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More Sufficient Conditions for A Graph to Have (g, f)-Factors *

Sizhong Zhou †
School of Mathematics and Physics
Jiangsu University of Science and Technology
Mengxi Road 2, Zhenjiang, Jiangsu 212003
People's Republic of China
e-mail: *zsz_cumt*@163.*com* and
Yang Xu
Department of Mathematics
Qingdao Agricultural University
Qingdao, Shandong 266109
People's Republic of China

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Abstract

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let g and f be integer-valued functions defined on $V(G)$ such that $0 \leq g(x) < f(x)$ for all $x \in V(G)$, and let H_1 and H_2 be two edge-disjoint subgraphs of G. We prove that G has a (g, f)-factor F such that $E(H_1) \subseteq E(F)$ and $E(H_2) \cap E(F) = \emptyset$ if G satisfies

(1) $g(x) + d_{H_2}(x) \leq d_G(x)$ and $f(x) \geq d_{H_1}(x)$ for all $x \in V(G)$, and

(2) $(f(x) - d_{H_1}(x))(d_G(y) - d_{H_2}(y)) \geq (d_G(x) - d_{H_1}(x))g(y)$ for all $x, y \in V(G)$.

Keywords: graph, subgraph, (g, f)-factor.

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1. Introduction

We consider a finite undirected graph G with vertex set $V(G)$ and edge set $E(G)$, which has neither loops or multiple edges. For a vertex $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G. For two disjoint vertex subsets S and T of G, the number of edges joining S to T is denoted by $e_G(S, T)$. For a subset $S \subseteq V(G)$, we denote by $G - S$ the subgraph obtained from G by deleting all the vertices of S together with the edges incident with the vertices of S.

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†Corresponding author

Let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) \le f(x)$ for all $x \in V(G)$. Then a (g, f)-factor of graph G is defined as a spanning subgraph F of G such that $g(x) \le d_F(x) \le f(x)$ for all $x \in V(G)$. Note that if $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then a (g, f)-factor is nothing but an $[a, b]$-factor. A matching in a graph G is a set of edges of G with the property that no two edges are adjacent. A k-matching is a matching of size k. The other terminologies and notations not given in this paper can be found in [1].

Many authors have investigated $[a, b]$-factors [2-8], and (g, f)-factors [9-17]. The following results on (g, f)-factors are known.

G. Liu [9] proved the following theorem.

Theorem 1. [9] *Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) < f(x)$ for all $x \in V(G)$. If $g(x) \le d_G(x)$ and $(f(x) - 1)d_G(y) \ge (d_G(x) - 1)g(y)$ for all $x, y \in V(G)$, then G has a (g, f)-factor containing any edge e of G.*

The above theorem was extended to the following results by S. Zhou [15].

Theorem 2. [15] *Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) < f(x)$ for all $x \in V(G)$, M is an $(rk - r + 1)$-matching of G. If $g(x) \le d_G(x)$ and $(f(x) - k)d_G(y) \ge (d_G(x) - k)g(y)$ for all $x, y \in V(G)$, then G has a (g, f)-factor containing M, where r and k are two positive integers.*

Theorem 3. [15] *Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) < f(x)$ for all $x \in V(G)$. If $g(x) \le d_G(x)$ and $(f(x) - k)d_G(y) \ge (d_G(x) - k)g(y)$ for all $x, y \in V(G)$, then G has a (g, f)-factor containing any k edges of G, where k is a nonnegative integer.*

In this paper, we prove the following results, which are an extension of Theorem 1 and 2 and 3.

Theorem 4. *Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) < f(x)$ for all $x \in V(G)$, and H is a subgraph of G. If $g(x) \le d_G(x)$ and $f(x) \ge d_H(x)$ and*

$$(f(x) - d_H(x))d_G(y) \ge (d_G(x) - d_H(x))g(y) \tag{1}$$

for all $x, y \in V(G)$, then G has a (g, f)-factor F such that $E(H) \subseteq E(F)$.

Theorem 5. *Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) < f(x)$ for all $x \in V(G)$, and H_1 and H_2 are two edge-disjoint subgraphs of G. If $g(x) + d_{H_2}(x) \le d_G(x)$ and $f(x) \ge d_{H_1}(x)$ and*

$$(f(x) - d_{H_1}(x))(d_G(y) - d_{H_2}(y)) \ge (d_G(x) - d_{H_1}(x))g(y) \tag{2}$$

for all $x, y \in V(G)$, then G has a (g, f)-factor F such that $E(H_1) \subseteq E(F)$ and $E(H_2) \cap E(F) = \emptyset$.

In Theorem 5, if $H_1 = \emptyset$, then we get the following corollary.

Corollary 1. *Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) < f(x)$ for all $x \in V(G)$, and H_2 is a subgraph of G. If $g(x) + d_{H_2}(x) \le d_G(x)$ and*

$$f(x)(d_G(y) - d_{H_2}(y)) \ge d_G(x)g(y)$$

for all $x, y \in V(G)$, then G has a (g,f)-factor F such that $E(H_2) \cap E(F) = \emptyset$.

Theorem 6. *Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) < f(x)$ for all $x \in V(G)$. If $g(x) + r \le d_G(x)$ and*

$$(f(x) - k)(d_G(y) - r) \ge (d_G(x) - k)g(y) \tag{3}$$

for all $x, y \in V(G)$, then G has a (g,f)-factor including any k edges of G and excluding any other r edges of G. Where r and k are two nonnegative integers.

In Theorem 6, if $k = 0$, then we get the following corollary.

Corollary 2. *Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) < f(x)$ for all $x \in V(G)$. If $g(x) + r \le d_G(x)$ and*

$$f(x)(d_G(y) - r) \ge d_G(x)g(y)$$

for all $x, y \in V(G)$, then G has a (g,f)-factor excluding any r edges of G. Where r is a nonnegative integers.

2. Proof of Main Theorems

In order to prove Theorem 4, we depend on the following theorem.

Theorem 7. [18] *Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) < f(x)$ for all $x \in V(G)$, and H is a subgraph of G. Then G has a (g,f)-factor F such that $E(H) \subseteq E(F)$ if and only if*

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \ge d_H(S) - e_H(S,T)$$

for all disjoint subsets S and T of $V(G)$.

Proof of Theorem 4. By Theorem 7, to prove Theorem 4 we need only to show that for all disjoint subsets S and T of $V(G)$

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \ge d_H(S) - e_H(S,T).$$

If $T = \emptyset$, then by $f(x) \ge d_H(x)$ for all $x \in V(G)$, we have $\delta_G(S, \emptyset) = f(S) \ge d_H(S) = d_H(S) - e_H(S, \emptyset)$. Thus we may assume that $T \ne \emptyset$. According to (1), we have

$$f(x)d_G(y) \ge d_G(x)g(y) + d_H(x)(d_G(y) - g(y)). \tag{4}$$

In view of (4), we obtain

$$\left(\sum_{x\in S} f(x)\right) d_G(y) \geq \left(\sum_{x\in S} d_G(x)\right) g(y) + \left(\sum_{x\in S} d_H(x)\right) (d_G(y) - g(y)) \,. \tag{5}$$

By (5), we get that

$$\left(\sum_{x\in S} f(x)\right)\left(\sum_{y\in T} d_G(y)\right) \geq \left(\sum_{x\in S} d_G(x)\right)\left(\sum_{y\in T} g(y)\right) + \left(\sum_{x\in S} d_H(x)\right)\left(\sum_{y\in T}(d_G(y) - g(y))\right),$$

that is,

$$f(S)d_G(T) \geq d_G(S)g(T) + d_H(S)(d_G(T) - g(T)). \tag{6}$$

By (6), we have

$$\begin{aligned} d_G(T)\delta_G(S,T) &= d_G(T)(f(S) + d_{G-S}(T) - g(T)) \\ &= d_G(T)f(S) + d_G(T)d_{G-S}(T) - d_G(T)g(T) \\ &\geq d_G(S)g(T) + d_H(S)(d_G(T) - g(T)) + d_G(T)d_{G-S}(T) - d_G(T)g(T) \\ &= g(T)(d_G(S) - d_G(T)) + d_G(T)d_{G-S}(T) + d_H(S)(d_G(T) - g(T)), \end{aligned}$$

i.e.

$$d_G(T)\delta_G(S,T) \geq g(T)(d_G(S) - d_G(T)) + d_G(T)d_{G-S}(T) + d_H(S)(d_G(T) - g(T)). \tag{7}$$

Note that $d_G(S) \geq e_G(S,T) + d_H(S) - e_H(S,T) \geq d_G(T) - d_{G-S}(T) + d_H(S) - e_H(S,T)$. Hence,

$$d_G(S) - d_G(T) \geq -d_{G-S}(T) + d_H(S) - e_H(S,T). \tag{8}$$

According to (7) and (8) and $d_G(x) \geq g(x)$ for all $x \in V(G)$, we obtain

$$\begin{aligned} d_G(T)\delta_G(S,T) &\geq g(T)(-d_{G-S}(T) + d_H(S) - e_H(S,T)) \\ &\quad + d_G(T)d_{G-S}(T) + d_H(S)(d_G(T) - g(T)) \\ &= d_{G-S}(T)(d_G(T) - g(T)) + d_H(S)d_G(T) - g(T)e_H(S,T) \\ &\geq d_H(S)d_G(T) - g(T)e_H(S,T) \\ &\geq d_H(S)d_G(T) - d_G(T)e_H(S,T) \\ &= d_G(T)(d_H(S) - e_H(S,T)) \end{aligned}$$

If $d_G(T) = 0$, then $d_{G-S}(T) = 0$ and $g(T) = 0$. Thus by $f(x) \geq d_H(x)$ for all $x \in V(G)$, we have $\delta_G(S,T) = f(S) \geq d_H(S) \geq d_H(S) - e_H(S,T)$. In the following, we may assume that $d_G(T) > 0$. Thus, we have

$$\delta_G(S,T) \geq d_H(S) - e_H(S,T).$$

In view of Theorem 7, G has a (g,f)-factor F such that $E(H)\subseteq E(F)$.
Completing the proof of Theorem 4. □

Proof of Theorem 5. Let $G'=G-H_2$. In this case, we have

$$d_G(x)\geq d_{G'}(x)\geq d_G(x)-d_{H_2}(x)\geq g(x) \tag{9}$$

for all $x\in V(G')$.

According to (2) and (9), we obtain

$$\begin{aligned}(f(x)-d_{H_1}(x))d_{G'}(y) &\geq (f(x)-d_{H_1}(x))(d_G(y)-d_{H_2}(y))\\ &\geq (d_G(x)-d_{H_1}(x))g(y)\\ &\geq (d_{G'}(x)-d_{H_1}(x))g(y)\end{aligned}$$

for all $x,y\in V(G^{'})$.

By Theorem 4, $G^{'}$ has a (g,f)-factor F such that $E(H_1)\subseteq E(F)$. That is, G has a (g,f)-factor F such that $E(H_1)\subseteq E(F)$ and $E(H_2)\cap E(F)=\emptyset$.
The proof is complete. □

Proof of Theorem 6. Let $e_i=u_iv_i, i=1,2,\cdots,r$ be any r edges of G, and let $G'=G-e_1-e_2-\cdots-e_r$. In this case, we have

$$d_G(x)\geq d_{G'}(x)\geq d_G(x)-r\geq g(x) \tag{10}$$

for all $x\in V(G')$.

In view of (3) and (10), we have

$$\begin{aligned}(f(x)-k)d_{G'}(y) &\geq (f(x)-k)(d_G(y)-r)\\ &\geq (d_G(x)-k)g(y)\\ &\geq (d_{G'}(x)-k)g(y)\end{aligned}$$

for all $x,y\in V(G^{'})$.

According to Theorem 3, G' has a (g,f)-factor including any k edges of $G^{'}$. That is to say, G has a (g,f)-factor including any k edges of G and excluding any other r edges of G.

This completes the proof. □

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