AKCE J. Graphs. Combin., 5, No. 1 (2008), pp. 7-14

Upper bounds for the Roman domination subdivision number of a graph

Abdollah Khodkar
Department of Mathematics
University of West Georgia
Carrollton, GA 30118.
e-mail: *akhodkar@westga.edu*
B.P. Mobaraky and S.M. Sheikholeslami [†] [*]
Department of Mathematics
Azarbaijan University of Tarbiat Moallem
Tabriz, I.R. Iran
e-mail: [†] *s.m.sheikholeslami@azaruniv.edu*

Communicated by: S. Arumugam
Received 12 January 2008; accepted 05 May 2008

Abstract

A Roman dominating function of a graph G is a labeling $f : V(G) \longrightarrow \{0,1,2\}$ such that every vertex with label 0 has a neighbor with label 2. The Roman domination number $\gamma_R(G)$ of G is the minimum of $\sum_{v\in V(G)} f(v)$ over such functions. The *Roman domination subdivision number* $\mathrm{sd}_{\gamma_R}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the Roman domination number. In this paper, we establish upper bounds for the Roman domination subdivision number of graphs.

Keywords: domination in graph, roman domination number, roman domination subdivision number

2000 Mathematics Subject Classification: 05C69

1. Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. Similarly, the *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$ and its *closed neighborhood* is $N[S] = N(S) \cup S$. The minimum and maximum vertex degrees in G are respectively

[*]Corresponding author

denoted by $\delta(G)$ and $\Delta(G)$. A matching M in a graph G is a set of edges having the property that no two edges in M have a vertex in common. The maximum cardinality of a matching in G is called the matching number of G and is denoted by $\beta_1(G)$. A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. For a more thorough treatment of domination parameters and for terminology not presented here see [5, 11].

A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is defined in [9, 10] as a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *weight* of an RDF is the value $\omega(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF on G. A $\gamma_R(G)$*-function* is a Roman dominating function of G with weight $\gamma_R(G)$. A Roman dominating function $f : V \longrightarrow \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) of V, where $V_i = \{v \in V \mid f(v) = i\}$.

Cockayne et al. [3] initiated the study of Roman domination, suggested originally in a Scientific American article by Ian Stewart [10]. Since $V_1 \cup V_2$ is a dominating set when f is an RDF, and since placing weight 2 at the vertices of a dominating set yields an RDF, they observed that

$$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G), \tag{1}$$

where $\gamma(G)$ is the domination number of G. In a sense, $2\gamma(G) - \gamma_R(G)$ measures "inefficiency" of domination, since the vertices with weight 1 in an RDF serve only to dominate themselves.

The *Roman domination subdivision number* of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the Roman domination number of G. The Roman domination subdivision number was introduced by Atapour et al. in [1] and denoted by $\mathrm{sd}_{\gamma_R}(G)$. Atapour et al. in [1] proved that:

Theorem A. *If G contains a matching M such that $\left\lfloor \frac{\gamma_R(G)}{2} \right\rfloor + 1 \leq |M|$, then $\mathrm{sd}_{\gamma_R}(G) \leq \lfloor \frac{\gamma_R(G)}{2} \rfloor + 1$.*

The purpose of this paper is first to generalize Theorem A. Then we present some new upper bounds for $\mathrm{sd}_{\gamma_R}(G)$.

We make use of the following results.

Theorem B. [2] *If G is an n-vertex graph, then $\gamma_R(G) \leq n - \Delta + 1$.*

The proofs of the following theorems can be found in [1].

Theorem C. *Let G be a simple connected graph of order $n \geq 3$ and $e = uv \in E(G)$. If G' is obtained from G by subdividing the edge e, then $\gamma_R(G') \geq \gamma_R(G)$.*

Theorem D. *Let G be a simple connected graph of order $n \geq 3$. If $\gamma_R(G) = 2$ or 3, then $\mathrm{sd}_{\gamma_R}(G) = 1$.*

Theorem E. *For every simple connected graph G of order $n \geq 3$ with $\delta(G) = 1$, $\mathrm{sd}_{\gamma_R}(G) \leq 2$.*

Theorem F. *Let G be a simple connected graph. If $v \in V(G)$ has degree at least two, then $\mathrm{sd}_{\gamma_R}(G) \leq \deg(v)$. Hence, if $\delta(G) \geq 2$, then $\mathrm{sd}_{\gamma_R}(G) \leq \delta(G)$.*

Theorem G. [6] *Let G be a simple connected graph of order n. If $\gamma_R(G) = 4$, then $\mathrm{sd}_{\gamma_R}(G) \leq 2$. Furthermore, this bound is sharp.*

Theorem H. [7] *If a graph G has $\mathrm{diam}(G) = 2$, then $\gamma_R(G) \leq 2\delta(G)$. Furthermore, this bound is sharp.*

2. A generalization of Theorem A

In this section we find an upper bound for $\mathrm{sd}_{\gamma_R}(G)$ in terms of $\gamma_R(G)$ for connected graphs G of order at least three. In fact we show that for each connected graph G of order at least 3, $\mathrm{sd}_{\gamma_R}(G) \leq \left\lfloor \frac{\gamma_R(G)}{2} \right\rfloor + 1$, generalizing Theorem A. For this purpose we need the following theorems.

Theorem 2.1. [5] *For any graph G without isolates, $\gamma(G) \leq \beta_1(G)$.*

The graphs with equal domination and matching numbers, but not containing a perfect matching, were characterized independently by Randerath and Volkmann [8] and Hare and McCuaig [4]. We will use the following result from [4].

Let $\mathcal{L}$ be the set of leaves of G; $\mathcal{N}$ be the set of support vertices of G; and define $\mathcal{I} = \{x \in V(G) - (\mathcal{N} \cup \mathcal{L}) : N(x) \subseteq \mathcal{N}\}$. Note that $\mathcal{I}$ is an independent set of vertices.

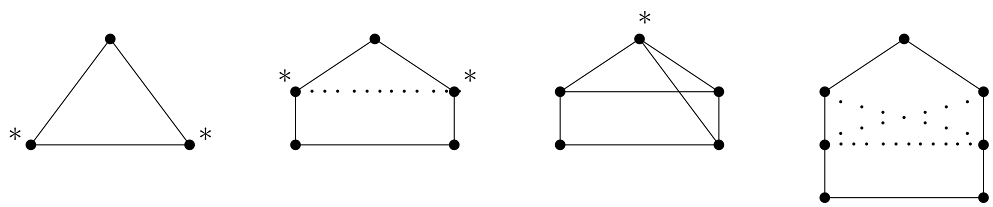
Let $\mathcal{G}$ be the class of graphs G without isolated vertices having the following properties.

1. If H_2 is the collection of the bipartite connected components of $G - (\mathcal{N} \cup \mathcal{L})$, then the vertices of H_2 can be partitioned into two independent sets A and B such that:

   For any two distinct vertices a_1 and a_2 in A with a common neighbor b in B, there exists a vertex $b_1 \in B - \{b\}$ such that $N_G(b_1) = \{a_1, a_2\}$. Furthermore, the only vertices of H_2 which have neighbors in $\mathcal{N}$ are vertices in B.

2. Every non-bipartite component H of $G - (\mathcal{N} \cup \mathcal{L})$ is one of the graphs shown in Figure 1, where each of the dashed edges may or may not be an edge of H. Furthermore, only the starred vertices can have neighbors in $\mathcal{N}$.

Theorem 2.2. [4] *A graph G with no isolated vertices and no perfect matching has $\gamma(G) = \beta_1(G)$ if and only if G is in $\mathcal{G}$.*



(a) (b) (c) (d)

Figure 1: Non-bipartite components of $G-(\mathcal{N}\cup\mathcal{L})$. Dashed line represents an optional edge, and a star indicates that a vertex may have a neighbor in $\mathcal{N}$.

Now we are ready to generalize Theorem A.

Theorem 2.3. *For any connected graph G of order $n\geq 3$,* $\mathrm{sd}_{\gamma_R}(G)\leq \left\lfloor \dfrac{\gamma_R(G)}{2}\right\rfloor+1$.

Proof. We may assume $\beta_1(G)\leq\lfloor\frac{\gamma_R(G)}{2}\rfloor$, otherwise the result follows by Theorem A. Then by Theorem 2.1 and (1) we have $\gamma(G)=\dfrac{\gamma_R(G)}{2}=\beta_1(G)$ and so $\gamma_R(G)$ is even. Since G is a connected graph of order at least three, $\gamma_R(G)\leq n-1$ by Theorem B. This implies that G does not have a perfect matching. Thus the only graphs that we need to consider are those having $\gamma(G)=\beta_1(G)<n/2$, that is, G has equal domination and matching numbers, but G does not have a perfect matching. These graphs are characterized in Theorem 2.2. If $\mathcal{L}\neq\emptyset$, then the result follows by Theorem E. Therefore we may suppose that $\mathcal{L}\cup\mathcal{N}\cup\mathcal{I}=\emptyset$. By Theorem 2.2 and the fact that G is connected, either G is a bipartite graph with minimum degree 2 or G is isomorphic to one of the graphs illustrated in Figure 1. In each case G has a vertex of minimum degree 2 and the result follows by Theorem E. □

3. Upper bounds for $\mathrm{sd}_{\gamma_R}(G)$

In this section we present some upper bounds for $\mathrm{sd}_{\gamma_R}(G)$. Our first lemma gives an upper bound on the Roman domination number of a regular graph.

Theorem 3.1. *For any r-regular connected graph G of order $n\geq 2$,*

$$\gamma_R(G)\leq \max\{2r, n-2r+2\}.$$

Proof. If G has two nonadjacent vertices u,v such that $N(u)\cap N(v)=\emptyset$, then obviously the ordered partition $((N(u)\cup N(v)), V(G)-(N[u]\cup N[v]), \{u,v\})$ is an RDF of G and so $\gamma_R(G)\leq n-2r+2$. Let for each two nonadjacent vertices u,v we have

$N(u) \cap N(v) \neq \emptyset$. Then obviously $\operatorname{diam}(G) = 2$ and the result follows by Theorem H. This completes the proof. □

Theorem 3.2. *For any connected graph G of order $n \geq 3$ except C_5,*

$$\operatorname{sd}_{\gamma_R}(G) \leq n - \gamma_R(G).$$

The bound is sharp for P_3, C_3, C_4 and C_8.

Proof. First let G not be a regular graph; that is $\Delta > \delta$. If $\Delta = 2$, then $G = P_n$. For $n = 3, 4, 5$, the statement is trivial and for $n \geq 6$ the result easily follows by Theorem E because $\gamma_R(G) \leq n - 2$. Let $\Delta \geq 3$ and $\deg(u) = \Delta$. Then the ordered partition $(N(u), V - N[u], \{u\})$ is an RDF for G, and so $\gamma_R(G) \leq n - \Delta + 1 \leq n - \delta$. Now the result follows by Theorem F. Assume G is an r-regular graph. Obviously, $r \geq 2$. If $r = 2$, then $G = C_n$ and $n \neq 5$. For $n = 3, 4$, the statement is trivial and for $n \geq 6$ the result easily follows because $\operatorname{sd}_{\gamma_R}(G) \leq 2$ by Theorem F and $\gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil$. Let $r \geq 3$. If G has two nonadjacent vertices u, v such that $N(u) \cap N(v) = \emptyset$, then $\gamma_R(G) \leq n - 2r + 2$, as in the proof of Theorem 3.1, and the result follows by Theorem F. If $\operatorname{diam}(G) = 1$, then G is a complete graph and the result follows. Therefore, from now on we assume $\operatorname{diam}(G) = 2$. Furthermore, we may assume $\gamma_R(G) \geq 5$, otherwise the result follows by Theorems D, G and B.

Let $u \in V(G)$, $N(u) = \{u_1, \ldots, u_r\}$ and $Y = V - N[u]$. If $Y = \emptyset$, then $\gamma_R(G) = 2$, a contradiction. Suppose that $Y \neq \emptyset$. If there is a vertex $v \in N(u)$ with at least three neighbors in Y, then the ordered partition $((N(u) \cup N(v)) - \{u, v\}, Y - N(v), \{u, v\})$ is an RDF for G, hence $\gamma_R(G) \leq n - r$. By Theorem F, $\operatorname{sd}_{\gamma_R}(G) \leq r \leq n - \gamma_R(G)$.

Now assume each vertex of $N(u)$ has at most two neighbors in Y. Then there are at most three vertices in Y with degree zero or one in $G[Y]$ because $r \geq 3$. Hence, if $|Y| \geq 4$, then there is a vertex w with $\deg_{G[Y]}(w) \geq 2$. Now the ordered partition $(N(u) \cup N(w), V(G) - (N[u] \cup N[w]), \{u, w\})$ is an RDF for G, hence $\gamma_R(G) \leq n - r$ and the result follows by Theorem F.

Finally, let $|Y| \leq 3$ and $\deg_{G[Y]}(w) \leq 1$ for each vertex $w \in Y$. Then $n \leq r + 4$. On the other hand, since $Y \neq \emptyset$, $n > r + 1$. If $n \in \{r + 2, r + 3\}$, then $\gamma_R(G) \leq 4$, a contradiction. We leave the case $n = r + 4$ for the reader.

It is straightforward to see that the bound is sharp for P_3, C_3, C_4 and C_8. □

We conclude this paper with two propositions. Recall that a vertex of G is *simplicial* if its neighborhood in G is a clique.

Proposition 3.3. *For any connected graph G of order $n \geq 3$ with two adjacent simplicial vertices,* $\operatorname{sd}_{\gamma_R}(G) \leq 2$.

Proof. Let u and v be two adjacent simplicial vertices of G. Since $n \geq 3$, we have $\min\{\deg(u), \deg(v)\} \geq 2$. On the other hand, $N[u] = N[v]$ since u and v are simplicial. Assume $w \in N(u) - \{v\}$. Let G' be the graph obtained from G by subdividing the edge uv, uw with subdivision vertex x, y, respectively. We claim that $\gamma_R(G') > \gamma_R(G)$. Let $g = (V_0, V_1, V_2)$ be a $\gamma_R(G')$-function. If $g(x) = 1$ or $g(y) = 1$, then the result follows by Theorem C. If $g(x) = g(y) = 2$, then obviously $g(u) \neq 1$ and the ordered partition $(V_0 - \{u\}, V_1, (V_2 - \{x, y\}) \cup \{u\})$ is an RDF of G with weight less than $\gamma_R(G')$.

Let $g(x) = 2$ and $g(y) = 0$ (the case $g(x) = 0$ and $g(y) = 2$ is similar). Then $g(u) = 2$ or $g(w) = 2$ and $g|_G$ is an RDF of G with weight less than $\gamma_R(G')$. Let $g(x) = g(y) = 0$. If $g(u) = 2$ and $g(v) \geq 1$, then $((V_0 - \{x, y\}) \cup \{v\}, V_1 - \{v\}, V_2 - \{v\})$ is an RDF of G with weight less than $\gamma_R(G')$. If $g(u) = 2$ and $g(v) = 0$, then in order to dominate v we have $V_2 \cap (N_G(v) - \{u\}) \neq \emptyset$ and so $((V_0 - \{x, y\}) \cup \{u\}, V_1, V_2 - \{u\})$ is an RDF of G with weight less than $\gamma_R(G')$. Let $g(u) \leq 1$. Then $g(v) = g(w) = 2$. Obviously, the ordered partition $((V_0 - \{x, y\}) \cup \{u, v\}, V_1 - \{u\}, V_2 - \{v\})$ is an RDF of G with weight less than $\gamma_R(G')$. Thus $\gamma_R(G') > \gamma_R(G)$ and the proof is complete. □

We now generalize Proposition 3.3.

Proposition 3.4. *Let G be a connected graph of order $n \geq 3$ with a simplicial vertex u of degree at least two. For each $v \in N(u)$,*

$$\mathrm{sd}_{\gamma_R}(G) \leq \deg(v) - \deg(u) + 2.$$

Proof. We may assume $N(v) - N[u] \neq \emptyset$, otherwise the result follows by Proposition 3.3. Suppose that $N(u) - \{v\} = \{u_1, \ldots, u_r\}$ and $N(v) - N[u] = \{v_1, \ldots, v_k\}$. Let G' be the graph obtained from G by subdividing the edge vv_i with subdivision vertex x_i, for $1 \leq i \leq k$, and the edges uv, uu_1 with subdivision vertices y_1, y_2, respectively. Define A to be the set of subdivision vertices and assume $g = (V_0, V_1, V_2)$ is a $\gamma_R(G')$-function.

We may assume $|V_2 \cap A| \leq 2$, otherwise $g' = (V_0 - (A \cup \{u, v\}), V_1 - (A \cup \{u, v\}), (V_2 - A) \cup \{u, v\})$ is an RDF for G with weight less than $\gamma_R(G')$. On the other hand, we may assume $V_1 \cap A = \emptyset$, otherwise the result follows by Theorem C.

First let $V_2 \cap A = \{w, z\}$. If $\{w, z\} \subseteq N_{G'}(v)$ (the case $\{w, z\} \subseteq N_{G'}(u)$ is similar), then the ordered partition $(V_0 - (A \cup \{v\}), V_1 - \{v\}, (V_2 - A) \cup \{v\})$ defines an RDF of G with weight less than $\gamma_R(G')$. Thus, without loss of generality, we may assume $V_2 \cap A = \{x_1, y_2\}$. Now in order to dominate y_1 we must have $g(u) = 2$ or $g(v) = 2$. Then the ordered partition $(V_0 - (A \cup \{u, v\}), V_1 - \{u, v\}, (V_2 - A) \cup \{u, v\})$ is an RDF for G with weight less than $\gamma_R(G')$.

Now let $V_2 \cap A = \{z\}$. If $z = y_1$, then $N_G(v) - N_G[u] \subseteq V_2$ and $g(u) = 2$ or $g(u_1) = 2$. Therefore $g|_G$ is an RDF of G with weight $\gamma_R(G') - 2$. Suppose that $z \neq y_1$. If $z = y_2$, then $N_G(v) - N_G[u] \subseteq V_2$ and $g(u) = 2$ or $g(v) = 2$. It is easy to see that the ordered partition $(V_0 - (A \cup \{u, v\}), V_1, (V_2 - \{y_2, u, v\}) \cup \{u_1\})$ is an RDF of G with weight $\gamma_R(G') - 2$. Now let $z \in N_{G'}(v) \setminus \{y_1\}$. Assume $z = x_1$. In order to dominate y_1 we must have $g(u) = 2$ or $g(v) = 2$. If $g(v) = 2$, then $g|_G$ is an RDF of G with weight $\gamma_R(G') - 2$. Let $g(u) = 2$. If $g(v_1) = 0$, then the ordered partition $(V_0 - (A \cup \{v_1\}), V_1 \cup \{v_1\}, V_2 - \{x_1\})$ is an RDF of G with weight $\gamma_R(G') - 1$. If $g(v_1) \geq 1$, then the ordered partition $(V_0 - A, V_1 - \{v_1\}, (V_2 - \{x_1\}) \cup \{v_1\})$ defines an RDF of G with weight at most $\gamma_R(G') - 1$.

Finally, if $|V_2 \cap A| = 0$, then $A \subseteq V_0$. In order to dominate y_1 we must have $g(u) = 2$ or $g(v) = 2$. If $g(u) = g(v) = 2$, then the ordered partition $((V_0 - A) \cup \{v\}, V_1, V_2 - \{v\})$ defines an RDF of G with weight $\gamma_R(G') - 2$. If $g(u) = 2$ and $g(v) = 0$, then in order to dominate v we have $w \in V_2 \cap (N(u) - \{v\}) \neq \emptyset$. Then the ordered partition $((V_0 - A) \cup \{u\}, V_1, V_2 - \{u\})$ defines an RDF of G with weight $\gamma_R(G') - 2$. If $g(u) = 0$ and $g(v) = 2$, then obviously $g(u_1) = 2$ and the ordered partition $(V_0 - A, V_1, V_2 - \{v\})$ is an RDF of G of wight $\gamma_R(G') - 2$. Finally, if $g(u) = 2$ and $g(v) = 1$ (the case $g(u) = 1$ and $g(v) = 2$ is similar), then the ordered partition $(V_0 - A, V_1 - \{v\}, V_2)$ is an RDF of G with weight $\gamma_R(G') - 1$. Thus $\gamma_R(G') > \gamma_R(G)$ and the proof is complete. □

References

[1] M. Atapour, A. Khodkar and S.M. Sheikholeslami, *Roman domination subdivision number of a graph*, (submitted).

[2] E.W. Chambers, B. Kinnersley, N. Prince and D.B. West, *Extremal problems for Roman domination* (submitted).

[3] E.J. Cockayne, P.A. Dreyer, S.M. Hedetniemi and S.T. Hedetniemi, On Roman domination in graphs, *Discrete Math.,* **278** (2004), 11–22.

[4] D. Hare and W. McCuaig, *A characterization of graphs whose domination and mathcing numbers are equal*, unpublished manuscript, 1998.

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in graphs*, Marcel Dekker, Inc., New york, 1998.

[6] A. Khodkar, B.P. Mobaraky and S.M. Sheikholeslami, *Roman domination subdivision number of a graph and its complement*, (submitted).

[7] B.P. Mobaraky and S.M. Sheikholeslami, *Bounds on the Roman domination numbers of a graph*, (submitted).

[8] B. Randerath and L. Volkmann, Characterization of graphs with equal domination and mathcing number, *Utilitas Math.,* **55** (1999), 65–72.

[9] C.S. Revelle and K.E. Rosing, Defendens imperium romanum: a classical problem in military strategy, *Amer. Math. Monthly,* **107** (2000), 585–594.

[10] I. Stewart, Defend the Roman Empire, *Sci. Amer.,* **281** (1999), 136-139.

[11] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Inc, 2000.