Upper bounds for the Roman domination subdivision number of a graph

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Abstract

A Roman dominating function of a graph $G$ is a labeling $f : V(G) \rightarrow \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. The Roman domination number $\gamma_R(G)$ of $G$ is the minimum of $\sum_{v \in V(G)} f(v)$ over such functions. The Roman domination subdivision number $sd_{\gamma_R}(G)$ is the minimum number of edges that must be subdivided (each edge in $G$ can be subdivided at most once) in order to increase the Roman domination number. In this paper, we establish upper bounds for the Roman domination subdivision number of graphs.

Keywords: domination in graph, roman domination number, roman domination subdivision number

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1. Introduction

In this paper, $G$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly $V$ and $E$). For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. Similarly, the open neighborhood of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$ and its closed neighborhood is $N[S] = N(S) \cup S$. The minimum and maximum vertex degrees in $G$ are respectively
denoted by $\delta(G)$ and $\Delta(G)$. A matching $M$ in a graph $G$ is a set of edges having the property that no two edges in $M$ have a vertex in common. The maximum cardinality of a matching in $G$ is called the matching number of $G$ and is denoted by $\beta_1(G)$. A subset $S$ of vertices of $G$ is a dominating set if $N[S] = V$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. For a more thorough treatment of domination parameters and for terminology not presented here see [5, 11].

A Roman dominating function (RDF) on a graph $G = (V,E)$ is defined in [9, 10] as a function $f : V \rightarrow \{0,1,2\}$ satisfying the condition that every vertex $v$ for which $f(v) = 0$ is adjacent to at least one vertex $u$ for which $f(u) = 2$. The weight of an RDF is the value $\omega(f) = \sum_{v \in V} f(v)$. The Roman domination number of a graph $G$, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF on $G$. A $\gamma_R(G)$-function is a Roman dominating function of $G$ with weight $\gamma_R(G)$. A Roman dominating function $f : V \rightarrow \{0,1,2\}$ can be represented by the ordered partition $(V_0,V_1,V_2)$ of $V$, where $V_i = \{v \in V \mid f(v) = i\}$.

Cockayne et al. [3] initiated the study of Roman domination, suggested originally in a Scientific American article by Ian Stewart [10]. Since $V_1 \cup V_2$ is a dominating set when $f$ is an RDF, and since placing weight 2 at the vertices of a dominating set yields an RDF, they observed that

$$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G),$$

where $\gamma(G)$ is the domination number of $G$. In a sense, $2\gamma(G) - \gamma_R(G)$ measures “inefficiency” of domination, since the vertices with weight 1 in an RDF serve only to dominate themselves.

The Roman domination subdivision number of a graph $G$ is the minimum number of edges that must be subdivided (where each edge in $G$ can be subdivided at most once) in order to increase the Roman domination number of $G$. The Roman domination subdivision number was introduced by Atapour et al. in [1] and denoted by $sd_{\gamma_R}(G)$. Atapour et al. in [1] proved that:

**Theorem A.** If $G$ contains a matching $M$ such that $\left\lfloor \frac{\gamma_R(G)}{2} \right\rfloor + 1 \leq |M|$, then $sd_{\gamma_R}(G) \leq |\frac{\gamma_R(G)}{2}| + 1$.

The purpose of this paper is first to generalize Theorem A. Then we present some new upper bounds for $sd_{\gamma_R}(G)$.

We make use of the following results.

**Theorem B.** [2] If $G$ is an $n$-vertex graph, then $\gamma_R(G) \leq n - \Delta + 1$.

The proofs of the following theorems can be found in [1].

**Theorem C.** Let $G$ be a simple connected graph of order $n \geq 3$ and $e = uv \in E(G)$. If $G'$ is obtained from $G$ by subdividing the edge $e$, then $\gamma_R(G') \geq \gamma_R(G)$. 
**Theorem D.** Let \( G \) be a simple connected graph of order \( n \geq 3 \). If \( \gamma_R(G) = 2 \) or \( 3 \), then \( \text{sd}_{\gamma_R}(G) = 1 \).

**Theorem E.** For every simple connected graph \( G \) of order \( n \geq 3 \) with \( \delta(G) = 1 \), \( \text{sd}_{\gamma_R}(G) \leq 2 \).

**Theorem F.** Let \( G \) be a simple connected graph. If \( v \in V(G) \) has degree at least two, then \( \text{sd}_{\gamma_R}(G) \leq \deg(v) \). Hence, if \( \delta(G) \geq 2 \), then \( \text{sd}_{\gamma_R}(G) \leq \delta(G) \).

**Theorem G.**[6] Let \( G \) be a simple connected graph of order \( n \). If \( \gamma_R(G) = 4 \), then \( \text{sd}_{\gamma_R}(G) \leq 2 \). Furthermore, this bound is sharp.

**Theorem H.**[7] If a graph \( G \) has \( \text{diam}(G) = 2 \), then \( \gamma_R(G) \leq 2\delta(G) \). Furthermore, this bound is sharp.

2. A generalization of Theorem A

In this section we find an upper bound for \( \text{sd}_{\gamma_R}(G) \) in terms of \( \gamma_R(G) \) for connected graphs \( G \) of order at least three. In fact we show that for each connected graph \( G \) of order at least 3, \( \text{sd}_{\gamma_R}(G) \leq \left\lfloor \frac{\gamma_R(G)}{2} \right\rfloor + 1 \), generalizing Theorem A. For this purpose we need the following theorems.

**Theorem 2.1.**[5] For any graph \( G \) without isolates, \( \gamma(G) \leq \beta_1(G) \).

The graphs with equal domination and matching numbers, but not containing a perfect matching, were characterized independently by Randerath and Volkmann [8] and Hare and McCuaig [4]. We will use the following result from [4].

Let \( L \) be the set of leaves of \( G \); \( N \) be the set of support vertices of \( G \); and define \( I = \{ x \in V(G) - (N \cup L) : N(x) \subseteq N \} \). Note that \( I \) is an independent set of vertices.

Let \( G \) be the class of graphs \( G \) without isolated vertices having the following properties.

1. If \( H_2 \) is the collection of the bipartite connected components of \( G - (N \cup L) \), then the vertices of \( H_2 \) can be partitioned into two independent sets \( A \) and \( B \) such that:

   For any two distinct vertices \( a_1 \) and \( a_2 \) in \( A \) with a common neighbor \( b \) in \( B \), there exists a vertex \( b_1 \in B - \{b\} \) such that \( N_G(b_1) = \{a_1, a_2\} \). Furthermore, the only vertices of \( H_2 \) which have neighbors in \( N \) are vertices in \( B \).

2. Every non-bipartite component \( H \) of \( G - (N \cup L) \) is one of the graphs shown in Figure 1, where each of the dashed edges may or may not be an edge of \( H \). Furthermore, only the starred vertices can have neighbors in \( N \).

**Theorem 2.2.**[4] A graph \( G \) with no isolated vertices and no perfect matching has \( \gamma(G) = \beta_1(G) \) if and only if \( G \) is in \( G \).
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**Theorem 2.3.** For any connected graph $G$ of order $n \geq 3$, $sd_{\gamma_R}(G) \leq \left\lfloor \frac{\gamma_R(G)}{2} \right\rfloor + 1$.

**Proof.** We may assume $\beta_1(G) \leq \left\lfloor \frac{\gamma_R(G)}{2} \right\rfloor$, otherwise the result follows by Theorem A. Then by Theorem 2.1 and (1) we have $\gamma(G) = \frac{\gamma_R(G)}{2} = \beta_1(G)$ and so $\gamma_R(G)$ is even. Since $G$ is a connected graph of order at least three, $\gamma_R(G) \leq n - 1$ by Theorem B. This implies that $G$ does not have a perfect matching. Thus the only graphs that we need to consider are those having $\gamma(G) = \beta_1(G) < n/2$, that is, $G$ has equal domination and matching numbers, but $G$ does not have a perfect matching. These graphs are characterized in Theorem 2.2. If $L \neq \emptyset$, then the result follows by Theorem E. Therefore we may suppose that $L \cup N \cup \mathcal{I} = \emptyset$. By Theorem 2.2 and the fact that $G$ is connected, either $G$ is a bipartite graph with minimum degree 2 or $G$ is isomorphic to one of the graphs illustrated in Figure 1. In each case $G$ has a vertex of minimum degree 2 and the result follows by Theorem E. \qed

3. Upper bounds for $sd_{\gamma_R}(G)$

In this section we present some upper bounds for $sd_{\gamma_R}(G)$. Our first lemma gives an upper bound on the Roman domination number of a regular graph.

**Theorem 3.1.** For any $r$-regular connected graph $G$ of order $n \geq 2$,

$$\gamma_R(G) \leq \max\{2r, n - 2r + 2\}.$$

**Proof.** If $G$ has two nonadjacent vertices $u, v$ such that $N(u) \cap N(v) = \emptyset$, then obviously the ordered partition $((N(u) \cup N(v)), V(G) - (N[u] \cup N[v]), \{u, v\})$ is an RDF of $G$ and so $\gamma_R(G) \leq n - 2r + 2$. Let for each two nonadjacent vertices $u, v$ we have

![Figure 1: Non-bipartite components of $G - (N \cup L)$](image-url)
\(N(u) \cap N(v) \neq \emptyset\). Then obviously \(\text{diam}(G) = 2\) and the result follows by Theorem H. This completes the proof. \(\square\)

**Theorem 3.2.** For any connected graph \(G\) of order \(n \geq 3\) except \(C_5\),

\[\text{sd}_{\gamma_R}(G) \leq n - \gamma_R(G).\]

The bound is sharp for \(P_3, C_3, C_4\) and \(C_8\).

**Proof.** First let \(G\) not be a regular graph; that is \(\Delta > \delta\). If \(\Delta = 2\), then \(G = P_n\). For \(n = 3, 4, 5\), the statement is trivial and for \(n \geq 6\) the result easily follows by Theorem E because \(\gamma_R(G) \leq n - 2\). Let \(\Delta \geq 3\) and \(\deg(u) = \Delta\). Then the ordered partition \((N(u), V - N[u], \{u\})\) is an RDF for \(G\), and so \(\gamma_R(G) \leq n - \Gamma - 1 \leq n - \delta\). Now the result follows by Theorem F. Assume \(G\) is an \(r\)-regular graph. Obviously, \(r \geq 2\). If \(r = 2\), then \(G = C_n\) and \(n \neq 5\). For \(n = 3, 4\), the statement is trivial and for \(n \geq 6\) the result easily follows because \(\text{sd}_{\gamma_R}(G) \leq 2\) by Theorem F and \(\gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil\).

Let \(r \geq 3\). If \(G\) has two nonadjacent vertices \(u, v\) such that \(N(u) \cap N(v) = \emptyset\), then \(\gamma_R(G) \leq n - 2r + 2\), as in the proof of Theorem 3.1, and the result follows by Theorem F. If \(\text{diam}(G) = 1\), then \(G\) is a complete graph and the result follows. Therefore, from now on we assume \(\text{diam}(G) = 2\). Furthermore, we may assume \(\gamma_R(G) \geq 5\), otherwise the result follows by Theorems D, G and B.

Let \(u \in V(G)\), \(N(u) = \{u_1, \ldots, u_r\}\) and \(Y = V - N[u]\). If \(Y = \emptyset\), then \(\gamma_R(G) = 2\), a contradiction. Suppose that \(Y \neq \emptyset\). If there is a vertex \(v \in N(u)\) with at least three neighbors in \(Y\), then the ordered partition \((|N(u) \cup N(v)| - \{u, v\}, Y - N(v), \{u, v\})\) is an RDF for \(G\), hence \(\gamma_R(G) \leq n - r\). By Theorem F, \(\text{sd}_{\gamma_R}(G) \leq r \leq n - \gamma_R(G)\).

Now assume each vertex of \(N(u)\) has at most two neighbors in \(Y\). Then there are at most three vertices in \(Y\) with degree zero or one in \(G[Y]\) because \(r \geq 3\). Hence, if \(|Y| \geq 4\), then there is a vertex \(w\) with \(\deg_{G[Y]}(w) \geq 2\). Now the ordered partition \((N(u) \cup N(w), V(G) - (N[u] \cup N[w]), \{u, w\})\) is an RDF for \(G\), hence \(\gamma_R(G) \leq n - r\) and the result follows by Theorem F.

Finally, let \(|Y| \leq 3\) and \(\deg_{G[Y]}(w) \leq 1\) for each vertex \(w \in Y\). Then \(n \leq r + 4\). On the other hand, since \(Y \neq \emptyset\), \(n > r + 1\). If \(n \in \{r + 2, r + 3\}\), then \(\gamma_R(G) \leq 4\), a contradiction. We leave the case \(n = r + 4\) for the reader.

It is straightforward to see that the bound is sharp for \(P_3, C_3, C_4\) and \(C_8\). \(\square\)

We conclude this paper with two propositions. Recall that a vertex of \(G\) is *simplicial* if its neighborhood in \(G\) is a clique.

**Proposition 3.3.** For any connected graph \(G\) of order \(n \geq 3\) with two adjacent simplicial vertices, \(\text{sd}_{\gamma_R}(G) \leq 2\).
Proof. Let \( u \) and \( v \) be two adjacent simplicial vertices of \( G \). Since \( n \geq 3 \), we have \( \min\{\deg(u),\deg(v)\} \geq 2 \). On the other hand, \( N[u] = N[v] \) since \( u \) and \( v \) are simplicial. Assume \( w \in N(u) \setminus \{v\} \). Let \( G' \) be the graph obtained from \( G \) by subdividing the edge \( uv, uw \) with subdivision vertex \( x, y \), respectively. We claim that \( \gamma_R(G') > \gamma_R(G) \). Let \( g = (V_0, V_1, V_2) \) be a \( \gamma_R(G') \)-function. If \( g(x) = 1 \) or \( g(y) = 1 \), then the result follows by Theorem C. If \( g(x) = g(y) = 2 \), then obviously \( g(u) \neq 1 \) and the ordered partition \( (V_0 - \{u\}, V_1, (V_2 - \{x, y\}) \cup \{u\}) \) is an RDF of \( G \) with weight less than \( \gamma_R(G') \).

Let \( g(x) = 2 \) and \( g(y) = 0 \) (the case \( g(x) = 0 \) and \( g(y) = 2 \) is similar). Then \( g(u) = 2 \) or \( g(v) = 2 \) and \( g|_G \) is an RDF of \( G \) with weight less than \( \gamma_R(G') \). Let \( g(x) = g(y) = 0 \). If \( g(u) = 2 \) and \( g(v) \geq 1 \), then \( ((V_0 - \{x, y\}) \cup \{v\}, V_1 - \{v\}, V_2 - \{v\}) \) is an RDF of \( G \) with weight less than \( \gamma_R(G') \). If \( g(u) = 2 \) and \( g(v) = 0 \), then in order to dominate \( v \) we have \( V_2 \cap (N_G(v) - \{u\}) \neq \emptyset \) and so \( ((V_0 - \{x, y\}) \cup \{u\}, V_1, V_2 - \{v\}) \) is an RDF of \( G \) with weight less than \( \gamma_R(G') \). Let \( g(u) \leq 1 \). Then \( g(v) = g(w) = 2 \). Obviously, the ordered partition \( ((V_0 - \{x, y\}) \cup \{u, v\}, V_1 - \{v\}, V_2 - \{v\}) \) is an RDF of \( G \) with weight less than \( \gamma_R(G') \). Thus \( \gamma_R(G') > \gamma_R(G) \) and the proof is complete. \( \Box \)

We now generalize Proposition 3.3.

**Proposition 3.4.** Let \( G \) be a connected graph of order \( n \geq 3 \) with a simplicial vertex \( u \) of degree at least two. For each \( v \in N(u) \),

\[
\text{sd}_R(G) \leq \deg(v) - \deg(u) + 2.
\]

Proof. We may assume \( N(v) - N[u] \neq \emptyset \), otherwise the result follows by Proposition 3.3. Suppose that \( N(u) - \{v\} = \{u_1, \ldots, u_r\} \) and \( N(v) - N[u] = \{v_1, \ldots, v_k\} \). Let \( G' \) be the graph obtained from \( G \) by subdividing the edge \( uv_i \) with subdivision vertex \( x_i \), for \( 1 \leq i \leq k \), and the edges \( uv, uu_1 \) with subdivision vertices \( y_1, y_2 \), respectively. Define \( A \) to be the set of subdivision vertices and assume \( g = (V_0, V_1, V_2) \) is a \( \gamma_R(G') \)-function.

We may assume \( |V_2 \cap A| \leq 2 \), otherwise \( g' = (V_0 - (A \cup \{u, v\}), V_1 - (A \cup \{u, v\}), (V_2 - A) \cup \{u, v\}) \) is an RDF for \( G \) with weight less than \( \gamma_R(G') \). On the other hand, we may assume \( V_1 \cap A = \emptyset \), otherwise the result follows by Theorem C.

First let \( V_2 \cap A = \{w, z\} \). If \( \{w, z\} \subseteq N_{G'}(v) \) (the case \( \{w, z\} \subseteq N_{G'}(u) \) is similar), then the ordered partition \( (V_0 - (A \cup \{v\}), V_1 - \{v\}, (V_2 - A) \cup \{v\}) \) defines an RDF of \( G \) with weight less than \( \gamma_R(G') \). Thus, without loss of generality, we may assume \( V_2 \cap A = \{x_1, y_2\} \). Now in order to dominate \( y_1 \) we must have \( g(u) = 2 \) or \( g(v) = 2 \). Then the ordered partition \( (V_0 - (A \cup \{u, v\}), V_1 - \{u, v\}, (V_2 - A) \cup \{u, v\}) \) is an RDF for \( G \) with weight less than \( \gamma_R(G') \).

Now let \( V_2 \cap A = \{z\} \). If \( z = y_1 \), then \( N_G(v) - N_G[u] \subseteq V_2 \) and \( g(u) = 2 \) or \( g(u_1) = 2 \). Therefore \( g|_G \) is an RDF of \( G \) with weight \( \gamma_R(G') - 2 \). Suppose that \( z \neq y_1 \). If \( z = y_2 \), then \( N_G(v) - N_G[u] \subseteq V_2 \) and \( g(u) = 2 \) or \( g(v) = 2 \). It is easy to see that the ordered partition \( (V_0 - (A \cup \{u, v\}), V_1, (V_2 - \{y_2, u, v\}) \cup \{u_1\}) \) is an RDF of \( G \) with weight \( \gamma_R(G') - 2 \). Now let \( z \in N_{G'}(v) \setminus \{y_1\} \). Assume \( z = x_1 \). In order to
dominate $y_1$ we must have $g(u) = 2$ or $g(v) = 2$. If $g(v) = 2$, then $g|_G$ is an RDF of $G$ with weight $\gamma_R(G') - 2$. Let $g(u) = 2$. If $g(v_1) = 0$, then the ordered partition $(V_0 - (A \cup \{v_1\}), V_1 \cup \{v_1\}, V_2 - \{x_1\})$ is an RDF of $G$ with weight $\gamma_R(G') - 1$. If $g(v_1) \geq 1$, then the ordered partition $(V_0 - A, V_1 - \{v_1\}, (V_2 - \{x_1\}) \cup \{v_1\})$ defines an RDF of $G$ with weight at most $\gamma_R(G') - 1$.

Finally, if $|V_2 \cap A| = 0$, then $A \subseteq V_0$. In order to dominate $y_1$ we must have $g(u) = 2$ or $g(v) = 2$. If $g(u) = g(v) = 2$, then the ordered partition $((V_0 - A) \cup \{v\}, V_1, V_2 - \{v\})$ defines an RDF of $G$ with weight $\gamma_R(G') - 2$. If $g(u) = 2$ and $g(v) = 0$, then in order to dominate $v$ we have $w \in V_2 \cap (N(u) - \{v\}) \neq \emptyset$. Then the ordered partition $((V_0 - A) \cup \{u\}, V_1, V_2 - \{v\})$ defines an RDF of $G$ with weight $\gamma_R(G') - 2$. If $g(u) = 0$ and $g(v) = 2$, then obviously $g(u_1) = 2$ and the ordered partition $(V_0 - A, V_1, V_2 - \{v\})$ is an RDF of $G$ with weight $\gamma_R(G') - 2$. Finally, if $g(u) = 2$ and $g(v) = 1$ (the case $g(u) = 1$ and $g(v) = 2$ is similar), then the ordered partition $(V_0 - A, V_1 - \{v\}, V_2)$ is an RDF of $G$ with weight $\gamma_R(G') - 1$. Thus $\gamma_R(G') > \gamma_R(G)$ and the proof is complete. 

\[\qed\]

References


