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Changing of the Number of Minimum Dominating Sets after Edge Addition: Critical Edges

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Abstract

Let $\gamma(G)$ and $\#\gamma(G)$ denote the domination number and the number of all distinct minimum dominating sets of a graph G, respectively. We show that if G is a graph without isolated vertices then for every edge $e \in E(\overline{G})$, $\gamma(G+e) < \gamma(G)$ if and only if $\#\gamma(G+e) < \#\gamma(G)$.

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1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes, et al. [3]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. For any vertex v of G, the *open neighborhood* of v is the set $N(v,G) = \{u \in V(G) : uv \in E(G)\}$, while the *closed neighborhood* of v is the set $N[v,G] = N(v,G) \cup \{v\}$. The degree of v is defined as $deg(v,G) = |N(v,G)|$. For a set of vertices $S \subseteq V(G)$, $N(S,G)$ is the union of $N(x,G)$ for all $x \in S$, and $N[S,G] = N(S,G) \cup S$. For $s \in S \subseteq V(G)$, $pn_G(s,S) = N[s,G] - N[S-\{s\},G]$ is the *private neighborhood of* s *relative to* S. A *dominating set* in a graph G is a set of vertices $D \subseteq V(G)$ such that every vertex of G is either in D or is adjacent to an element of D. A dominating set D of a graph G is a *minimal dominating set* if no set $D^{'} \subsetneq D$ is a dominating set. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. Any dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$*-set*, or just γ-set when the graph G is clear from the context. The set of all γ-sets of a graph G is denoted by $\mathcal{D}(G)$. If $U \subseteq V(G)$, then denote $\mathcal{D}(U,G) = \{M \in \mathcal{D}(G) : U \subseteq M\}$. The number of distinct $\gamma(G)$-sets is denoted

$\#\gamma(G)$ ([6]). The number of all $\gamma(G)$-sets each of which has $U \subseteq V(G)$ as a subset is denoted by $\#\gamma(U, G)$.

It is often of interest to known how the value of a graph parameter is affected when a small change is made in a graph, for instance vertex or edge removal, edge addition and edge contraction. In this connection, in this paper we consider this question in the case $\#\gamma(G)$ when an edge from $\overline{G}$ is added to G.

If $e \in \overline{G}$ and $\gamma(G+e) < \gamma(G)$ then e is called a $\gamma(G)$-*critical edge*. A graph G is γ-*edge-addition-critical* if all edges of $\overline{G}$ are $\gamma(G)$-critical. This concept was introduced by Sumner et al. [7]. The study of effects on domination related parameters when a graph is modified by adding an edge is classical; see for instance [2, 4, 5, 10] and for surveys [3, Chapter 5] and [1, 8]. Note that $\gamma(G+e)+1 \geq \gamma(G) \geq \gamma(G+e)$ ([3]).

Definition 1.1. *Let G be a graph. An edge $e \in E(\overline{G})$ is $\#\gamma(G)$-critical if $\#\gamma(G+e) < \#\gamma(G)$. A graph G is $\#\gamma$-edge-addition-critical if all edges of $\overline{G}$ are $\#\gamma(G)$-critical.*

Our main results are:

Theorem 1.2. *Let x_1 and x_2 be two distinct, nonadjacent and nonisolated vertices of a graph G. Then x_1x_2 is $\gamma(G)$-critical if and only if x_1x_2 is $\#\gamma(G)$-critical.*

Corollary 1.3. *Let G be a graph with no isolated vertex. Then G is γ-edge-addition-critical if and only if G is $\#\gamma$-edge-addition-critical.*

2. Proofs

We need the following notation and results.

Let u and v be nonadjacent vertices of a graph G. We write $u \mapsto v$ whenever $\gamma(G-v) < \gamma(G)$ and u belongs to at least one γ-set of $G-v$.

Lemma 2.1. *Let G be a graph, $x_1, x_2 \in V(G)$, $x_1x_2 \in E(G)$ and let $G_1 = G + x_1x_2$.*

(i) ([10] Theorem 3; [9] Theorem 2.8) $\gamma(G_1) < \gamma(G)$ *if and only if either* $x_1 \in pn_{G_1}(x_2, M)$ *or* $x_2 \in pn_{G_1}(x_1, M)$ *for each γ-set M of G_1.*

(ii) ([4], Lemma 5(2)) $\gamma(G_1) < \gamma(G)$ *if and only if at least one of* $x_1 \mapsto x_2$ *or* $x_2 \mapsto x_1$ *holds.*

(iii) ([4], Theorem 3) *If* $\gamma(G_1) < \gamma(G)$ *and* $x_2 \not\mapsto x_1$ *then* x_2 *belongs to no γ-set of* G_1.

Lemma 2.2. *Let G be a graph, $x_1, x_2 \in V(G)$, $x_1 \mapsto x_2$ and let $G_1 = G + x_1x_2$. Then:*

(i) $\mathcal{D}(G_1)$ *is disjoint union of* $\mathcal{D}(\{x_1\}, G_1)$ *and* $\mathcal{D}(\{x_2\}, G_1)$*;*

(ii) $\mathcal{D}(\{x_1\}, G_1) = \mathcal{D}(\{x_1\}, G - x_2)$ *;*

(iii) *if* $x_2 \mapsto x_1$ *then* $\mathcal{D}(\{x_1\}, G - x_2)$ *and* $\mathcal{D}(\{x_2\}, G - x_1)$ *form a partition of* $\mathcal{D}(G_1)$ *;*

(iv) *if* $x_2 \not\mapsto x_1$ *then* $\mathcal{D}(G_1) = \mathcal{D}(\{x_1\}, G - x_2)$ *;*

(v) $\#\gamma(\{x_1, x_2\}, G) \geq \#\gamma(\{x_1\}, G - x_2) \neq 0$.

Proof. By Lemma 2.1 (ii), $\gamma(G_1) = \gamma(G) - 1 = \gamma(G - x_2)$.

(i) By Lemma 2.1 (i), $|\{x_1, x_2\} \cap M| = 1$ for each γ-set M of G_1.

(ii) If $M \in \mathcal{D}(\{x_1\}, G - x_2)$ then M is a dominating set of G_1 with $|M| = \gamma(G - x_2) = \gamma(G_1)$. Hence M is a γ-set of G_1 which implies $\mathcal{D}(\{x_1\}, G - x_2) \subseteq \mathcal{D}(\{x_1\}, G_1)$.

Now assume $M \in \mathcal{D}(\{x_1\}, G_1)$. By (i), $x_2 \notin M$ and then M is a dominating set of $G - x_2$ of cardinality $\gamma(G_1) = \gamma(G - x_2)$ which implies M is a γ-set of $G - x_2$. Thus $\mathcal{D}(\{x_1\}, G_1) \subseteq \mathcal{D}(\{x_1\}, G - x_2)$.

(iii) The result follows immediately by (i) and (ii).

(iv) By Lemma 2.1(iii), $\mathcal{D}(\{x_2\}, G_1) = \emptyset$ and the result follows by (i) and (ii).

(v) Since $x_1 \mapsto x_2$, $\mathcal{D}(\{x_1\}, G - x_2)$ is not empty and $M \cup \{x_2\} \in \mathcal{D}(\{x_1, x_2\}, G)$ for each $M \in \mathcal{D}(\{x_1\}, G - x_2)$. □

Proposition 2.3. *Let* G *be a graph,* $x_1, x_2 \in V(G)$ *and let* $G_1 = G + x_1x_2$.

(i) *If* $x_1 \mapsto x_2$, $x_2 \mapsto x_1$ *and* $deg(x_2, G) \geq deg(x_1, G) \geq 1$ *then* $\#\gamma(G_1)deg(x_1, G) + \#\gamma(\{x_1\}, G_1)(deg(x_2, G) - deg(x_1, G)) + \#\gamma(\{x_1, x_2\}, G) = \#\gamma(\{x_2\}, G - x_1)deg(x_1, G) + \#\gamma(\{x_1\}, G - x_2)deg(x_2, G) + \#\gamma(\{x_1, x_2\}, G) \leq \#\gamma(G)$.

(ii) *If* $x_1 \mapsto x_2$, $x_2 \not\mapsto x_1$ *and* $deg(x_s, G) \geq 1$, $s = 1, 2$, *then* $\#\gamma(G_1)(deg(x_2, G)+1) \leq \#\gamma(\{x_1\}, G) \leq \#\gamma(G)$.

(iii) *Let* $x_1 \mapsto x_2$, $deg(x_1, G) = 0$ *and* $deg(x_2, G) \geq 2$. *Then* $\#\gamma(G_1) + \#\gamma(G - x_2)(deg(x_2, G) - 1) \leq \#\gamma(G)$.

(iv) *Let* $x_1 \mapsto x_2$, $deg(x_1, G) = 0$ *and* $deg(x_2, G) = 1$. *Let* $N(x_2, G) = \{x_3\}$ *and* $A = \{M : M$ *is a minimal dominating set of* $G - x_2$, $x_3 \in M$ *and* $|M| = \gamma(G)\}$. *Then* $\#\gamma(G) = \#\gamma(G_1) + |A|$.

(v) *Let* $x_1 \not\mapsto x_2$, $x_2 \mapsto x_1$ *and* $deg(x_1, G) = 0$. *Then* $\#\gamma(G_1) = \#\gamma(\{x_2\}, G)$. *If* x_2 *is in each* $\gamma(G)$*-set then* $\#\gamma(G_1) = \#\gamma(G)$. *If* x_2 *belongs to some* $\gamma(G)$*-set but not to all* $\gamma(G)$*-sets then* $\#\gamma(G_1) < \#\gamma(G)$.

(vi) *If* $deg(x_1, G) = deg(x_2, G) = 0$ *then* $\#\gamma(G_1) = 2\#\gamma(G)$.

Proof. Let $u, v \in V(G)$ with $u \mapsto v$ and $deg(v, G) \geq 1$. Then each neighbor of v belongs to no γ-set of $G - v$. Define $B(u, v) = \{M \cup \{z\} : M \in \mathcal{D}(\{u\}, G - v)$ and $z \in N(v, G)\}$. Hence $B(u, v) \subseteq \mathcal{D}(\{u\}, G) - \mathcal{D}(\{u, v\}, G)$ and $|B(u, v)| = \#\gamma(\{u\}, G - v) deg(v, G)$.

(i) $\#\gamma(G) \geq |\mathcal{D}(\{x_1\}, G) \cup \mathcal{D}(\{x_2\}, G)| = |\mathcal{D}(\{x_1\}, G) - \mathcal{D}(\{x_1, x_2\}, G)| + |\mathcal{D}(\{x_2\}, G) - \mathcal{D}(\{x_1, x_2\}, G)| + \#\gamma(\{x_1, x_2\}, G) \geq |B(x_1, x_2))| + |B(x_2, x_1)| + \#\gamma(\{x_1, x_2\}, G) = \#\gamma(\{x_1\}, G - x_2) deg(x_2, G) + \#\gamma(\{x_2\}, G - x_1) deg(x_1, G) + \#\gamma(\{x_1, x_2\}, G)$. It remains to note that $\#\gamma(\{x_2\}, G - x_1) = \#\gamma(G_1) - \#\gamma(\{x_1\}, G - x_2)$ by Lemma 2.2(iii) and $\#\gamma(\{x_1\}, G - x_2) = \#\gamma(\{x_1\}, G_1)$ by Lemma 2.2 (ii).

(ii) $\#\gamma(G) \geq \#\gamma(\{x_1\}, G) \geq \#\gamma(\{x_1, x_2\}, G) + |\mathcal{D}(\{x_1\}, G) - \mathcal{D}(\{x_1, x_2\}, G)| \geq \#\gamma(\{x_1, x_2\}, G) + |B(x_1, x_2)| = \#\gamma(\{x_1, x_2\}, G) + \#\gamma(\{x_1\}, G - x_2) deg(x_2, G)$ Since $\#\gamma(\{x_1, x_2\}, G) \geq \#\gamma(\{x_1\}, G - x_2) = \#\gamma(G_1)$ (by Lemma 2.2 (v) and Lemma 2.2 (iv), respectively), we have the result.

(iii) Clearly, $x_2 \mapsto x_1$. We have, $\#\gamma(G) = \#\gamma(\{x_1\}, G) = \#\gamma(\{x_1, x_2\}, G) + |\mathcal{D}(\{x_1\}, G) - \mathcal{D}(\{x_1, x_2\}, G)| \geq \#\gamma(\{x_1, x_2\}, G) + |B(x_1, x_2)| = \#\gamma(\{x_1, x_2\}, G) + \#\gamma(\{x_1\}, G - x_2) deg(x_2, G) = \#\gamma(\{x_2\}, G - x_1) + \#\gamma(G - x_2) deg(x_2, G)$. It remains to note that by Lemma 2.2(iii), $\#\gamma(\{x_2\}, G - x_1) = \#\gamma(G_1) - \#\gamma(\{x_1\}, G - x_2) = \#\gamma(G_1) - \#\gamma(G - x_2)$.

(iv) Clearly $\mathcal{D}(\{x_3\}, G)$ is disjoint union of A and $\{M \cup \{x_3\} : M$ is a γ-set of $G - x_2\}$. Hence $\#\gamma(\{x_3\}, G) = |A| + \#\gamma(G - x_2) = |A| + \#\gamma(\{x_1\}, G - x_2)$. Since $\mathcal{D}(G)$ is disjoint union of $\mathcal{D}(\{x_2\}, G)$ and $\mathcal{D}(\{x_3\}, G)$, we have $\#\gamma(G) = \#\gamma(\{x_2\}, G) + \#\gamma(\{x_3\}, G) = \#\gamma(\{x_2\}, G - x_1) + |A| + \#\gamma(\{x_1\}, G - x_2)$ and the result now follows by Lemma 2.2 (iii).

(v) By Lemma 2.2 (iv), $\#\gamma(G_1) = \#\gamma(\{x_2\}, G - x_1) = \#\gamma(\{x_2\}, G)$.

(vi) Obvious. □

Proof of Theorem 1.2.

Sufficiency: – Let x_1x_2 be a $\#\gamma(G)$-critical edge. Assume $\gamma(G) = \gamma(G + x_1x_2)$. Then each $\gamma(G)$-set is a $\gamma(G + x_1x_2)$-set. This implies $\#\gamma(G) \leq \#\gamma(G + x_1x_2)$, a contradiction.

Necessity: – Let x_1x_2 be a $\gamma(G)$-critical edge. By Lemma 2.1(ii), at least one of $x_1 \mapsto x_2$ or $x_2 \mapsto x_1$ holds. Since $\gamma(\{x_1, x_2\}, G) \neq 0$ (by Lemma 2.2(v)), it follows from Proposition 2.3(i)–(ii) that $\#\gamma(G + x_1x_2) < \#\gamma(G)$. □

Proof of Corollary 1.3. The result immediately follows by Theorem 1.2. □

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