.

The following result gives a necessary and sufficient condition for a pair of integral sequences to be the signed degree sequences of some complete signed bipartite graph. This is analogous to Theorem 5 [1] on signed degree sequences in complete signed graphs.

**Theorem 2.5.** *Let $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ be standard sequences and let $r = \frac{d_1+q}{2}$. Let $\alpha'$ be obtained from $\alpha$ by deleting $d_1$ and $\beta'$ be obtained from $\beta$ by reducing $r$ greatest entries of $\beta$ by 1 each and adding remaining entries of $\beta$ by 1 each. Then $\alpha$ and $\beta$ are the signed degree sequences of some complete signed bipartite graph if and only if $\alpha'$ and $\beta'$ are also.*

*Proof.* Let $G'(U', V')$ be a complete signed bipartite graph with signed degree sequences $\alpha'$ and $\beta'$. Let $U' = \{u_2, u_3, \cdots, u_p\}$ and $V' = \{v_1, v_2, \cdots, v_q\}$ such that $sdeg(u_i) = d_i$, for $2 \leq i \leq p$, and

$$sdeg(v_i) = \begin{cases} e'_i - 1, & \text{for } 1 \leq i \leq r, \\ e'_i + 1, & \text{for } r+1 \leq i \leq q, \end{cases}$$

where $e'_i = e_j$ for $1 \leq i, j \leq q$ such that $e'_1 \geq e'_2 \geq \cdots \geq e'_q$.

Now, construct a new complete signed bipartite graph $G(U, V)$ as follows.

Let $U = \{u_1\} \cup U'$, $V = V'$ with $\{u_1\} \cap U' = \phi$, and join $u_1$ by positive edges to the vertices $v_1$, $v_2, \cdots, v_r$ and join $u_1$ by negative edges to the vertices $v_{r+1}, v_{r+2}, \cdots, v_q$. Then, $sdeg(u_1) = r - (q - r) = 2r - q = d_1$, $sdeg(u_i) = d_i$, for $2 \leq i \leq p$ and

$$sdeg(v_i) = \begin{cases} e'_i - 1 + 1, & \text{for } 1 \leq i \leq r, \\ e'_i + 1 - 1, & \text{for } r+1 \leq i \leq q, \end{cases}$$

$$= e'_i, \quad \text{for } 1 \leq i \leq q,$$
$$= e_j, \quad \text{for } 1 \leq j \leq q.$$

Hence, $G(U,V)$ is a complete signed bipartite graph with signed degree sequences $\alpha$ and $\beta$.

Conversely, let $\alpha$ and $\beta$ be the signed degree sequences of a complete signed bipartite graph. Let $e'_i = e_j$ for $1 \leq i,j \leq q$ such that $e'_1 \geq e'_2 \geq \cdots \geq e'_q$. Then, $\beta_1 = [e'_1, e'_2, \cdots, e'_q]$ is non-increasing sequence. Suppose the vertex sets of the complete signed bipartite graph be $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$ such that $sdeg(u_i) = d_i$, for $1 \leq i \leq p$, and $sdeg(v_j) = e'_j$, for $1 \leq j \leq q$.

Among all the complete signed bipartite graphs with $\alpha$ and $\beta_1$ as the signed degree sequences, let $G(U,V)$ be one with the property that the sum $S$ of the signed degrees of the vertices of $V$ joined to $u_1$ by positive edges is maximum. Let $d_1^+$ and $d_1^-$ be respectively the number of positive edges and number of negative edges incident with $u_1$. Then, $sdeg(u_1) = d_1 = d_1^+ - d_1^-$ and $deg(u_1) = d_1^+ + d_1^- = q$. Therefore, $d_1^+ = \frac{d_1+q}{2} = r$. We claim that $u_1$ must be joined by positive edges to the vertices of $V$ having signed degrees $e'_1, e'_2, \cdots, e'_r$. If this is not true, then there exist vertices $v_i$ and $v_j$ in $V$ with $j > i$ such that the edge $u_1v_i$ is negative and the edge $u_1v_j$ is positive. Since $\beta_1$ is in non-increasing order, then $sdeg(v_i) \geq sdeg(v_j)$, that is, $e'_i \geq e'_j$. Thus, there is a vertex $u_n$ ($\neq u_1$) of $U$ such that $u_nv_i$ is a positive edge and $u_nv_j$ is a negative edge. Now, change the signs of these edges so that $u_1v_j$ and $u_nv_i$ are negative and $u_1v_i$ and $u_nv_j$ are positive. Thus, we obtain a complete signed bipartite graph with signed degree sequences $\alpha$ and $\beta_1$ in which the sum of the signed degrees of the vertices of $V$ joined to $u_1$ by positive edges exceeds $S$, which is a contradiction.

Hence, we may assume that $u_1$ is joined by positive edges to the vertices $v_1, v_2, \cdots, v_r$ and by negative edges to the vertices $v_{r+1}, v_{r+2}, \cdots, v_q$. Then $G(U,V) - u_1$ is a complete signed bipartite graph with $\alpha'$ and $\beta'$ as the signed degree sequences. □

Theorem 2.5 provides an algorithm for determining whether or not the standard sequences $\alpha$ and $\beta$ are the signed degree sequences, and for constructing a corresponding complete signed bipartite graph. Suppose $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ be the standard signed degree sequences of a complete signed bipartite graph with parts $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$. Deleting $d_1$ and reducing $r = \frac{d_1+q}{2}$ greatest entries of $\beta$ by 1 each and adding remaining entries of $\beta$ by 1 each to form $\beta' = [e'_1, e'_2, \cdots, e'_q]$. Then edges are defined by $u_1v_i^+$ for which $e'_j = e_i - 1$ and $u_1v_i^-$ for which $e'_j = e_i + 1$. For $-\alpha$ and $-\beta$, the edges are defined by $u_1v_i^-$ for which $e'_j = e_i - 1$ and $u_1v_i^+$ for which $e'_j = e_i + 1$. Now, if (A) is not satisfied then we use (B) in which we delete $e_1$ and reduce $r = \frac{e_1+p}{2}$ greatest entries of $\alpha$ by 1 each and adding remaining entries of $\alpha$ by 1 each to form $\alpha' = [d'_1, d'_2, \cdots, d'_p]$. In this case, edges are defined by $v_1u_i^+$ for which $d'_j = d_i - 1$ and $v_1u_i^-$ for which $d'_j = d_i + 1$. For $-\alpha$ and $-\beta$, the edges are defined by $v_1u_i^-$ for which $d'_j = d_i - 1$ and $v_1u_i^+$ for which $d'_j = d_i + 1$. If this method is applied successively, then (i) it tests whether or not $\alpha$ and $\beta$ are signed degree sequences, and if $\alpha$ and $\beta$ are signed degree sequences (ii) a complete signed bipartite graph $\Delta(\alpha, \beta)$ with signed degree sequences $\alpha$ and $\beta$ is constructed.

We illustrate this algorithm with the following example, beginning with standard sequences $\alpha_1$ and $\beta_1$ .
$\alpha_1 = [4, 4, 0, -2]$ , $\beta_1 = [4, 2, 2, 0, 0, -2]$.
Here $d_1 = 4$ , $q = 6$ , so $r = \frac{4+6}{2} = 5$ .
$\alpha_2 = [4, 0, -2]$ , $\beta_2 = [3, 1, 1, -1, -1, -1]$
$u_1v_1^+$ , $u_1v_2^+$ , $u_1v_3^+$ , $u_1v_4^+$ , $u_1v_5^+$ , $u_1v_6^-$ .
Here $d_2 = 4$ , $q = 6$ , so $r = \frac{4+6}{2} = 5$ .
$\alpha_3 = [0, -2]$ , $\beta_3 = [2, 0, 0, -2, -2, 0]$
$u_2v_1^+$ , $u_2v_2^+$ , $u_2v_3^+$ , $u_2v_4^+$ , $u_2v_5^+$ , $u_2v_6^-$ .
Write $\alpha_3$ and $\beta_3$ in standard form, we have
- $\alpha_3 = [2, 0]$ , $-\beta_3 = [-2, 0, 0, 2, 2, 0]$ .
Here $d_4 = 2$ , $q = 6$ , so $r = \frac{2+6}{2} = 4$ .
- $\alpha_4 = [0]$ , $-\beta_4 = [-1, 1, -1, 1, 1, -1]$
$u_4v_3^-$ , $u_4v_4^-$ , $u_4v_5^-$ , $u_4v_6^-$ , $u_4v_1^+$ , $u_4v_2^+$ .
Here $-\alpha_4$ is zero sequence and - $\beta_4$ is non-zero sequence.
Write $-\alpha_4$ and $-\beta_4$ in standard form, we have
$\alpha_4 = [0]$ , $\beta_4 = [1, -1, 1, -1, -1, 1]$.
Here $e_1 = 1$ , $p = 1$ , so $r = \frac{1+1}{2} = 1$ .
$\alpha_5 = [-1]$ , $\beta_5 = [-1, 1, -1, -1, 1]$
$v_1u_3^+$.
Write $\alpha_5$ and $\beta_5$ in standard form, we have
$-\alpha_5 = [1]$ , $-\beta_5 = [1, -1, 1, 1, -1]$.
Here $d_1 = 1$ , $q = 5$ , so $r = \frac{1+5}{2} = 3$ .
$-\alpha_6 = \phi$ , $-\beta_6 = [0, 0, 0, 0, 0]$
$u_3v_2^-$ , $u_3v_4^-$ , $u_3v_5^-$ , $u_3v_3^+$ , $u_3v_6^+$ .

The resulting complete signed bipartite graph will have vertex sets $U = \{u_1, u_2, u_3, u_4\}$ , $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ , and edges $u_1v_1^+$ , $u_1v_2^+$ , $u_1v_3^+$ , $u_1v_4^+$ , $u_1v_5^+$ , $u_1v_6^-$ , $u_2v_1^+$ , $u_2v_2^+$ , $u_2v_3^+$ , $u_2v_4^+$ , $u_2v_5^+$ , $u_2v_6^-$ , $u_3v_1^+$ , $u_3v_2^-$ , $u_3v_3^+$ , $u_3v_4^-$ , $u_3v_5^-$ , $u_3v_6^+$ , $u_4v_1^+$ , $u_4v_2^+$ , $u_4v_3^-$ , $u_4v_4^-$ , $u_4v_5^-$ , $u_4v_6^-$ .

The next result gives a necessary and sufficient condition for a pair of integral sequences to be the signed degree sequences of some signed bipartite graph. This result is similar to Theorem 1.1.

**Theorem 2.6.** *Let $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ be standard sequences. Then $\alpha$ and $\beta$ are the signed degree sequences of a signed bipartite graph if and only if there exist integers $r$ and $s$ with $d_1 = r - s$ and $0 \le s \le \frac{q-d_1}{2}$ such that $\alpha'$ and $\beta'$ are the signed degree sequences of a signed bipartite graph, where $\alpha'$ is obtained from $\alpha$ by deleting $d_1$ and $\beta'$ is obtained from $\beta$ by reducing $r$ greatest entries of $\beta$ by 1 each and adding $s$ least entries of $\beta$ by 1 each.*

*Proof.* Suppose $r$ and $s$ be integers with $d_1 = r - s$ and $0 \le s \le \frac{q-d_1}{2}$ such that $\alpha'$ and $\beta'$ are the signed degree sequences of a signed bipartite graph $G'(U', V')$ . Let

$U' = \{u_2, u_3, \cdots, u_p\}$ and $V' = \{v_1, v_2, \cdots, v_q\}$ such that $sdeg(u_i) = d_i$, for $2 \le i \le p$, and

$$sdeg(v_i) = \begin{cases} e'_i - 1, & \text{for } 1 \le i \le r, \\ e'_i, & \text{for } r+1 \le i \le q-s, \\ e'_i + 1, & \text{for } q-s+1 \le i \le q, \end{cases}$$

where $e'_i = e_j$ for $1 \le i, j \le q$ such that $e'_1 \ge e'_2 \ge \cdots \ge e'_q$.

Now, construct a new signed bipartite graph $G(U, V)$ as follows.

Let $U = \{u_1\} \cup U'$, $V = V'$ with $\{u_1\} \cap U' = \phi$, and join $u_1$ by positive edges to the vertices $v_1, v_2, \cdots, v_r$ and join $u_1$ by negative edges to the vertices $v_{q-s+1}, v_{q-s+2}, \cdots, v_q$. Then, $sdeg(u_1) = r - s = d_1$, $sdeg(u_i) = d_i$, for $2 \le i \le p$, and

$$sdeg(v_i) = \begin{cases} e'_i - 1 + 1, & \text{for } 1 \le i \le r, \\ e'_i, & \text{for } r+1 \le i \le q-s, \\ e'_i + 1 - 1, & \text{for } q-s+1 \le i \le q, \end{cases}$$

$$= e'_i, \quad \text{for } 1 \le i \le q,$$

$$= e_j, \quad \text{for } 1 \le j \le q.$$

Hence, $G(U, V)$ is a signed bipartite graph with signed degree sequences $\alpha$ and $\beta$.

Conversely, let $\alpha$ and $\beta$ be the signed degree sequences of a signed bipartite graph. Let $e'_i = e_j$ for $1 \le i, j \le q$ such that $e'_1 \ge e'_2 \ge \cdots \ge e'_q$. Then, $\beta_1 = [e'_1, e'_2, \cdots, e'_q]$ is a non-increasing sequence. Suppose the vertex sets of the signed bipartite graph be $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$ such that $sdeg(u_i) = d_i$, for $1 \le i \le p$, and $sdeg(v_j) = e'_j$, for $1 \le j \le q$.

Among all the signed bipartite graphs with $\alpha$ and $\beta_1$ as the signed degree sequences, let $G(U, V)$ be one with the property that the sum $S$ of the signed degrees of the vertices of $V$ joined to $u_1$ by positive edges is maximum. Let $d_1^+ = r$ and $d_1^- = s$ be respectively the number of positive edges and number of negative edges incident with $u_1$. Then, $sdeg(u_1) = d_1 = d_1^+ - d_1^- = r - s$ and $deg(u_1) = d_1^+ + d_1^- = r + s \le q$. Therefore, $0 \le s \le \frac{q-d_1}{2}$. We claim that $u_1$ must be joined by positive edges to the vertices of $V$ having signed degrees $e'_1, e'_2, \cdots, e'_r$. If this is not true, then there exist vertices $v_i$ and $v_j$ in $V$ with $j > i$ such that the edge $u_1v_j$ is positive and either (i) $u_1v_i$ is negative or (ii) $u_1$ and $v_i$ are not adjacent in $G(U, V)$. Since $\beta_1$ is in non-increasing order, then $sdeg(v_i) \ge sdeg(v_j)$, that is, $e'_i \ge e'_j$. Here we consider only (i), since (ii) is similar to (i).

We note that if there exists a vertex $u_n$ ($\ne u_1$) of $U$ such that $u_nv_i$ is a positive edge and $u_nv_j$ is a negative edge, then we change the signs of these edges so that $u_1v_j$ and $u_nv_i$ are negative and $u_1v_i$ and $u_nv_j$ are positive. Then we have a signed bipartite graph with signed degree sequences $\alpha$ and $\beta_1$ in which the sum of the signed degrees of the vertices of $V$ joined to $u_1$ by positive edges exceeds $S$, a contradiction. So, assume that no such vertex $u_n$ of $U$ exists.

Now, suppose that $v_i$ is not incident to any positive edges. Since $e'_i \geq e'_j$, there exist at least two vertices $u_n$ and $u_k$ (both distinct from $u_1$) of $U$ such that $u_n v_j$ and $u_k v_j$ are negative edges and both $u_n$ and $u_k$ are not adjacent to $v_i$. Then, by changing the edges so that $u_1 v_i$ is positive edge and $u_1 v_j$, $u_n v_i$ and $u_k v_i$ are negative edges, we again get a contradiction. Hence, $v_i$ must be incident to at least one positive edge.

Claim that there exists at least one vertex $u_m$ of $U$ such that $u_m v_i$ is a positive edge and $u_m$ is not adjacent to $v_j$. Suppose on contrary that whenever $v_i$ is joined to a vertex of $U$ by a positive edge, then $v_j$ is also joined to this vertex of $U$ by a positive edge. Since $e'_i \geq e'_j$, then again we have the same situation as above, from which we get a contradiction. Thus, there exists a vertex $u_m$ of $U$ such that $u_m v_i$ is a positive edge and $u_m$ is not adjacent to $v_j$. Similarly, it is easy to see that there exists a vertex $u_n$ of $U$ such that $u_n v_j$ is a negative edge and $u_n$ is not adjacent to $v_i$. By changing the edges so that $u_1 v_i$ and $u_m v_j$ are positive edges and $u_1 v_j$ and $u_n v_i$ are negative edges, we again get a contradiction. Therefore, $u_1$ must be joined by positive edges to the vertices of $V$ having signed degrees $e'_1, e'_2, \cdots, e'_r$.

In a similar way, it can be shown that $u_1$ is joined by negative edges to the vertices of $V$ having signed degrees $e'_{q-s+1}, e'_{q-s+2}, \cdots, e'_q$. Hence, $G(U,V) - u_1$ is a signed bipartite graph with $\alpha'$ and $\beta'$ as the signed degree sequences. □

Theorem 2.6 also provides an algorithm for determining whether or not the standard sequences $\alpha$ and $\beta$ are the signed degree sequences, and for constructing a corresponding signed bipartite graph. Suppose $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ be the standard signed degree sequences of a signed bipartite graph with parts $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$. Let $d_1 = r - s$ and $0 \leq s \leq \frac{q-d_1}{2}$. Deleting $d_1$ and reducing $r$ greatest entries of $\beta$ by 1 each and adding $s$ least entries of $\beta$ by 1 each to form $\beta' = [e'_1, e'_2, \cdots, e'_q]$. Then edges are defined by $u_1 v_i^+$ for which $e'_j = e_i - 1$; $u_1 v_i^-$ for which $e'_j = e_i + 1$ and $u_1$ and $v_i$ are not adjacent for which $e'_j = e_i$. For $-\alpha$ and $-\beta$, the edges are defined by $u_1 v_i^-$ for which $e'_j = e_i - 1$; $u_1 v_i^+$ for which $e'_j = e_i + 1$ and $u_1$ and $v_i$ are not adjacent for which $e'_j = e_i$. Now, if (A) is not satisfied then we use (B) in which we take $e_1 = r - s$ and $0 \leq s \leq \frac{p-e_1}{2}$. Deleting $e_1$ and reduce $r$ greatest entries of $\alpha$ by 1 each and adding $s$ least entries of $\alpha$ by 1 each to form $\alpha' = [d'_1, d'_2, \cdots, d'_p]$. Then edges are defined by $v_1 u_i^+$ for which $d'_j = d_i - 1$; $v_1 u_i^-$ for which $d'_j = d_i + 1$ and $v_1$ and $u_i$ are not adjacent for which $d'_j = d_i$. For $-\alpha$ and $-\beta$, the edges are defined by $v_1 u_i^-$ for which $d'_j = d_i - 1$; $v_1 u_i^+$ for which $d'_j = d_i + 1$ and $v_1$ and $u_i$ are not adjacent for which $d'_j = d_i$. If this method is applied successively, then (i) it tests whether or not $\alpha$ and $\beta$ are signed degree sequences, and if $\alpha$ and $\beta$ are signed degree sequences (ii) a signed bipartite graph $\Delta(\alpha, \beta)$ with signed degree sequences $\alpha$ and $\beta$ is constructed.

We illustrate this algorithm with the following example, beginning with standard sequences $\alpha_1$ and $\beta_1$.

$\alpha_1 = [3, 2, 1, -2]$ , $\beta_1 = [3, 2, -1]$
Here $d_1 = 3$ , $q = 3$ , so $0 \le s \le \frac{3-3}{2}$ . Then $s = 0$ and $r = 3 + 0 = 3$ .
$\alpha_2 = [2, 1, -2]$ , $\beta_2 = [2, 1, -2]$
$u_1v_1^+$ , $u_1v_2^+$ , $u_1v_3^+$ .
Here $d_2 = 2$ , $q = 3$ , so $0 \le s \le \frac{3-2}{2}$ . Then $s = 0$ and $r = 2 + 0 = 2$ .
$\alpha_3 = [1, -2]$ , $\beta_3 = [1, 0, -2]$
$u_2v_1^+$ , $u_2v_2^+$ .
Write $\alpha_3$ and $\beta_3$ in standard form, we have $-\alpha_3 = [2, -1]$ , $-\beta_3 = [-1, 0, 2]$.
Here $d_4 = 2$ , $q = 3$ , so $0 \le s \le \frac{3-2}{2}$ . Then $s = 0$ and $r = 2 + 0 = 2$ .
$-\alpha_4 = [-1]$ , $-\beta_4 = [-1, -1, 1]$
$u_4v_2^-$ , $u_4v_3^-$ .
Write $-\alpha_4$ and $-\beta_4$ in standard form, we have
$\alpha_4 = [1]$ , $\beta_4 = [1, 1, -1]$.
Here $d_3 = 1$ , $q = 3$ , so $0 \le s \le \frac{3-1}{2}$ . Then $s = 0$ or $1$ . Choose $s = 1$ . Then $r = 1 + 1 = 2$ .
$\alpha_5 = \phi$ , $\beta_5 = [0, 0, 0]$
$u_3v_1^+$ , $u_3v_2^+$ , $u_3v_3^-$ .

The resulting signed bipartite graph will have vertex sets $U = \{u_1, u_2, u_3, u_4\}$ , $V = \{v_1, v_2, v_3\}$ , and edges $u_1v_1^+$ , $u_1v_2^+$ , $u_1v_3^+$ , $u_2v_1^+$ , $u_2v_2^+$ , $u_3v_1^+$ , $u_3v_2^+$ , $u_3v_3^-$ , $u_4v_2^-$ , $u_4v_3^-$ .

Since $r = d_1 + s$ , Theorem 2.6 can be re-stated as follows.

**Theorem 2.7.** *Standard sequences $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ are $s$-graphical if and only if $\alpha' = [d_2, d_3, \cdots, d_p]$ and $\beta' = [e_1' - 1, e_2' - 1, \cdots, e'_{d_1+s} - 1, e'_{d_1+s+1}, \cdots, e'_{q-s}, e'_{q-s+1} + 1, \cdots, e'_q + 1]$ are $s$-graphical for some $s$, $0 \le s \le \frac{q-d_1}{2}$, where $e'_i = e_j$ for $1 \le i, j \le q$ such that $e'_1 \ge e'_2 \ge \cdots \ge e'_q$.*

When $s$ is choosen large in Theorem 2.7, then it is difficult to device an efficient algorithm. The next result which gives necessary and sufficient conditions for a pair of integral sequences to be the signed degree sequences of some signed bipartite graph, provides a better choice for parameter $s$ in Theorem 2.7. We note that Theorem 2.8 is similar to Theorem 1.2.

**Theorem 2.8.** *Standard sequences $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ are $s$-graphical if and only if $\alpha'_m = [d_2, d_3, \cdots, d_p]$ and $\beta'_m = [e'_1 - 1, e'_2 - 1, \cdots, e'_{d_1+m} - 1, e'_{d_1+m+1}, \cdots, e'_{q-m}, e'_{q-m+1} + 1, \cdots, e'_q + 1]$ are $s$-graphical, where $m$ is the maximum non-negative integer $e'_{d_1+m} > e'_{q-m+1}$ and $e'_i = e_j$ for $1 \le i, j \le q$ such that $e'_1 \ge e'_2 \ge \cdots \ge e'_q$.*

*Proof.* Let $G'(U', V')$ be a signed bipartite graph with signed degree sequences $\alpha'_m$ and $\beta'_m$. Let $U' = \{u_2, u_3, \cdots, u_p\}$ and $V' = \{v_1, v_2, \cdots, v_q\}$ such that $sdeg(u_i) = d_i$, for $2 \le i \le p$, and

$$sdeg(v_i) = \begin{cases} e'_i - 1, & \text{for } 1 \leq i \leq d_1 + m, \\ e'_i, & \text{for } d_1 + m + 1 \leq i \leq q - m, \\ e'_i + 1, & \text{for } q - m + 1 \leq i \leq q. \end{cases}$$

Now, construct a new signed bipartite graph $G(U, V)$ as follows.

Let $U = \{u_1\} \cup U'$, $V = V'$ with $\{u_1\} \cap U' = \phi$, and join $u_1$ by positive edges to the vertices $v_1, v_2, \cdots, v_{d_1+m}$ and join $u_1$ by negative edges to the vertices $v_{q-m+1}, v_{q-m+2}, \cdots, v_q$. Then, $sdeg(u_1) = d_1 + m - m = d_1$, $sdeg(u_i) = d_i$, for $2 \leq i \leq p$,
and

$$sdeg(v_i) = \begin{cases} e'_i - 1 + 1, & \text{for } 1 \leq i \leq d_1 + m, \\ e'_i, & \text{for } d_1 + m + 1 \leq i \leq q - m, \\ e'_i + 1 - 1, & \text{for } q - m + 1 \leq i \leq q, \end{cases}$$

$$= e'_i, \quad \text{for } 1 \leq i \leq q,$$

$$= e_j, \quad \text{for } 1 \leq j \leq q.$$

Hence, $G(U, V)$ is a signed bipartite graph with signed degree sequences $\alpha$ and $\beta$.

Conversely, let $\alpha$ and $\beta$ be the signed degree sequences of a signed bipartite graph $G(U, V)$. Let $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$ such that $sdeg(u_i) = d_i$, for $1 \leq i \leq p$ and $sdeg(v_j) = e'_j$, for $1 \leq j \leq q$.

Now, for each $s$, $0 \leq s \leq \frac{q-d_1}{2}$, consider the sequences $\alpha'_s = [d_2, d_3, \cdots, d_p]$ and $\beta'_s = [e'_1 - 1, e'_2 - 1, \cdots, e'_{d_1+s} - 1, e'_{d_1+s+1}, \cdots, e'_{q-s}, e'_{q-s+1} + 1, \cdots, e'_q + 1]$.

By Theorem 2.7, $\alpha'_s$ and $\beta'_s$ are $s$-graphical for some $s$. Choosing $s$ such that $|s - m|$ is minimum. Let $G'(U', V')$ be a signed bipartite graph with signed degree sequences $\alpha'_s$ and $\beta'_s$. Let $U' = \{u_2, u_3, \cdots, u_p\}$ and $V' = \{v_1, v_2, \cdots, v_q\}$ such that $sdeg(u_i) = d_i$, for $2 \leq i \leq p$ and

$$sdeg(v_i) = \begin{cases} e'_i - 1, & \text{for } 1 \leq i \leq d_1 + s, \\ e'_i, & \text{for } d_1 + s + 1 \leq i \leq q - s, \\ e'_i + 1, & \text{for } q - s + 1 \leq i \leq q. \end{cases}$$

First note that if $s < m$, then $e'_a > e'_b$ by the choice of $m$, where $a = d_1 + s + 1$ and $b = q - s$. Since $e'_a > e'_b$, therefore there exists a vertex $u_k$ of $U'$ such that

(i) $v_a u_k$ is a positive edge and $v_b u_k$ is a negative edge, or

(ii) $v_a u_k$ is a positive edge and $v_b$ is not adjacent to $u_k$, or

(iii) $v_a$ is not adjacent to $u_k$ and $v_b u_k$ is a negative edge.

For (i), delete the positive edge $v_a u_k$ and the negative edge $v_b u_k$ from $G'(U', V')$; for (ii), delete the positive edge $v_a u_k$ from $G'(U', V')$ and add a new positive edge $v_b u_k$ to $G'(U', V')$; for (iii), delete the negative edge $v_b u_k$ from $G'(U', V')$ and add a new negative edge $v_a u_k$ to $G'(U', V')$. Then in each case, we obtain a signed bipartite graph $G''(U'', V'')$ with signed degree sequences $\alpha'_{s+1}$ and $\beta'_{s+1}$, which contradicts the minimality of $|s - m|$.

Now, if $s > m$ then $e'_{d_1+s} = e'_{q-s+1}$ and hence $e'_{d_1+s} - 1 < e'_{q-s+1} + 1$. As in above, we again get a contradiction.

Thus, $s = m$ and $\alpha'_m$ and $\beta'_m$ are $s$-graphical. □

Theorem 2.8 provides an efficient algorithm for recognizing signed degree sequences. Suppose $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ be the standard signed degree sequences of a signed bipartite graph with parts $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$. Further, assume that $\beta$ is in non-increasing order such that $sdeg(u_i) = d_i$, for $1 \le i \le p$ and $sdeg(v_j) = e_j$, for $1 \le j \le q$. Let $m$ be the maximum non-negative integer such that $e_{d_1+m} > e_{q-m+1}$. Deleting $d_1$ and reducing $d_1 + m$ largest entries of $\beta$ by 1 each and adding $m$ least entries of $\beta$ by 1 each to form $\beta' = [e'_1, e'_2, \cdots, e'_q]$. Now, if (A) is not satisfied then we use (B) in which we assume that $\alpha$ is in non-increasing order such that $sdeg(u_i) = d_i$, for $1 \le i \le p$ and $sdeg(v_j) = e_j$, for $1 \le j \le q$. Let $m$ be the maximum non-negative integer such that $d_{e_1+m} > d_{p-m+1}$. Deleting $e_1$ and reducing $e_1 + m$ largest entries of $\alpha$ by 1 each and adding $m$ least entries of $\alpha$ by 1 each to form $\alpha' = [d'_1, d'_2, \cdots, d'_p]$. If this method is applied successively, then it tests whether or not $\alpha$ and $\beta$ are signed degree sequences.

We illustrate this algorithm with the following example, beginning with standard sequences $\alpha_1$ and $\beta_1$.

$\alpha_1 = [3, 3, -1, -2]$, $\beta_1 = [2, 2, 1, 0, 0, -2]$
Here $d_1 = 3$, $q = 6$ and $m = 1$ is maximum such that $e_4 > e_6$.
$\alpha_2 = [3, -1, -2]$, $\beta_2 = [1, 1, 0, -1, 0, -1]$
Write $\beta_2$ in non-increasing order, we have
$\alpha_2 = [3, -1, -2]$, $\beta_2 = [1, 1, 0, 0, -1, -1]$
Here $d_2 = 3$, $q = 6$ and $m = 1$ is maximum such that $e'_4 > e'_6$.
$\alpha_3 = [-1, -2]$, $\beta_3 = [0, 0, -1, -1, -1, 0]$
Write $\alpha_3$ and $\beta_3$ in standard form and then $\beta_3$ in non-increasing order, we have
$-\alpha_3 = [2, 1]$, $-\beta_3 = [1, 1, 1, 0, 0, 0]$
Here $d_4 = 2$, $q = 6$ and $m = 1$ is maximum such that $e''_4 > e''_5$.
$-\alpha_4 = [1]$, $-\beta_4 = [0, 0, 0, 0, 0, 1]$

Clearly, $-\alpha_4$ and $-\beta_4$ are the signed degree sequences of a signed bipartite graph. Hence, by Theorem 2.8, $\alpha_1$ and $\beta_1$ are the signed degree sequences of a signed bipartite graph.

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