AKCE J. Graphs. Combin., 5, No. 1 (2008), pp. 83-97

Geometric Labeled Graphs

S.M. Hegde
Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka
Surathkal, Srinivasanagar-575025
Karnataka, India.
e-mail: *smhegde@nitk.ac.in*
and
P. Shankaran
Department of Mathematics
N.M.A.M. Institute of Technology
Nitte-574110, Karnataka, India.
email: *shankarpbharath@yahoo.co.in*

Communicated by: S. Arumugam
Received 13 June 2006; accepted 7 December 2007

Abstract

Let $G = (V, E)$ be a (p, q)-graph. The graph G is said to be (a, r)-*geometric* if its vertices can be assigned distinct positive integers so that the values of the edges, obtained as the products of the numbers assigned to their end vertices, can be arranged as a geometric progression $a, ar, ar^2, \ldots, ar^{q-1}$. In this paper we present results on geometric labelings of some classes of graphs and give some necessary conditions for graphs to admit geometric labelings.

Keywords: Geometric labelings / graphs, planar grid, generalized n-crown, Tp-trees.

2000 Mathematics Subject Classification: 05C78.

1. Introduction

We consider only finite graphs. For all notations in graph theory we follow Harary [4] and West [8].

Several practical problems in real life situations have motivated the study of labelings of a graph $G = (V, E)$, which are required to obey variety of conditions. There is an enormous literature dealing with several kinds of labelings of graphs over the past three decades or so and for a dynamic survey of various graph labeling problems, we refer to Gallian [3].

Graph labeling, where the vertices are assigned values subject to certain conditions, have often been motivated by practical problems, but they are also of interest in their own right.

Most interesting graph labeling problems have three ingredients:

(i) a set of numbers S from which the labels are chosen;

(ii) a rule that assigns a value to each edge;

(iii) a condition that these values must satisfy.

In this paper we are interested in the study of multiplicative functions.

Let $G = (V, E)$ be a graph. A function $f : V(G) \to N$ is called a multiplicative function if the induced function $f^{\times} : E(G) \to N$ is defined as $f^{\times}(xy) = f(x) \cdot f(y)$, for all $xy \in E(G)$. A multiplicative function f is called a multiplicative labeling if both f and $f^{\times}$ are injective.

The following result gives a general property of multiplicative functions, which can be proved by easy counting arguments.

Theorem 1.1. [1] *For any graph G and for any multiplicative function $f : V(G) \to \mathbb{N}$*

$$\prod_{e \in E(G)} f^{\times}(e) = \prod_{u \in V(G)} f(u)^{d(u)}. \qquad (1)$$

We adopt the following notations throughout this paper:

$M(G) =$ The set of all multiplicative labelings of G.

$f(G) = \{f(u) : u \in V(G)\}$; $\qquad f^{\times}(G) = \{f^{\times}(e) : e \in E(G)\}$;

$f_{min}(G) = \min\limits_{u \in V(G)} f(u)$; $\qquad f_{max}(G) = \max\limits_{u \in V(G)} f(u)$;

$f^{\times}_{max}(G) = \max\limits_{e \in E(G)} f^{\times}(e)$; $\qquad \Theta(G) = \min\limits_{f \in M(G)} f_{max}(G)$.

2. Geometric Labelings

Hegde [5] has initiated a study on geometric graphs. Given a (p, q)-graph G and an $f \in M(G)$, we say that f is (a, r)*-geometric*, if $f^{\times}(G) = \{a, ar, ar^2, \ldots, ar^{q-1}\}$, where a and r are positive integers ≥ 2. Let $M_{a,r}(G)$ denote the set of all (a, r)-geometric labelings of G. A graph G is called *geometric* if $M_{a,r}(G) \neq \emptyset$ and it is called (a, r)*-geometric* if it is (a, r)-geometric for at least one pair of values (a, r).

Theorem 2.1. *Let G be a connected (a, a)-geometric (p, q)-graph. Then for any $f \in M_{a,a}(G)$, $1 \in f(G)$ if and only if $a | f(u)$ for every $u \in V(G)$ with $f(u) \neq 1$.*

Proof. Let $f \in M_{a,a}(G)$. Then $f^{\times}(G) = \{a^j : 1 \leq j \leq q\}$. Suppose $a|f(u)$ for all $u \in V(G)$, with $f(u) \neq 1$. Then for the edge $xy \in E(G)$ with $f^{\times}(xy) = a$, we must have either $f(x)$ or $f(y)$ to be 1, whence $1 \in f(G)$.

For the converse, let $v \in V(G)$ be such that $f(v) = 1$. Then for all $w \in N(v) = \{u \in V(G) : uv \in E(G)\}$ we must have $f(v) \cdot f(w) = f^{\times}(vw) = a^t$ where t is a positive integer. This yields $f(w) = a^t$ as $f(v) = 1$. Thus $a|f(w)$ for all $w \in N(v)$. Now, fix any $w_0 \in N(v)$. Then for any $w_1 \in N(w_0) - N(v)$ we have $f(w_0) \cdot f(w_1) = f^{\times}(w_0 w_1) = a^j$ for some positive integer $j \geq 1$, so that $f(w_1) = a^{|j-t|}$, whence $a|f(w_1)$. Since w_1 was arbitrary by choice we get that $a|f(w)$ for all $w \in N(v) \bigcup_{u \in N(v)} N(u)$. Continuing this process, we get $a|f(w)$ for all $w \in V - \{v\}$. □

From the above proof one can see that for any (a, r)-geometric graph G with a, a prime number or the square of a prime number, $1 \in f(G)$ for every $f \in M_{a,r}(G)$.

Theorem 2.2. *Let G be a connected bipartite graph which is not a star. If G is an (a, a)-geometric graph, then for any $f \in M_{a,a}(G)$, $1 \notin f(G)$.*

Proof. Consider any $f \in M_{a,a}(G)$ and let $A = \{u_1, u_2, \ldots, u_a\}$ and $B = \{v_1, v_2, \ldots, v_b\}$ be a bipartition of G. Suppose that $1 \in f(G)$. Without loss of generality, assume that $f(u_1) = 1$. Since $f^{\times}(G) = \{a, a^2, \ldots, a^q\}$ we must have $f(v_j) = a$ for some $v_j \in N(u_1)$. Without loss of generality, let $j = 1$. Next let $xy \in E(G)$ be such that $a^2 = f^{\times}(xy) = f(x) \cdot f(y)$. By Theorem 2.1, a divides both $f(x)$ and $f(y)$, or one of $f(x)$ and $f(y)$. Thus $f(x) = 1$ and $x = u_1$. Also $f(y) = a^2$ and without loss of generality, we may take $y = v_2$. Next let $xy \in E(G)$ be such that $a^3 = f^{\times}(xy) = f(x) \cdot f(y)$. Again by Theorem 2.1, a divides both $f(x)$ and $f(y)$ or one of $f(x)$ and $f(y)$. Therefore, we see that either one of $f(x)$ and $f(y)$ is a and the other is a^2 or that one of $f(x)$ and $f(y)$ is 1 and the other is a^3. Clearly, the first possibility cannot arise as f is injective and vertices labeled a and a^2 have already occurred in B. So the latter case must hold. Then, without loss of generality, we may assume that $f(x) = 1$ and $f(y) = a^3$. But then injectivity of f forces $x = u_1$, so that $y = v_3$. Continuing this way, we see that if $N(u_1) = \{v_1, v_2, \ldots, v_t\}$ where t is the degree of u_1 in G, then $f(v_j) = a^j$ for each $j, 1 \leq j \leq t$. Since G is not a star and connected, it follows that $|A| \geq 2$ so that $a^{t+1} \in f^{\times}(G)$. Let $xy \in E(G)$ be such that $a^{t+1} = f^{\times}(xy) = f(x) \cdot f(y)$. This yields $x \neq u_1$ and $f(x) = (a^{t+1})/f(y) \leq a^t$ or $f(y) = (a^{t+1})/f(x) \leq a^t$ as $f(x) \geq a$ and $f(y) \geq a$, a contradiction to the fact that $f(N(u_1)) = \{a^j : 1 \leq j \leq t\}$ and that f is injective. □

Corollary 2.3. *No connected bipartite graph, except the star is (a, a)-geometric when a is a prime number or square of a prime number.*

Corollary 2.4. *Any connected (a, a)-geometric graph when a is a prime number or square of a prime number, is either a star or has a triangle.*

Remark 2.5. *In Corollary 2.3 it is possible to relax the condition of connectedness of* G *if* $a = 2, 3$ *or* 5. *However, if* $a = 4$ *then it is not possible to do so as* $G = K_2 \cup K_{1,3}$ *has (4, 4)-geometric labeling, which can be easily verified.*

A function g is called a *graceful labeling* of a (p,q)-graph $G = (V,E)$, if g is an injection from $V(G)$ to the set $\{0,1,\ldots,q\}$ such that, when each edge uv is assigned the label $|g(u) - g(v)|$, the resulting edge labels are distinct. Rosa [6] introduced this concept and also defined a balanced labeling of a graph G as a graceful labeling g of G such that for each edge uv of G either $g(u) \le m < g(v)$ or $g(v) \le m < g(u)$ for some integer m, called the characteristic of g. Such a labeling g is also called an α-valuation.

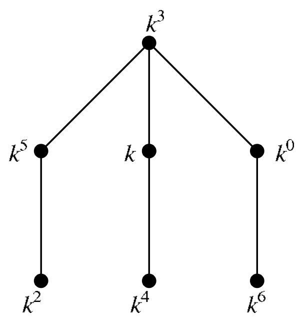
Theorem 2.6. *If* T *is a balanced tree with* p *vertices and* q *edges, characterized by* m, *then* T *is* (a^{m+1}, a)*-geometric, where* a *is any integer greater than 1.*

Proof. Let g be an α-valuation of T, characterized by m. Construct a new valuation f on T as follows:

$$f(v) = \begin{cases} a^{g(v)} & \text{if } 0 \le g(v) \le m \\ a^{q+m+1-g(v)} & \text{if } m+1 \le g(v) \le q. \end{cases}$$

One can check that f thus defined represents a geometric labeling and $f^{\times}(T)$ contains the numbers $k^{m+1}, k^{m+2}, \ldots, k^{m+q}$. Hence the proof. □

Remark 2.7. *Note that the converse is not true. We know that* $S(K_{1,3})$ *(subdivision graph of* $K_{1,3}$*) is not a balanced graph. However* $S(K_{1,3})$ *admits a geometric labeling, as shown in Figure 1.*



A tree which admits (k^3, k)-geometric labeling but not balanced
Figure 1

Theorem 2.8. *Let* $K_{a,b}$ *be the complete bipartite graph with* $min(a,b) \ge 2$. *Let* f *be a* (t^k, t)*-geometric labeling of* $K_{a,b}$ *with* $1 \in f(G)$. *Let* $A = \{u_1, u_2, \ldots, u_a\}$ *and* $B = \{v_1, v_2, \ldots, v_b\}$ *be the bipartition of* $K_{a,b}$. *Then there exists a divisor* i *of* a *or* b, *say* *a, and sets of integers,* $A' = \{1, t, \ldots, t^{i-1}, t^{bi}, t^{bi+1}, \ldots, t^{((a/i)-1)bi}, \ldots, t^{((a/i)-1)bi+i-1}\}$, $B' = \{1, t^i, \ldots, t^{(b-1)i}\}$, *such that* $f(A) = t^k \cdot A'$ *and* $f(B) = B'$ *or* $f(A) = A'$ *and* $f(B) = t^k \cdot B'$.

Proof. Since $1 \in f(G)$, any vertex adjacent to the vertex with f-value 1 will have f-value of the form $t^k \cdot t^i$. Thus, the vertices of G can be partitioned into two sets with f-values of the form $t^k \cdot C$ and D respectively. Here C and D denote sets of non-negative integers with the property that $C + D = \{0, 1, \ldots, ab-1\}$.

Thus, we have to determine the nature of two sets of positive integers $A' = \{1 = t^{i_1} < t^{i_2} < \cdots < t^{i_a}\}$ and $B' = \{1 = t^{j_1} < t^{j_2} < \cdots < t^{j_b}\}$ with the property that $A' \cdot B' = \{1, t, \ldots, t^{(ab-1)}\}$.

Now, t must belong to one of the two sets A' or B', say A'. Let t^i be the first missing integer from A', i.e. $1, t, \ldots, t^{i-1}$ belongs to A' and t^i does not belong to A'. Then, we can easily verify that A' and B' are of the form given in the theorem. The f-values are then got by multiplying t^k to A' or B' as the case may be and then taking other set as it is. This completes the proof. □

For example: Consider the complete bipartite graph $K_{4,4}$. Here $a = b = 4$. Let $i = 2, t = 3, k > 9$. Then $A' = \{3^0, 3, 3^8, 3^9\}$ and $B' = \{3^0, 3^2, 3^4, 3^6\}$. One can see that both $t^k \cdot A', B'$ as well as $A', t^k \cdot B'$ give (t^k, t)-geometric labelings of $K_{4,4}$.

Definition 2.9. [7] *Let* u *be a vertex of* $P_m \times P_n$ *such that* $deg(u) = 2$. *Introduce an edge between every pair of distinct vertices* v, w *with* $deg(v)$, $deg(w) \neq 4$ *and* $d(u,v) = d(u,w)$, *where* $d(u,v)$ *is the distance between* u *and* v. *The graph so obtained is defined as the level joined planar grid and is denoted as* $LJ : P_m \times P_n$.

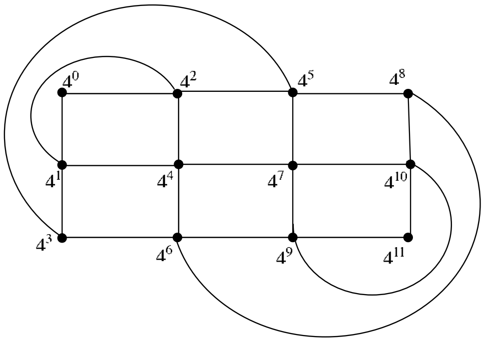
Theorem 2.10. *The graph* $LJ : P_m \times P_n$ *is* (a,a)*-geometric.*

Proof. Denote the vertex of i^{th} row and j^{th} column of $P_m \times P_n$ as $v_{ij}, 1 \leq i \leq m$, $1 \leq j \leq n$. Without loss of generality we can assume that $m \leq n$. Construct the graph $LJ : P_m \times P_n$ as explained above. Note that it contains $2mn - 3$ edges. Define the map $f : V(LJ : P_m \times P_n) \to \{a^0, a, \ldots, a^{mn-1}\}$ by

$$f(v_{1j}) = \begin{cases} a^{\frac{j(j+1)}{2}-1} & 1 \leq j \leq m \\ a^{\frac{m(2j+1-m)}{2}} & m < j \leq n. \end{cases}$$

$$f(v_{ij}) = \begin{cases} a^{\frac{f(v_{i-1,j+1})}{k}} & 2 \leq i \leq m, 1 \leq j \leq n-1 \\ f(v_{i-1,j}) \cdot a^{m+n+1-i-j} & 2 \leq i \leq m, j = n. \end{cases}$$

We can easily verify that the map f so defined is injective and $f^{\times}(LJ : P_m \times P_n)$ is $\{a, a^2, \ldots, a^q\}$. □



A (4,4)-gemetric labeling of $LJ : P_3 \times P_4$.
Figure 2

Theorem 2.11. *For any two integers k and m, both greater than 1 and m odd, the graph mP_n is (a^r, a)-geometric, where $r = \frac{mn+3}{2}$ or $r = \frac{m(n+1)+3}{2}$ according as n is odd or even.*

Proof. Denote $V(mP_n)$ as $V_1 \cup V_2 \cup \cdots \cup V_m$, where $V_i = \{v_i^1, v_i^2, \ldots, v_i^n\}$. Note that mP_n has $m(n-1)$ edges. We have two cases.

Case 1. m and n are odd.

For $i = 1, 2, \ldots, \frac{(m-1)}{2}$, define

$$f(v_i^j) = \begin{cases} a^{\frac{(n+j-1)m+1}{2}+i} & \text{for } j = 1, 3, \ldots, n \\ a^{\left(\frac{j}{2}-1\right)m+i} & \text{for } 2, 4, \ldots, n-1. \end{cases}$$

For $i = (m+1/2), (m+3/2), \ldots, m$, define

$$f(v_i^j) = \begin{cases} a^{\frac{(n+j-3)m+1}{2}+i} & \text{for } j = 1, 3, \ldots, n \\ a^{\left(\frac{j}{2}-1\right)m+i} & \text{for } 2, 4, \ldots, n-1. \end{cases}$$

It can be easily verified that f is injective and $f^{\times}(mP_n) = \{a^r, a^{r+1}, \ldots, a^{r+q-1}\}$, where $r = \frac{(mn+3)}{2}$. Hence mP_n is (a^r, a)-geometric where m and n are odd.

Case 2. m is odd and n is even.

For $i = 1, 2, \ldots, \frac{m-1}{2}$, define

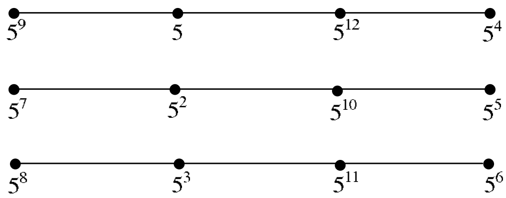
$$f(v_i^j) = \begin{cases} a^{\frac{(n+j)m+1}{2}+i} & \text{for } j = 1, 3, \ldots, n-1 \\ a^{\left(\frac{j}{2}-1\right)m+i} & \text{for } 2, 4, \ldots, n. \end{cases}$$

For $i = \frac{m+1}{2}, \frac{m+3}{2}, \ldots, m$, define

$$f(v_i^j) = \begin{cases} a^{\frac{(n+j-2)m+1}{2}+i} & \text{for } j = 1, 3, \ldots, n-1 \\ a^{(\frac{j}{2}-1)m+i} & \text{for } 2, 4, \ldots, n. \end{cases}$$

Then one can check that f is injective and $f^{\times}(mP_n) = \{a^r, a^{r+1}, \ldots, a^{r+q-1}\}$, where $r = \frac{(m(n+1)+3)}{2}$. Hence mP_n is (a^r, a)-geometric for m odd and n even. □

Given below is an example for Case 2.



A $(5^9, 5)$-geometric labeling of $3P_4$
Figure 3

3. Cycle Related Geometric Graphs

In this section we prove that some cycle related graphs are geometric.

Definition 3.1. *The Helm H_n is the graph obtained from a wheel by attaching a pendent edge at each vertex of the n-cycle. Let $CH(2,n)$ be the graph obtained from H_n by joining the pendant vertices to form a cycle. The graph obtained by iterating t times the process of adding pendent vertices and joining them to form a cycle is called the generalized closed helm and is denoted by $CH(t,n)$.*

Theorem 3.2. *For all positive integers $k > 1, d \geq 1$, odd n, the generalized closed helm $CH(t,n)$ is (k^r, k^d)-geometric, where $r = (\frac{n-1}{2})d$.*

Proof. Denote the central vertex of $CH(t,n)$ as $v_{0,0}$, the successive vertices of the innermost cycle as $v_{1,1}, v_{1,2}, \ldots, v_{1,n}$ and the vertex adjacent to $v_{1,\frac{n+1}{2}}$ as $v_{2,1}$. Next, denote the vertices adjacent to $v_{1,\frac{n+3}{2}}, v_{1,\frac{n+5}{2}}, \ldots, v_{1,n}, v_{1,1}, \ldots, v_{1,\frac{n-1}{2}}$ on the second cycle as $v_{2,2}, v_{2,3}, \ldots, v_{2,\frac{n+1}{2}}, \ldots, v_{2,n}$. Then the vertex adjacent to $v_{2,\frac{n+1}{2}}$ of the third cycle as $v_{3,1}$ etc. Define the map $f : V(CH(t,n)) \to N$ by

$f(v_{i,j}) = k^{((i-1)n+j-1)d}$, $1 \leq i \leq t$, $1 \leq j \leq n$ and

$f(v_{0,0}) = f(v_{t,n}) \cdot f(v_{t,\frac{n+1}{2}}) \cdot k^d$.

Then one can see that f defined above is injective and it is not difficult to verify that $f^{\times}(CH(t,n)) = \{k^r, k^{r+d}, \ldots, k^{r+(q-1)d}\}$, where $r = (\frac{n-1}{2})d$. □

Definition 3.3. *The graph obtained by joining the pendent vertices of a helm to form a cycle and then adding a single pendent edge to each vertex of this outer cycle is called a web. The graph obtained by iterating* t *times the process of adding pendent edges and joining them to form a cycle and then adding pendent edges to the new cycle is called the generalized web and is denoted by* $W(t,n)$.

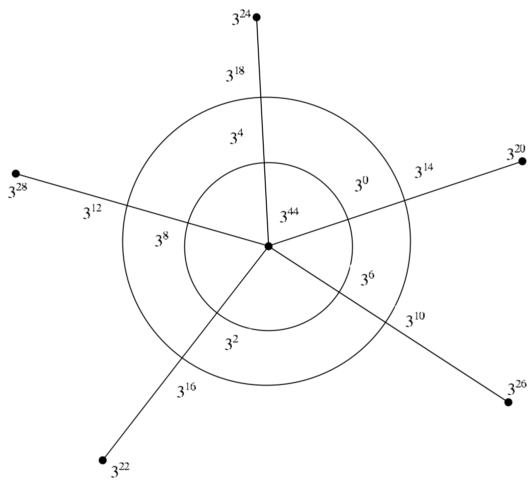
Theorem 3.4. *For all positive integers* $k > 1, d \geq 1$, *odd* n, *the generalized web,* $W(t,n)$ *is* (k^r, a)*-geometric, where* $a = k^d, r = (\frac{n-1}{2})d$.

Proof. Denote the vertices of the innermost cycle of $W(t,n)$ successively as $v_{1,1}, v_{1,2}, \ldots, v_{1,n}$, the vertices adjacent to $v_{1,1}, v_{1,2}, \ldots, v_{1,n}$ of the second cycle as $v_{2,1}, v_{2,2}, \ldots, v_{2,n}, \ldots$, the vertices on the t^{th} cycle as $v_{t,1}, v_{t,2}, \ldots, v_{t,n}$, the pendant vertices adjacent to $v_{t,1}, v_{t,2}, \ldots, v_{t,n}$ as $v_{t+1,1}, v_{t+1,2}, \ldots, v_{t+1,n}$ respectively and the centre as $v_{0,0}$. Define a labeling $f : V(W(t,n)) \to N$ by

$$f(v_{i,j}) = \begin{cases} k^{\left((i-1)n+\frac{j-1}{2}\right)d} & \text{for } i,j \text{ odd, } 1 \leq i \leq t+1,\ 1 \leq j \leq n \\ k^{\left((i-1)n+\frac{n+j-1}{2}\right)d} & \text{for } i \text{ odd, } j \text{ even } 1 \leq i \leq t+1, 2 \leq j \leq n-1 \\ k^{\left((i-2)n+\frac{3n+j-2}{2}\right)d} & \text{for } i \text{ even, } j \text{ odd, } 2 \leq i \leq t+1,\ 1 \leq j \leq n \\ k^{\left((i-1)n+\frac{j-2}{2}\right)d} & \text{for } i,j \text{ even, } 2 \leq i \leq t+1,\ 2 \leq j \leq n-1 \\ k^{\left(2tn+\frac{n-1}{2}\right)d} & \text{for } i=j=0. \end{cases} \quad (2)$$

Then one can see that f defined above is injective and $f^{\times}(W(t,n)) = \{k^r, k^r a, k^r a^2, \ldots, k^r a^{q-1}\}$, where $a = k^d, r = (\frac{n-1}{2})d$. □

Given below is a $(3^4, 3^2)$-geometric labeling of $W(2,5)$.



A $(3^4, 3^2)$-geometric labeling of $W(2,5)$
Figure 4

Corollary 3.5. *For n odd, the helm $H_n = W_n + K_1$ is (k^r, a)-geometric, where $a = k^d, r = \left(\frac{n-1}{2}\right) d$.*

Proof. Take $t = 1$ in (2). □

Corollary 3.6. *For n odd, the web $W(2,n)$ is (k^r, a)-geometric, where $a = k^d, r = \left(\frac{n-1}{2}\right) d$.*

Proof. Take $t = 2$ in (2). □

Corollary 3.7. *For n odd, the generalized web without centre $W_0(t,n)$ is (k^r, a)-geomtric, where $a = k^d, r = \left(\frac{n-1}{2}\right) d$.*

Proof. By deleting the last definition from (2), the proof follows. □

Corollary 3.8. *For n odd, the generalized P-webcone, $W_0(t,n) + \overline{K}_p$ is (k^r, a)-geometric, where $a = k^d, r = \left(\frac{n-1}{2}\right) d$.*

Proof. Denote the vertices of the innermost cycle of $W_0(t,n)$ as $v_{1,1}, v_{1,2}, \ldots, v_{1,n}$, the vertices adjacent to $v_{1,1}, v_{1,2}, \ldots, v_{1,n}$ of the second cycle as $v_{2,1}, v_{2,2}, \ldots, v_{2,n}, \ldots$, the vertices of the t^{th} cycle as $v_{t,1}, v_{t,2}, \ldots, v_{t,n}$ and the pendant vertices adjacent to $v_{t,1}, v_{t,2}, \ldots, v_{t,n}$ as $v_{t+1,1}, v_{t+1,2}, \ldots, v_{t+1,n}$ respectively. Let $V(\overline{K}_p) = \{v_{t+2,j} : 1 \leq j \leq p\}$. Replace the last definition in (2) by

$$f(v_{ij}) = k^{\left(n\left(t(j+1)+j-\frac{1}{2}\right)-\frac{1}{2}\right)d} \text{ for } i = t+2, 1 \leq j \leq p.$$

Then the proof follows. □

Definition 3.9. *Consider the graph $G_m = P_m \times K_3$, where $\times$ stands for the cartesian product. The graph $G_m \odot K_{1,n}$ obtained by introducing n new pendant edges at each vertex of the outermost K_3 in G_m is called the generalized n-crown.*

Theorem 3.10. *For all positive integers, $k > 1$, $d \geq 1$, the generalized n-crown $G_m \odot K_{1,n}$ is (a,a)-geometric, where $a = k^d$.*

Proof. Denote the vertices of the innermost cycle as $v_{1,1}, v_{1,2}, v_{1,3}$, the vertices adjacent to $v_{1,3}, v_{1,1}, v_{1,2}$ on the next cycle as $v_{2,1}, v_{2,2}, v_{2,3}, \ldots$, the vertices adjacent to $v_{m-1,3}, v_{m-1,1}, v_{m-1,2}$ on the outermost cycle as $v_{m,1}, v_{m,2}, v_{m,3}$ respectively. Also denote the pendant vertices introduced at each vertex $v_{m,j}$ as $v^l_{m,j}$, $1 \leq j \leq 3$, $1 \leq l \leq n$. Note that the generalized n-crown contains $3(2m+n-1)$ edges. Define

$$f(v_{i,j}) = \begin{cases} k^{\left(3i+\frac{j-7}{2}\right)d} & 1 \leq j \leq m, j = 1,3 \\ k^{(3(i-1)+j)d} & 1 \leq i \leq m, j = 2 \end{cases}$$

$$f(v^l_{m,j}) = k^{(3(m+l)-j)d}, 1 \leq l \leq n, j = 1, 2, 3.$$

One can easily verify that f so defined is injective and $f^\times(G_m \odot K_{1,n}) = \{k^d, k^{2d}, \ldots, k^{qd}\}$. Hence the generalized n-crown is (a, a)-geometric, where $a = k^d$. □

Theorem 3.11. *For any cycle C_n, $n \geq 4$, the following statements hold:*

(i) *For any positive integer $t \geq 1$, the cycle C_{4t} is (a, a)-geometric if and only if a is neither a prime number nor the square of a prime number.*

(ii) *For any positive integer $t \geq 1$ and $r \geq 2$, C_{4t+1} is (r^{2t}, r)-geometric.*

(iii) *C_{4t+2} is not geometric for any positive integer $t \geq 1$.*

(iv) *For any positive integer $t \geq 1$ and $r \geq 2$, C_{4t+3} is (r^{2t+1}, r)-geometric.*

Proof. (i) Suppose C_{4t} is (a, a)-geometric. Then by Theorem 2.2 we get $1 \notin f(C_{4t})$. This means the number a is obtained as the product of two distinct numbers a_1 and a_2 such that neither of them is one and hence a is neither a prime nor a square of any prime number.

Conversely, suppose that a is a positive integer which is neither a prime number nor a square of any prime number. Let $a = a_1 \cdot a_2$, $2 \leq a_1 < a_2$.

Define $f : V(C_{4t}) \to \mathbb{N}$ by

$$f(u_i) = \begin{cases} a_1 r^{(i-1)/2} & \text{if } i \text{ is odd} \\ a_2 r^{(i-2)/2} & \text{if } i \text{ is even, } 2 \leq i \leq 2t \\ a_2 r^{i/2} & \text{if } i \text{ is even, } 2t+2 \leq i \leq 4t. \end{cases}$$

Then f is the required geometric labeling of C_{4t}.

(ii) Under the hypothesis, the map $f : V(C_{4t+1}) \to \mathbb{N}$ defined by

$$f(u_i) = \begin{cases} r^{(i-1)/2} & \text{if } i \text{ is odd} \\ f^{(4t+i)/2} & \text{if } i \text{ is even} \end{cases}$$

is a geometric labeling of C_{4t+1}.

(iii) Suppose that C_{4t+2} has an (a, r)-geometric labeling f. Let $f(u_i) = x_i, 1 \leq i \leq 4t + 2$. Without loss of generality, we may assume that $x_1 \cdot x_2 = a$. Then by Theorem 1.1

we get

$$(x_1x_2)\prod_{i=3}^{4t+2} x_i^2 = a^{4t+2}r^{(2t+1)(4t+1)}.$$

$$\text{Hence } a^2((x_3x_4)(x_5x_6)\dots(x_{4t+1}x_{4t+2}))^2 = a^{4t+2}r^{(2t+1)(4t+1)}.$$

$$\text{Therefore, } a^2(a^{2t}r^m)^2 = a^{4t+2}r^{(2t+1)(4t+1)}.$$

$$\text{Thus, } r^{2m} = r^{(2t+1)(4t+1)}.$$

Hence $2m = (2t+1)(4t+1)$, which is a contradiction.

(iv) Under the hypothesis the map $f : V(C_{4t+3}) \to \mathbb{N}$ defined by

$$f(u_i) = \begin{cases} r^{(i-1)/2} & \text{for odd } i\text{'s} \\ r^{(4t+2+i)/2} & \text{for even } i\text{'s} \end{cases}$$

is a required geometric labeling of C_{4t+3}. □

Definition 3.12. *The graph obtained from the cycle C_n by taking n copies of the path P_3 and joining the i^{th} vertex of C_n to all the three vertices of the i^{th} copy of P_3 is denoted by $C_n \odot P_3$.*

Theorem 3.13. *For all positive integers $k > 1$, odd n, the graph $C_n \odot P_3$ is (k^r, k) - geometric, where $n = 2r + 1$.*

Proof. Let $V(C_n) = \{v_i : 1 \le i \le n\}$. Construct $C_n \odot P_3$ as explained above. Note that $C_n \odot P_3$ contains $4n$ vertices and $6n$ edges. Define

$$f(v_i) = k^{(1-i)r(mod\ n)},\ if 1 \le i \le n,\ n = 2r+1.$$

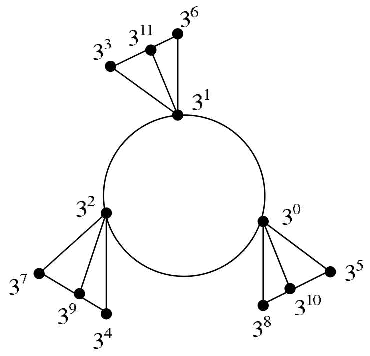
Among the n f-values, r of them are with odd powers and $n-r$ are with even powers. Next, we relabel the rim vertices in the following manner. First, consider all the vertices whose f-values are with odd powers and arrange them so as to form a decreasing sequence. Next, consider the vertices with even powered f-values and arrange them so as to form a decreasing sequence. Write the vertices in the new order by taking the vertices in the former group first, those in the latter group second and the new sequence is labeled as $w_1, w_2, \dots, w_n$. Denote the three vertices adjacent to each w_i as $a_i, b_i, c_i, 1 \le i \le n$. Define,

$$f(a_i) = k^{n+i-1}, 1 \le i \le n$$

$$f(bi) = \begin{cases} k^{3n+i+r} & 1 \le i \le r \\ k^{3n+i-r-1} & r+1 \le i \le n \end{cases}$$

$$f(c_i) = k^{2n+i-1}, 1 \le i \le n.$$

It can be easily seen that all induced edge values are distinct and $f^{\times}(C_n \odot P_3) = \{k^r, k^{r+1}, \dots, k^{r+q-1}\}$ for all $k > 1$. Hence the proof. □



A (3,3)-geometric labeling of $C_3 \odot P_3$
Figure 5

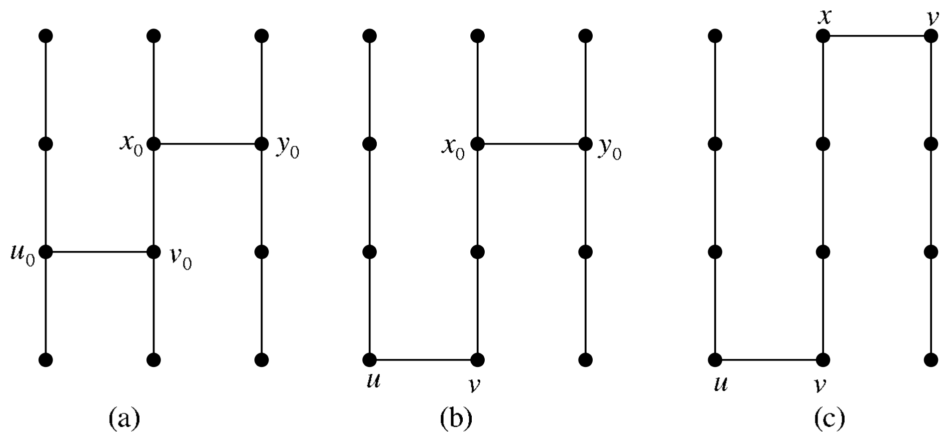
4. Transformed Trees (T_p -Trees)

In this section we prove that a class of trees called T_p*-trees* (*transformed trees*) (see Acharya [2]) are geometric. Also, we prove that the subdivision $S(T)$ of a T_p-tree T, obtained by subdividing every edge of T exactly once is geometric. Note that the subdivision $S(T)$ of a T_p-tree T is not necessarily a T_p- tree.

Let T be a tree and u_0 and v_0 be two adjacent vertices in T. Let there be two pendant vertices u and v in T such that the length of $u_0 - u$ path is equal to the length of $v_0 - v$ path. If the edge u_0v_0 is deleted from T and u, v are joined by an edge uv, then such a transformation of T is called an elementary parallel transformation (or an *ept*) and the edge u_0v_0 is called a *transformable edge.*

If by a sequence of *ept*'s T can be reduced to a path, then T is called a T_p*-tree* (*transformed tree*) and any such sequence regarded as a composition of mappings (*ept*'s) denoted by P, is called a parallel transformation of T. The path, the image of T under P, is denoted as $P(T)$.

A T_p-tree and a sequence of two *ept*'s reducing it to a path are illustrated in Figure 6.



T_p-tree T An ept $P_1(T)$ Second ept $P_2(T)$

Figure 6

Theorem 4.1. *Every T_p-tree T is geometric.*

Proof. Let T be a T_p-tree with $n+1$ vertices, where n is a positive integer. By definition there exists a parallel transformation P of T such that for the path $P(T)$, we have

$$V(P(T)) = V(T)$$
$$E(P(T)) = (E(T) - E_d) \cup E_p,$$

where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the *ept*'s P_i used to arrive at the path $P(T)$. Clearly E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \ldots, v_{n+1}$ starting from one pendant vertex of $P(T)$ right up to the other. Define a function $f : V(P(T)) \to N$ by

$$f(v_i) = \begin{cases} a^{[(i-1)/2]d} & \text{for odd } i, 1 \le i \le n+1 \\ a^{k+(q-1)d+[(i-2)/2]d} & \text{for even } i, 2 \le i \le n+1 \end{cases}$$

where k and d are positive integers and q is the number of edges of T.

Let v_iv_j be an edge in T, $1 < i < j \le n+1$ and let P_1 be the *ept* obtained by deleting this edge and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *ept*'s. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore i and j are of opposite parity. The value of the edge v_iv_j is

$$f^{\times}(v_iv_j) = f^{\times}(v_iv_{i+2t+1}) = f(v_i) \cdot f(v_{i+2t+1}) \tag{2}$$

If i is odd and $1 \le i \le n$, then

$$\begin{aligned} f(v_i)\cdot f(v_{i+2t+1}) &= a^{[(i-1)/2]d}\cdot a^{k+(q-1)d+[(i+2t+1-2)/2]d} \\ &= a^{k+(q-1)d+(i+t-1)d}. \end{aligned} \tag{3}$$

If i is even and $2 \le i \le n$, then

$$\begin{aligned} f(v_i)\cdot f(v_{i+2t+1}) &= a^{k+(q-1)d+[(i-2)/2]d}\cdot a^{[(i+2t+1)/2]d} \\ &= a^{k+(q-1)d+(i+t-1)d}. \end{aligned} \tag{4}$$

Therefore, from (3), (4), (5), we get

$$f^{\times}(v_iv_j) = a^{k+(q-1)d+(i+t-1)d} \text{ for all } i. \tag{5}$$

The value of the edge $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} f^{\times}(v_{i+t}v_{j-t}) &= f(v_{i+t})\cdot f(v_{j-t}) \\ &= f(v_{i+t})\cdot f(v_{i+t+1}). \end{aligned} \tag{6}$$

If $i+t$ is odd, then

$$\begin{aligned} f(v_{i+t})\cdot f(v_{i+t+1}) &= a^{[(i+t-1)/2]d}\cdot a^{k+(q-1)d+[(i+t+1-2)/2]d} \\ &= a^{k+(q-1)d+(i+t-1)d}. \end{aligned} \tag{7}$$

If $i+t$ is even, then

$$\begin{aligned} f(v_{i+t})\cdot f(v_{i+t+1}) &= a^{[(i+t+1-1)/2]d}\cdot a^{k+(q-1)d+[(i+t-2)/2]d} \\ &= a^{k+(q-1)d+(i+t-1)d}. \end{aligned} \tag{8}$$

Therefore, from (7), (8) and (9), we get

$$f^{\times}(v_{i+t}v_{j-t}) = a^{k+(q-1)d+(i+t-1)d}. \tag{9}$$

Thus, from (6) and (10), we get $f^{\times}(v_iv_j) = f^{\times}(v_{i+t}v_{j-t})$.

Also, one can verify that the labeling as defined by f is a $(a^{k+(q-1)d}, a)$-geometric labeling of T for all positive integers k, d. ☐

Figure 7 given below is an illustrative example of a $(2^{14}, 2)$-geometric labeling of a T_p-tree, using the labeling given in the proof of the theorem.

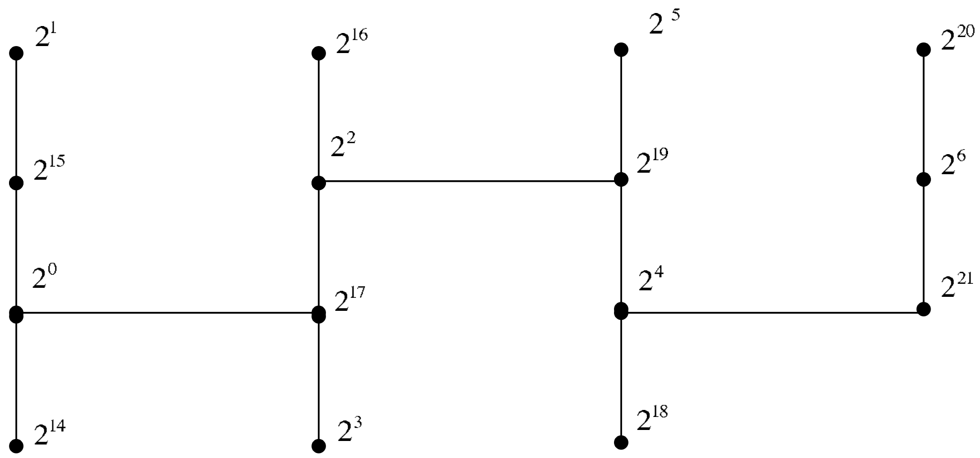


Figure 7

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