RESTRAINED DOUBLE DOMINATION NUMBER OF A GRAPH

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Abstract

A set $S \subseteq V(G)$ is a restrained double dominating set for $G$ if every vertex in $V$ is dominated by at least two vertices in $S$ and $\langle V - S \rangle$ has no isolated vertices. The minimum cardinality of a minimal restrained double dominating set is the restrained double domination number and is denoted by $\gamma_{2r}(G)$. In this paper we initiate a study of this parameter and obtain some bounds for $\gamma_{2r}(G)$ and characterize the graphs attaining these bounds. We also derive Nordhaus–Gaddum type results for $\gamma_{2r}(G)$.

Keywords: Domination; Double domination; Restrained domination; Restrained double domination; Restrained double domination number.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. For graph theoretical terms we refer to Harary [4] and for terms related to domination we refer to Haynes et al. [5].

A subset $S$ of $V$ is said to be a dominating set in $G$ if every vertex in $V - S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$.

The concept of restrained domination was introduced by Telle and Proskurowski [7] indirectly as a vertex partitioning problem. Cyman and Raczek [1] introduced the concept
of total restrained domination. A restrained dominating set is a set \( S \subseteq V \) where every vertex in \( V - S \) is adjacent to a vertex in \( S \) as well as to another vertex in \( V - S \). The smallest cardinality of a restrained dominating set \( S \) of \( G \) is called the restrained domination number of \( G \) and is denoted by \( \gamma_r(G) \). A restrained dominating set \( S \) with \( |S| = \gamma_r \) is called a \( \gamma_r \)-set.

Harary and Haynes [3] introduced the concept of double domination number of a graph. A set \( S \subseteq V \) is a double dominating set for \( G \) if every vertex in \( V \) is dominated by at least two vertices of \( S \). The minimum cardinality of a double dominating set is the double domination number of \( G \) and is denoted by \( dd(G) \).

In this paper we define restrained double domination number \( \gamma_2r(G) \) and initiate a study of this parameter. We obtain some bounds for \( \gamma_2r(G) \) and characterize the graphs attaining these bounds. We also derive Nordhaus–Gaddum type results for \( \gamma_2r(G) \). We need the following.

**Definition 1.1.** The graph obtained by joining the centres of two stars \( K_{1,r} \) and \( K_{1,s} \) by an edge is defined to be a bistar and is denoted by \( B(r, s) \).

**Theorem 1.2.** [2] Let \( G \) be a connected graph of order \( p \). Then \( \gamma_r(G) = p \) if and only if \( G \) is a star and for any graph \( G \), \( \gamma_r(G) = p \) if and only if \( G \) is a galaxy (disjoint union of stars).

**Theorem 1.3.** [2] If \( T \) is a tree of order \( p \geq 3 \) then \( \gamma_r(T) = p - 2 \) if and only if \( T \) is obtained from \( P_4, P_5 \) or \( P_6 \) by adding zero or more pendent vertices to the supports.

**Theorem 1.4.** [2] Let \( G \) be a connected graph of order \( p \) containing a cycle. Then \( \gamma_r(G) = p - 2 \) if and only if \( G \) is \( C_4, C_5 \), or \( G \) can be obtained from \( C_3 \) by attaching zero or more pendent vertices to at most two of the vertices of the cycle.

**Theorem 1.5.** [6] If \( G \) is any connected graph with \( \delta(G) = 1 \) such that \( G \not\cong K_{1,p-1} \), then \( \gamma_r(G) \) is equal to the number of pendent vertices of \( G \) if and only if every nonpendent vertex in \( G \) is a support.

**Theorem 1.6.** [2] If \( k \geq 1 \) is an integer and \( r \in \{1, 2, 3\} \) then \( \gamma_r(C_{3k+r}) = k + r \).

### 2. Main Results

**Definition 2.1.** A set \( S \subseteq V(G) \) is a restrained double dominating set for \( G \) if every vertex in \( V \) is dominated by at least two vertices in \( S \) and \( (V - S) \) has no isolated vertices. The minimum cardinality of a minimal restrained double dominating set is called the restrained double domination number of \( G \) and is denoted by \( \gamma_2r(G) \).

**Example 2.2.**

(i) \( \gamma_2r(K_p) = 2 \) for \( p \neq 3 \) and \( \gamma_2r(K_3) = 3 \).

(ii) \( \gamma_2r(K_{1,p-1}) = p \).
(iii) For the Petersen graph $G$, $\gamma_{2r}(G) = 6$.

(iv) $\gamma_{2r}(C_p) = p$ for all $p$.

(v) $\gamma_{2p}(W_p) = p - 2k$ when $p = 3k + r$, $0 \leq r \leq 2$. Let $u$ be the central vertex of $W_p$ and $v_1, v_2, \ldots v_{p-1}$ be the vertices of $C_{p-1}$. By Theorem 1.6, $\gamma_r(C_p) = k + r$. Since $u$ is adjacent to every other vertex $\gamma_{2r}(W_p) = \gamma_r(C_p) = k + r = p - 2k$.

(vi) $\gamma_{2r}(K_{m,n}) = 4$ when $3 \leq m \leq n$.

Remarks 2.3.

(i) Every graph $G$ without isolated vertices has a restrained double dominating set, as $V(G)$ is such a set for $G$.

(ii) If $S$ is any restrained double dominating set of $G$ then $S$ contains all pendent vertices, supports and vertices of degree two. Hence it follows from Theorem 1.5 if $\gamma_r(G)$ is equal to the number of pendent vertices then $\gamma_{2r}(G) = p$.

(iii) For any graph $G$ with $\delta(G) \geq 3$, $\gamma_{2r}(G) \leq p - 2$.

(iv) It follows from Theorems 1.3 and 1.4 that there is no graph $G$ with $\gamma_r(G) = \gamma_{2r}(G) = p - 2$.

(v) For a graph $G$ without isolated vertices, $\gamma_r(G) \leq \gamma_{2r}(G)$ and by Theorem 1.2, $\gamma_r(G) = \gamma_{2r}(G) = p$ if and only if $G$ is a galaxy.

(vi) If $G$ is any graph without isolated vertices and $G$ is not a galaxy, then $\gamma_r(G) \neq \gamma_{2r}(G)$. By Theorem 1.2, if $\gamma_{2r}(G) = p$ then $\gamma_r(G) < p$. Suppose $\gamma_{2r}(G) \leq p - 2$. Let $S$ be a $\gamma_{2r}$-set of $G$ and let $u \in V - S$. For any $w \in N(v) \cap S$, $S - \{w\}$ is a restrained dominating set of $G$ so that $\gamma_r(G) < \gamma_{2r}(G)$.

Theorem 2.4. Let $G$ be a graph without isolated vertices and let $P$ be the set of all pendent vertices and supports of $G$ ($P$ may be empty). Then $2 \leq \gamma_{2r}(G) \leq p$. Lower bound is attained if and only if $G \cong K_2$ or $G$ has at least two vertices with full degree and $\delta(G) \geq 3$. Upper bound is attained if and only if for every edge $(u, v)$ in $(V(G) - P)$, either of $\deg u$, $\deg v$ is two or $\delta((V(G) - \{u, v\})) = 0$.

Proof. Clearly $2 \leq \gamma_{2r}(G) \leq p$. Suppose $\gamma_{2r}(G) = 2$ and let $S = \{u, v\}$ be a $\gamma_{2r}$-set. Then $\deg u = \deg v = p - 1$ and $(V - S)$ has no isolated vertices and so $\delta(G) \geq 3$. Converse is obvious.

Assume $\gamma_{2r}(G) = p$. If there exists an edge $(u, v)$ in $(V(G) - P)$ with $\deg u \geq 3, \deg v \geq 3$ and $\delta((V(G) - \{u, v\})) > 0$ then $V(G) - \{u, v\}$ is a restrained double dominating set which is a contradiction.

Conversely let $S$ be a $\gamma_{2r}$-set of $G$. We claim that $S = V$. If not there exists $u, v \in V - S$ such that $u$ and $v$ are adjacent in $G$. Then $\deg u \geq 3$, $\deg v \geq 3$ and $\delta((V(G) - \{u, v\})) \neq 0$ which is a contradiction. Hence $\gamma_{2r}(G) = p$. □
We now prove the following theorem whose proof technique is similar to that in [1].

**Theorem 2.5.** Let $G$ be a graph without isolated vertices. Then $\gamma_{2r}(G) \geq \frac{5p-2q}{4}$ and the bound is attained for the graph $G_1$ given in Figure 1.

![Figure 1](image)

**Proof.** Let $S$ be a $\gamma_{2r}$-set. Every vertex in $V - S$ is adjacent to at least two vertices in $S$ and one vertex in $V - S$. Also every vertex in $S$ must have at least one neighbor in $S$. Hence

$$q \geq 2|V - S| + \frac{|V - S| + \gamma_{2r}(G)}{2}$$

$$= \frac{5}{2}|V - S| + \frac{\gamma_{2r}(G)}{2}$$

$$= \frac{5}{2}(p - \gamma_{2r}(G)) + \frac{\gamma_{2r}(G)}{2}.$$

Thus $2q \geq 5p - 4\gamma_{2r}(G)$, so that $\gamma_{2r}(G) \geq \frac{5p-2q}{4}$. The bound is attained if $G \cong G_1$. \qed

**Theorem 2.6.** If $G$ has no isolated vertices, then $\gamma_{2r}(G) \geq \frac{2p}{\Delta(G)+1}$.

**Proof.** Let $S$ be a $\gamma_{2r}$-set of $G$. Let $s$ be the number of edges with one end in $S$ and the other end in $V - S$. Since every vertex in $S$ has at least one neighbor in $S$,

$$s \leq (\Delta(G) - 1)|S| = (\Delta(G) - 1)\gamma_{2r}(G).$$

Also every vertex in $V - S$ is adjacent to at least two vertices in $S$ and so $s \geq 2|V - S| = 2(p - \gamma_{2r}(G))$.

Thus $2p - 2\gamma_{2r}(G) \leq (\Delta(G) - 1)\gamma_{2r}(G)$ and hence $\gamma_{2r}(G) \geq \frac{2p}{\Delta(G)+1}$.

The bound is attained for the graphs $mK_2$ and $K_p (p \geq 2)$. So the bound is sharp. \qed

**Theorem 2.7.** Let $G = (X, Y)$ be a connected bipartite graph with $|X| = m$ and $|Y| = n$, $2 \leq m \leq n$. Then $\gamma_{2r}(G) = p$ if and only if either $\Delta(G) \leq 2$ or for every $v \in V(G)$ with $\deg v \geq 3$, whenever $u \in N(v)$ either $\deg u \leq 2$ or $u$ is a support.
Proof. Let $\gamma_2^r(G) = p$ and $\Delta(G) \geq 3$. If there exist vertices $u, v \in V(G)$ such that $v$ is not a support, $u \in N(v)$, $\deg v \geq 3$ and $\deg u \geq 3$, then $V(G) - \{u, v\}$ is a restrained double dominating set which is a contradiction.

Converse follows by Remark 2.3 (ii).

Theorem 2.8. Let $G$ be a cubic connected graph. Then $\gamma_2^r(G) = 2$ if and only if $G \cong K_4$ and $\gamma_2^r(G) = 4$ if and only if $G \cong G_i (1 \leq i \leq 5)$ where $G_i$ are given in Figure 2.

![Figure 2](image-url)

Proof. Suppose $\gamma_2^r(G) = 2$. Since $G$ is a cubic graph, by Theorem 2.4, $G \cong K_4$. Converse is obvious.

Suppose $\gamma_2^r(G) = 4$. Let $S = \{u, v, w, x\}$ be a $\gamma_2^r$-set.

Case (i) $\langle S \rangle$ contains a path of length 3.

In $\langle S \rangle$, $\deg v = \deg w = 2$. Let $y$ and $z$ be the neighbors of $v$ and $w$ in $V - S$ respectively. Since $G$ is cubic, $|V - S| = 2$. Hence $u$ and $x$ are adjacent. Obviously $y$ and $z$ are adjacent. If $y$ is adjacent to $u$ and $z$ is adjacent to $x$ then $G \cong G_1$. If $y$ is adjacent to $x$ and $z$ is adjacent to $u$ then $G \cong G_2$.

Case (ii) $\langle S \rangle$ does not contain a path of length 3.
In this case \( \langle S \rangle \cong 2K_2 \). Without loss of generality, let \( u \) and \( v \) be adjacent and \( w \) and \( x \) be adjacent. Clearly \( |V - S| = 4 \). If \( u \) and \( v \) have two common neighbors in \( V - S \) then \( w \) and \( x \) also have two common neighbors and so \( G \cong G_3 \).

If \( u \) and \( v \) have only one common neighbor say \( s \), then \( w \) and \( x \) have only one common neighbor \( t \).

Let \( w_1 \in N(u) \cap (V - S) \) (\( w_1 \) is different from \( t \)) and \( v_1 \in N(v) \cap (V - S) \) (\( v_1 \) is different from \( s \)). Without loss of generality let \( w_1 \in N(w) \) and \( v_1 \in N(x) \). If \( s \in N(w_1) \) and \( v_1 \in N(t) \), then \( G \cong G_3 \). If \( s \in N(v_1) \) and \( w_1 \in N(t) \), then also \( G \cong G_3 \). If \( s \in N(t) \) and \( w_1 \in N(v_1) \), then \( G \cong G_4 \).

If \( u \) and \( v \) do not have common neighbors, then \( w \) and \( x \) also do not have common neighbors.

Let \( u_1 \in N(u) \cap (V - S), v_1 \in N(v) \cap (V - S), w_1 \in N(w) \cap (V - S) \) and \( x_1 \in N(x) \cap (V - S) \). Without loss of generality let \( u_1 \in N(u), v_1 \in N(x), w_1 \in N(w) \) and \( x_1 \in N(v) \).

If \( u_1 \) and \( v_1 \) are adjacent and \( w_1 \) and \( x_1 \) are adjacent, then \( G \cong G_5 \). If \( u_1 \) and \( w_1 \) are adjacent and \( v_1 \) and \( x_1 \) are adjacent, then \( G \cong G_3 \). If \( u_1 \) and \( x_1 \) are adjacent and \( v_1 \) and \( w_1 \) are adjacent then \( G \cong G_5 \). Converse is obvious.

**Proposition 2.9.** There exists no connected cubic graph \( G \), with \( \gamma_{2r}(G) = 3 \).

**Proof.** Let \( S \) be a \( \gamma_{2r} \)-set with \( |S| = 3 \). Clearly \( \langle S \rangle \cong P_3 = (u, v, w) \). Let \( y \) be the neighbor of \( v \) in \( V - S \). Without loss of generality let \( y \in N(u) \). Since \( y \) has a neighbor in \( V - S \), there exists \( x \in V - S \) such that \( x \in N(u) \cap N(w) \). But then \( \deg w = 2 \) and so there exists no connected cubic graph with \( \gamma_{2r}(G) = 3 \).

**Theorem 2.10.** If \( T \) is a tree such that \( \overline{T} \) has no isolated vertices, then \( \gamma_{2r}(T) + \gamma_{2r}(\overline{T}) \leq 2p \). Equality holds if and only if \( T \cong P_4, P_5 \) or \( B(2, 1) \).

**Proof.** Obviously, \( \gamma_{2r}(T) + \gamma_{2r}(\overline{T}) \leq 2p \). Suppose \( \gamma_{2r}(T) + \gamma_{2r}(\overline{T}) = 2p \).

We claim that \( \text{diam}(T) = 3 \) or \( 4 \). Clearly \( \text{diam}(T) \geq 3 \). Suppose \( \text{diam}(T) = d \geq 5 \) and let \( v_1, v_2, \ldots, v_{d+1} \) be the diametrical path in \( T \). Now \( V(T) - \{v_1, v_{d+1}\} \) is a restrained double dominating set of \( T \) which is a contradiction and so \( \text{diam}(T) = 3 \) or \( 4 \).

Suppose \( \text{diam}(T) = 3 \). Let \( (u, u_1, v_1, v) \) be the diametrical path. If \( \deg u_1 = \deg v_1 = 2, T \cong P_4 \). If \( \deg u_1 \geq 4 \) or if \( \deg u_1 = 3 \) and \( \deg v_1 \geq 3 \), then we get a restrained double dominating set of \( T \) with cardinality \( p - 2 \) and so \( T \cong B(2, 1) \).

If \( \text{diam}(T) = 4 \), then \( T \cong P_5 \), since in all other cases we get a restrained dominating set of \( T \) with cardinality \( p - 2 \).

**Theorem 2.11.** Let \( G \) be a connected unicyclic graph such that \( \overline{G} \) has no isolated vertices. Then \( \gamma_{2r}(G) + \gamma_{2r}(\overline{G}) \leq 2p \) and equality holds if and only if \( G \cong C_4, C_5 \) or \( G_i(1 \leq i \leq 4) \) where \( G_i \) are given in Figure 3.
Proof. Clearly \( \gamma_{2r}(G) + \gamma_{2r}(\bar{G}) \leq 2p \). Suppose \( \gamma_{2r}(G) + \gamma_{2r}(\bar{G}) = 2p \) and let \( C_n = (v_1, v_2, \ldots, v_n = v_1) \) be the cycle in \( G \). If \( n \geq 6 \), \( V(G) - \{v_2, v_5\} \) is a restrained double dominating set of \( \bar{G} \) and so \( n \leq 5 \). If \( n = 5 \) and if \( u \in V(G) - C_n \), then for any \( v \in C_n \cap (V - N(u)) \), \( V(G) - \{u, v\} \) is a restrained double dominating set of \( \bar{G} \) and so \( G \cong C_5 \). Suppose \( n = 4 \). If there exists a pendent vertex \( u \in V(G) - C_n \) such that \( d(u, v_i) \geq 2 \) for some \( i (1 \leq i \leq 4) \), then \( V(G) - \{u, v_{i-1}\} \) is a restrained double dominating set of \( \bar{G} \). If \( G \) contains two pendent vertices say \( u \) and \( v \), then \( V(G) - \{u, v\} \) is a restrained double dominating set of \( \bar{G} \). Hence \( G \cong C_4 \) or \( G_1 \). If \( n = 3 \), as above we can show that every vertex not on \( C_3 \) is at most at distance two from a vertex of \( C_3 \). If there exists two vertices at distance two from a vertex of \( C_3 \), we get a similar contradiction. Also the degree of every vertex of \( C_3 \) is either two or three, since otherwise either \( \bar{G} \) has isolated vertices or \( \bar{G} \) has a restrained double dominating set of cardinality \( p - 2 \). Hence \( G \cong G_2, G_3 \) or \( G_4 \). Converse is obvious. \( \square \)

**Theorem 2.12.** If \( T \) is a tree different from the star \( K_{1,p-1} \), then

\[
\gamma_{2r}(T) = \begin{cases} 
5 & \text{if } T \cong P_3, B(r, s) \text{ with } r + s = 3 \\
4 & \text{if } T \cong B(r, s) \text{ with } r + s \neq 3 \text{ or } T_1 \\
3 & \text{otherwise.}
\end{cases}
\]

**Proof.** Let \( S \) be the set of all supports of \( T \). Since \( G \) is different from star, \( |S| \geq 2 \).
Case (i) $|S| \geq 4$.

Let $u_1, v_1, w_1, x_1$ be four distinct supports with pendent vertices, $u \in N(u_1), v \in N(v_1), w \in N(w_1)$ and $x \in N(x_1)$. Since $T$ is a tree, at least two of these four supports are nonadjacent. Without loss of generality we can assume that $x_1$ is nonadjacent to $u_1$. Let $D = \{u, v, w\}$. Clearly $D$ is a double dominating set in $T$ and by choice of $x_1, \langle V - D \rangle$ has no isolated vertices. Hence $D$ is a restrained double dominating set of $\bar{T}$. Also no set of cardinality two can be a restrained double dominating set of $\bar{T}$ and so $\gamma_{2r}(\bar{T}) = 3$.

Case (ii) $|S| = 3$.

Let $S = \{u_1, v_1, w_1\}$ where $u, v$ and $w$ are pendent vertices with $u \in N(u_1), v \in N(v_1), w \in N(w_1)$. Suppose $\langle S \rangle \cong P_3$. If $\deg u_1 \geq 3$ or $\deg w_1 \geq 3$, then $D = \{u, v, w\}$ is a restrained double dominating set in $T$. Also no set of cardinality two can be a restrained double dominating set of $\bar{T}$ and so $\gamma_{2r}(\bar{T}) = 3$. If $\deg u_1 = \deg v_1 = 2$ and $\deg v_1 \geq 3$, then $\gamma_{2r}(\bar{T}) = 3$ or $T$ is isomorphic to the tree $T_1$ given in Figure 4 and $\gamma_{2r}(\bar{T}) = 4$. If $\langle S \rangle \cong K_2 \cup K_1$ or $3K_1$ then there exists a vertex $x$ which is neither a pendent vertex nor a support and $x$ is nonadjacent to at least one of $\{u_1, v_1, w_1\}$. In this case, $D = \{u, v, w\}$ is a $\gamma_{2r}$-set in $T$ and so $\gamma_{2r}(\bar{T}) = 3$.

Case (iii) $|S| = 2$.

Let $S = \{u_1, v_1\}$ and let $u$ and $v$ be pendent vertices with $u \in N(u_1)$ and $v \in N(v_1)$. If $u_1$ and $v_1$ are adjacent then $T \cong B(r, s)$. It can be easily verified that $\gamma_{2r}(\bar{T}) = 5$ if $r + s = 3$ and $\gamma_{2r}(\bar{T}) = 4$ otherwise.

Suppose $u_1$ and $v_1$ are nonadjacent. If there exists a vertex $x$ adjacent to both $u_1$ and $v_1$, either $T \cong P_5$ in which case $\gamma_{2r}(\bar{T}) = 5$ or $\deg u_1(\deg v_1) \geq 3$ and $\{u, u_1, v\}$ is a $\gamma_{2r}$-set in $\bar{T}$ so that $\gamma_{2r}(\bar{T}) = 3$. Otherwise $\{u, u_1, v\}$ is a $\gamma_{2r}$-set and so $\gamma_{2r}(\bar{T}) = 3$.

Corollary 2.13. If $T$ is a tree different from the star $K_{1, p-1}$, then $\gamma_{2r}(T) + \gamma_{2r}(\bar{T}) \leq p + 5$ and equality holds if and only if $T$ is $P_5$, $B(r, s)$, where $r + s = 3$. If $T \not\cong P_5$, $B(r, s)$, where $r + s = 3$, then $\gamma_{2r}(T) + \gamma_{2r}(\bar{T}) \leq p + 4$ and equality holds if and only if $T \cong T_1, T_2, B(r, s)$, where $r + s \neq 3$. In all other cases, $\gamma_{2r}(T) + \gamma_{2r}(\bar{T}) \leq p + 3$ and this bound is sharp.

Proof. Follows from Theorem 2.12. Sharpness is exhibited by the graph given in Figure 5.
Theorem 2.14. Let $G$ be a connected bipartite graph with bipartition $X, Y$, $|X| = m$, $|Y| = n$ ($2 \leq m \leq n$) such that $\overline{G}$ has no isolated vertices. If $G \not\cong K_{2,3}$ or $K_{3,n}$, then $\gamma_{2r}(\overline{G}) = 3$ or 4. Furthermore $\gamma_{2r}(\overline{G}) = 4$ if and only if every pair of vertices $u, v$ in $X$, $N(u) \cap N(v) \cap Y \neq \emptyset$ and vice versa.

Proof. Clearly $\gamma_{2r}(\overline{G}) \geq 2$. If $\{u, v\}$ is a $\gamma_{2r}$-set in $\overline{G}$, then $u$ and $v$ are isolated vertices in $G$ and so $\gamma_{2r}(\overline{G}) \geq 3$. Since $G \not\cong K_{2,3}$ or $K_{3,n}$, there exists a set $S$ with $|S \cap X| \geq 2$ and $|S \cap Y| \geq 2$ such that $S$ is a restrained double dominating set of $\overline{G}$ and hence $\gamma_{2r}(\overline{G}) \leq 4$.

Suppose $\gamma_{2r}(\overline{G}) = 4$. If there exists $u, v$ in $X$ with $N(u) \cap N(v) \cap Y = \emptyset$, then for any $w \in Y, \{u, v, w\}$ is a $\gamma_{2r}$-set in $\overline{G}$, which is a contradiction. Converse is obvious.

Corollary 2.15. Let $G$ be a connected bipartite graph such that $\overline{G}$ has no isolated vertices. Then $\gamma_{2r}(G) + \gamma_{2r}(\overline{G}) \leq p + 4$ and the bound is sharp.

Proof. Follows from Theorem 2.14. Sharpness is exhibited by the graph given in Figure 6.

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