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On Equality in an Upper Bound for the Domination Number with Respect to Nondegenerate Properties

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Abstract

For a graph property $\mathcal{P}$ and a graph G, a subset S of vertices of G is a $\mathcal{P}$-set if the subgraph induced by S has the property $\mathcal{P}$. The domination number with respect to the property $\mathcal{P}$, denoted by $\gamma_{\mathcal{P}}(G)$, is the minimum cardinality of a dominating $\mathcal{P}$-set. Any property $\mathcal{P}$ satisfied by all edgeless graphs is called nondegenerate. For any graph G with n vertices and maximum degree $\Delta(G)$, $\gamma_{\mathcal{P}}(G) \leq n - \Delta(G)$ where $\mathcal{P}$ is nondegenerate and closed under union with K_1. In this paper we characterize the connected graphs and the connected triangle-free graphs which achieve this upper bound.

Keywords: domination; nondegenerate/induced-hereditary graph property.

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1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes, et al. [6]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G \rangle$. For a vertex x of G, $N(x,G)$ denote the set of all neighbors of x in G, $N[x,G] = N(x,G) \cup \{x\}$ and the degree of x is $deg(x,G) = |N(x,G)|$. We denote by $\Delta(G)$ and $n(G)$, respectively, the maximum degree and the order $|V(G)|$ of a graph G. For a set of vertices $S \subseteq V(G)$, $N(S,G)$ is the union of $N(x,G)$ for all $x \in S$. When no confusion is possible, we may denote any parameter $f(G)$ of G by f.

Let $\mathcal{G}$ denote the set of all mutually nonisomorphic graphs. A *graph property* is any non-empty subset of $\mathcal{G}$. We say that a *graph G has property $\mathcal{P}$* whenever there exists a graph $H \in \mathcal{P}$ which is isomorphic to G. For example we list some graph properties:

- $\mathcal{I} = \{H \in \mathcal{G} \,:\, H$ is totally disconnected$\}$;
- $\mathcal{E}_1 = \{H \in \mathcal{G} \,:\, |E(H)| \leq 1\}$;
- $\mathcal{C} = \{H \in \mathcal{G} \,:\, H$ is connected$\}$;
- $\mathcal{F} = \{H \in \mathcal{G} \,:\, H$ is a forest$\}$;
- $\mathcal{I}_k = \{H \in \mathcal{G} \,:\, \Delta(H) \leq k\}$, where k is nonnegative integer.

A property $\mathcal{P}$ is *induced-hereditary* if the fact that G satisfies $\mathcal{P}$ implies that H satisfies $\mathcal{P}$ for all induced subgraphs H of G. A property $\mathcal{P}$ is called *nondegenerate* if $\mathcal{I} \subseteq \mathcal{P}$. Note that $\mathcal{I}$, $\mathcal{E}_1$, $\mathcal{I}_k$ and $\mathcal{F}$ are nondegenerate and induced-hereditary whereas $\mathcal{C}$ is neither nondegenerate nor induced-hereditary.

A *dominating set* for a graph G is a set of vertices $D \subseteq V(G)$ such that every vertex of G is either in D or is adjacent to an element of D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality taken over all dominating sets of G.

Any set $S \subseteq V(G)$ such that the subgraph $\langle S, G\rangle$ satisfies property $\mathcal{P}$ is called a $\mathcal{P}$-set. The concept of domination with respect to any property $\mathcal{P}$ was introduced by Goddard et al. [5]. The *domination number with respect to the property* $\mathcal{P}$, denoted by $\gamma_{\mathcal{P}}(G)$, is the smallest cardinality of a dominating $\mathcal{P}$-set of G. Note that there may be no dominating $\mathcal{P}$-set of G at all. For example, all graphs having at least two components are without dominating $\mathcal{C}$-sets. On the other hand, if a property $\mathcal{P}$ is nondegenerate then every maximal independent set is a $\mathcal{P}$-set and thus $\gamma_{\mathcal{P}}(G)$ exists. Observe that if $\mathcal{I} \subseteq \mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{G}$ then [5] $i(G) \geq \gamma_{\mathcal{P}_1}(G) \geq \gamma_{\mathcal{P}_2}(G) \geq \gamma(G)$. Michalak [8] has considered these parameters where the property is additive (i.e., closed under taking disjoint union of graphs) and induced-hereditary.

Note that:

(a) in the case $\mathcal{P} = \mathcal{G}$ we have $\gamma_{\mathcal{G}}(G) = \gamma(G)$;

(b) in the case $\mathcal{P} = \mathcal{I}$, $\gamma_{\mathcal{I}}(G)$ is the well-known *independent domination number* $i(G)$;

(c) in the case $\mathcal{P} = \mathcal{C}$, $\gamma_{\mathcal{C}}(G)$ is denoted by $\gamma_c(G)$ and is called the *connected domination number*;

(d) in the case $\mathcal{P} = \mathcal{F}$, $\gamma_{\mathcal{F}}(G)$ is denoted by $\gamma_a(G)$ and is called the *acyclic domination number* [7];

(e) in the case $\mathcal{P} = \mathcal{I}_k$, $\gamma_{\mathcal{I}_k}(G)$ is denoted by $\gamma^k(G)$ and is called the *k-dependent domination number* [3].

A classical result by Berge [1] states that for any graph G, $\gamma(G) \leq n(G) - \Delta(G)$. Domke et al. [2] noted that $i(G) \leq n(G) - \Delta(G)$ and they were the first to consider the problem of characterization the graphs with $\mu(G) = n(G) - \Delta(G)$, $\mu \in \{\gamma, i\}$. Favaron and Mynhardt [4] and Pedersen [9] continued the study of the problem, and gave necessary and sufficient conditions for $\mu(G) = n(G) - \Delta(G)$ where $\mu \in \{\gamma, i\}$ and $\mu = \gamma_c$ respectively.

Let $\mathcal{P}$ be a nondegenerate property. Since $\gamma_{\mathcal{P}}(G) \leq i(G)$, it follows that $\gamma_{\mathcal{P}}(G) \leq n(G) - \Delta(G)$. This paper deals with graphs G satisfying $\gamma_{\mathcal{P}}(G) = n(G) - \Delta(G)$.

We need the following notation and results. Let G be a graph, $x \in V(G)$, $deg(x,G) = \Delta(G)$. Let $B = N(x,G), C = V(G) - N[x,G]$ and $R = B - N(C,G)$. For each $c \in C$, let $B_c = N(c,G) \cap B$. Now we define the following properties, the first two of which were also given in [2, 4] and the third, when $\mathcal{P} = \mathcal{G}$, in [4]:

$P_1(x)$**:** C is independent;

$P_2(x)$**:** Every vertex of B has at most one neighbour in C;

$P_3(x;\mathcal{P})$**:** For every non-empty subset C' of C, the subset $B' = (\cup_{u \in C'} B_u) \cup R$ of B is not dominated by a $\mathcal{P}$-set consisting of exactly one vertex of each B_u, $u \in C'$.

Theorem 1.1. [2, 4] *Let G be a connected graph of order n and maximum degree Δ. Let $x \in V(G)$ and $deg(x,G) = \Delta$.*

(1) *Then $i(G) + \Delta \leq n$. If $i(G) + \Delta = n$ then $P_1(x)$ holds.*

(2) *Let $P_1(x)$ and $P_2(x)$ hold.*

  (a) *Then $n - \Delta - 1 \leq \gamma(G)$.*

  (b) *If $y \in B$ with $N[y,G] \subseteq R \cup \{x\}$ then $\gamma(G) = n - \Delta$.*

If G is a disconnected graph with $k \geq 2$ components, $\Delta(G) \geq 1$ and $i(G) + \Delta(G) = n(G)$, then all but one component of G are K_1-components because of Theorem 1.1(1). This shows that it is sufficient to consider the connected graphs G with $\gamma_{\mathcal{P}}(G) + \Delta(G) = n(G)$.

2. Graphs which satisfies $\gamma_{\mathcal{P}}(G) = n(G) - \Delta(G)$

Our main result is the following theorem.

Theorem 2.1. (Favaron and Mynhardt [4] when $\mathcal{H} = \mathcal{G}$) *Let G be a connected graph of order n and maximum degree Δ.*

(i) *Let $\mathcal{E}_1 \subseteq \mathcal{H} \subseteq \mathcal{G}$ and $\mathcal{H}$ closed under union with K_1. If $\gamma_{\mathcal{H}}(G) = n - \Delta$ then $P_1(x), P_2(x)$ and $P_3(x, \mathcal{H})$ hold for every vertex x of degree Δ.*

(ii) *Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and induced hereditary. If $P_1(x), P_2(x)$ and $P_3(x, \mathcal{H})$ hold for some vertex x of degree Δ then $\gamma_{\mathcal{H}}(G) = n - \Delta$.*

Proof. We use the ideas of the proof of Theorem 4 [4].

(i) Let $x \in V(G)$, $deg(x) = \Delta$ and $\gamma_{\mathcal{H}}(G) = n - \Delta$. Since $\gamma_{\mathcal{H}}(G) \leq i(G)$ we have $i(G) = n - \Delta$. By Theorem 1.1(1), $P_1(x)$ is satisfied. Let $y \in N(x)$ and suppose y is adjacent to r vertices in C with $r > 1$. Then x and y together with the $n - \Delta - 1 - r$ vertices of C that are not in $N(y, G)$, form a dominating $\mathcal{E}_1$-set of G, say M. Then we have $\gamma_{\mathcal{H}}(G) \le \gamma_{\mathcal{E}_1}(G) \le |M| = (n - \Delta - 1 - r) + 2 < n - \Delta$, a contradiction. Hence $P_2(x)$ is also satisfied.

Note that, if $u \in C$ then B_u is non-empty since G is connected. Moreover, the B_u's form a partition of $N(C, G) = B - R$ by $P_2(x)$. Suppose $P_3(x, \mathcal{H})$ does not hold and $C^{'}$ is a non-empty subset of C such that there exists a dominating $\mathcal{H}$-set D of $\langle(\cup_{u \in C^{'}} B_u) \cup R, G\rangle$ consisting of exactly one vertex of each B_u, $u \in C^{'}$. Then $M = D \cup (C - C^{'})$ is a dominating set of G of cardinality $|M| = |C| = n - \Delta - 1 = \gamma_{\mathcal{H}}(G) - 1$. Since $C - C^{'}$ is independent, $\mathcal{H}$ is closed under union with K_1 and $N(D, G) \cap (C - C^{'}) = \emptyset$, it follows that M is a dominating $\mathcal{H}$-set of G of cardinality $\gamma_{\mathcal{H}}(G) - 1$, a contradiction. Hence $P_3(x, \mathcal{H})$ holds.

(ii) Let $x \in V(G)$, $deg(x) = \Delta$ and $P_1(x)$, $P_2(x)$ and $P_3(x, \mathcal{H})$ hold. Since $\mathcal{H}$ is nondegenerate, there is a $\gamma_{\mathcal{H}}$-set of G, say D. Let $C^{''} = D \cap C$ and $C^{'} = C - C^{''}$. If $B \cap D$ is empty then clearly $x \in D$ and by $P_1(x)$, $C \subseteq D$; hence $n - \Delta \ge \gamma_{\mathcal{H}}(G) = |D| \ge |C| + 1 = n - \Delta$. So, assume $B \cap D$ is not empty. In order to dominate C, D contains at least one vertex of each of the $|C|$ disjoint sets $\{u\} \cup B_u$, $u \in C$ (by $P_1(x)$ and $P_2(x)$). Hence the set $D \cap B$ contains at least $|C^{'}|$ vertices, one vertex in each B_u with $u \in C^{'}$.

Assume $|D \cap B| = |C^{'}|$. Since $\mathcal{H}$ is induced hereditary, $D \cap B$ is an H-set. Then by $P_3(x, \mathcal{H})$ it follows that $D \cap B$ does not dominate $B^{'} = B - \cup_{u \in C^{''}} B_u$. Hence in all cases we have $n - \Delta = |C^{'}| + |C^{''}| + 1 \le |D| = \gamma_{\mathcal{H}}(G) \le n - \Delta$. □

Observe that any induced-hereditary and closed under union with K_1 graph-property is nondegenerate. Hence, Theorem 2.1 holds for any induced-hereditary and closed under union with K_1 property $\mathcal{H}$ such that $\mathcal{E}_1 \subseteq \mathcal{H} \subseteq \mathcal{G}$; in particular, when $\mathcal{H} \in \{\mathcal{F}, \mathcal{I}_1, \mathcal{I}_2, \dots\}$.

The next lemma improves Corollary 6 [4].

Lemma 2.2. *Let G be a connected graph, $x \in V(G)$ and $deg(x, G) = \Delta(G)$. Let $P_1(x)$ and $P_2(x)$ hold. If $R = \emptyset$ and C contains a vertex of degree one then $i(G) = \gamma(G) = n - \Delta - 1$.*

Proof. Let $R = \emptyset$, $u \in C$ and $N(u, G) = \{v\}$. By $P_1(x)$, $C - \{u\}$ is independent and by $P_2(x)$, $M = \{v\} \cup (C - \{u\})$ is also independent. Since $R = \emptyset$, M is a dominating set of G. Hence M is an independent dominating set of G of cardinality $|M| = |C| = n - \Delta - 1$. Since $\gamma(G) \le i(G) \le |M|$, the result follows by Theorem 1.1 (2)(a). □

Connected triangle-free graphs with $\gamma_{\mathcal{P}}(G) = n(G) - \Delta(G)$ where $\mathcal{P} = \mathcal{G}$ were characterized by Favaron and Mynhardt [4]. Here we present an improvement of their result by relaxing the condition of a graph being triangle-free to a graph containing a vertex of degree Δ which does not lie on a triangle.

Theorem 2.3. *Let G be a connected nontrivial graph, $\mathcal{E}_1 \subseteq \mathcal{H} \subseteq \mathcal{G}$ and let $\mathcal{H}$ be closed under union with K_1. If $x \in V(G)$ has maximum degree and $N(x,G)$ is an independent set then the following are equivalent.*

(i) $\gamma(G) = n(G) - \Delta(G)$;

(ii) $\gamma_{\mathcal{H}}(G) = n(G) - \Delta(G)$;

(iii) *G is bipartite with bipartition $A \cup B$, where $|A| \leq |B|$, $\Delta(G) = |B|$, $deg(u,G) \leq 2$ for each $u \in B$, and if $deg(u,G) = 2$ for all $u \in B$, then $deg(v,G) \geq 2$ for each $v \in A$.*

Proof. (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): If $\gamma_{\mathcal{H}}(G) = 1$ then the result is obvious. So, let $2 \leq \gamma_{\mathcal{H}}(G) = n(G) - \Delta(G)$. By Theorem 2.1(i) the properties $P_1(x), P_2(x)$ and $P_3(x,\mathcal{H})$ hold. By $P_1(x)$, C is independent. Hence G is bipartite with bipartition $A \cup B$, where $A = \{x\} \cup C$ and $B = N(x,G)$. Since A and B are independent dominating sets of G (and therefore $\mathcal{H}$-dominating sets of G), and $|A| = n - \Delta = \gamma_{\mathcal{H}}(G)$ we have $|A| \leq |B|$. Moreover, $|B| = \Delta$, by the choice of x. By $P_2(x)$, $deg(u) \leq 2$ for each $u \in B$. Suppose $deg(u) = 2$ for each $u \in B$. Then $deg(x) \geq 2$, $R = \emptyset$ and if some vertex u of $A - \{x\} = C$ has degree 1, then $\gamma_{\mathcal{H}}(G) < n - \Delta$ because of Lemma 2.2, a contradiction.

(iii) $\Rightarrow$ (i): This is Corollary 7 in [4]. □

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