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Resistance matrices of blocks in a graph

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Abstract

A well-known result due to Graham, Hoffman and Hosoya asserts that the determinant of the distance matrix of a graph remains invariant if the blocks of the graph are reassembled in a different manner. We observe that this result is true for a larger class of distance functions which we term as additive distance functions with respect to blocks. The resistance distance is shown to be such a distance. This facilitates the calculation of the determinant and the inverse of the resistance matrix of a graph if we know the corresponding information for the blocks of the graph. The technique is illustrated with examples.

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1. Introduction

A *weighted* graph is a graph in which each edge is assigned a weight, which is a real number. Let G be a directed, strongly connected, weighted graph. The (classical) *distance* between vertices i and j is defined to be the minimum weight of a directed path from i to j, where the weight of a path is the sum of the weights of the edges on the path. The *resistance distance* between two vertices i and j is the effective electrical resistance

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between i and j when we place a resistor along every edge (with the resistance offered by the resistor equal to the weight of the edge) and a battery is attached at i and j. For more information on resistance distance see [13, 7, 2, 12, 17, 16].

Resistance is an important physical entity with many applications in physics, electrical engineering and chemistry. Hence it has been a widely studied topic in these fields. The resistance distance in a graph was first studied by Klien and Randić [13]. Since then many properties of the resistance distance have been studied. The classical distance in a graph has been studied for a long time, albeit the resistance distance is more structured and has many attractive properties. The classical distance and resistance distance of a graph coincide if and only if the graph is acyclic, i.e., a forest. This explains some very attractive properties of distance in trees, which do not carry over to arbitrary graphs.

The determinant of the distance matrix of a tree has been enunciated in Graham and Pollack [11], where it was observed that the determinant depends only on the number of vertices. An extension to the weighted case was obtained in [5] where it was shown that the determinant of the distance matrix of a weighted tree is independent of the way in which the vertices are connected. More precisely, it was shown that the determinant is $(-1)^{n-1}2^{n-2}(\prod_{i=1}^{n-1} w_i)(\sum_{i=1}^{n-1} w_i)$ for a weighted tree with edge weights $w_1, w_2, \cdots, w_{n-1}$.

An important result due to Graham, Hoffman and Hosoya [9] asserts that the determinant of the distance matrix of a graph remains the same irrespective of the way in which the blocks of the graph are joined. (For the definition of a block and for other graph theoretic preliminaries, we refer to [15].) As a starting point of the present paper, we make the observation that the result of Graham, Hoffman and Hosoya continues to hold for a more general class of distances, namely those which are additive with respect to the blocks in the graph. This notion is made precise in Section 2. It turns out that resistance distance is an additive function with respect to the blocks of the graph, and this was indeed our motivation.

The resistance matrix of a graph is a matrix indexed by the vertices, with the $(i,j)-$ entry equal to the resistance distance between i and j, if $i \neq j$ and 0, if $i = j$. A formula for the determinant and the inverse of the resistance matrix of a graph was obtained in [3], generalizing the formulas known for a tree. In Section 3 we obtain some results relating the resistance matrix of a graph and of its blocks. When the graph is a tree, its blocks are just the edges and the results capture the known results for a tree in this case.

In Section 4 we provide some specific applications of the results in Section 3. In particular, the determinant of the resistance matrix of a cactus (a graph in which each block is either a cycle or an edge) is obtained.

2. Additive distance functions

Let G be a directed, strongly connected, weighted graph with vertex set V. A real valued function on $V \times V$ will be referred to as a distance function on G. Thus classical distance and resistance distance are both examples of distance functions. A distance function f on G will be called additive (with respect to the blocks) if it satisfies

(i) $f(i,i)=0,$ for all $i \in V$,

(ii) If $i,j \in V, i \neq j,$ and if a directed path from i to j passes through a cut-vertex k, then $f(i,j)=f(i,k)+f(k,j)$.

If A is an $n \times n$ matrix then $cof A$ will denote the sum of all cofactors of A. If f is an additive function on the graph G with n vertices, then $D_f(G)$ will denote the $n \times n$ matrix with rows and columns indexed by the vertices of G and with the (i,j)-entry equal to $f(i,j)$. We now state the following result.

Theorem 2.1. *Let G be a directed, strongly connected, weighted graph and let $G_1, G_2, \cdots, G_r$ be the strong blocks of G. Let f be an additive function on G. Then the following assertions hold:*

(i) $cof D_f(G) = \prod_{i=1}^{r} cof D_f(G_i)$

(ii) $det D_f(G) = \sum_{i=1}^{r} det D_f(G_i) \cdot \prod_{j \neq i} cof D_f(G_j)$

Theorem 2.1 has been proved for the classical distance by Graham, Hoffman and Hosoya[9] and an examination of their proof reveals that it goes through verbatim for the case of an additive function on G. We therefore omit the proof of Theorem 2.1.

From this point onwards we shall be concerned with the resistance distance. We therefore restrict ourselves to undirected graphs. The results stated thus far continue to hold since we may change an undirected graph into a directed graph by replacing each edge by two directed edges, one in either direction.

We now introduce some notation. Let $G=(V,E)$ be a weighted graph with n vertices. We assume that the weights are all nonzero. The Laplacian matrix $L(G)$, or simply L, is the $n \times n$ matrix defined as follows. The rows and columns of L are indexed by V. If $i \neq j$, then the (i,j)-entry equals -1 times the reciprocal of the weight of the edge (i,j), or 0 according as i and j are adjacent or nonadjacent respectively. The diagonal elements of L are defined so that the row sums are zero. For basic properties of the Laplacian see [14,1].

Let G be a connected, weighted graph with Laplacian L. If S and T are subsets of the vertex set V then $L(S|T)$ will denote the submatrix of L formed by rows indexed by $V \backslash S$ and columns indexed by $V \backslash T$. It is well-known (see, for example, [2]) that the resistance distance r_{ij} between i and j admits the formula

$$r_{ij} = \frac{detL(i,j|i,j)}{detL(i|i)}.$$

The weight of a spanning tree is defined as the product of the reciprocals of the weights of its edges. Recall that by the Matrix-Tree Theorem, $detL(i|i)$ equals the sum of the weights of the spanning trees of G. A 2-forest is a forest with two components. We say that a spanning 2-forest of G separates vertices i and j if i and j lie in different components of the forest. By the "all minors" Matrix-Tree Theorem [8,6], $detL(i,j|i,j)$ equals the sum of the weights of all spanning 2-forests of G which separate i and j.

Theorem 2.2. *Let $G=(V,E)$ be a connected, weighted graph with n vertices and for $i,j \in V$, let r_{ij} denote the resistance distance between i and j. Then r is an additive function on G.*

Proof. Let i and j be distinct vertices of G and suppose an (i,j)-path passes through the cut-vertex k. A spanning 2-forest of G which separates i and j must separate either i and k or k and j. Similarly a spanning 2-forest of G which separates i and k or k and j necessarily separates i and j. Therefore in view of the observation preceding the statement of the Theorem,

$$detL(i,j|i,j) = detL(i,k|i,k) + detL(k,j|k,j).$$

Therefore $r_{ij} = r_{ik} + r_{kj}$ and the proof is complete. □

We remark that Theorem 2.2 may also be proved using the electrical network interpretation of resistance distance.

We denote the resistance matrix of the graph G by $R(G)$. The next result is an immediate consequence of Theorems 2.1 and 2.2.

Corollary 2.3. *Let $G=(V,E)$ be a connected, weighted graph and let $G_1,\dots,G_r$ be the blocks of G. Then the following assertions hold:*

(i) $cofR(G) = \prod_{i=1}^{r} cofR(G_i)$

(ii) $detR(G) = \sum_{i=1}^{r} detR(G_i) \cdot \prod_{j\neq i} cofR(G_j)$

3. Resistance matrix of a graph and its blocks

Let G be a connected, weighted graph with vertex set $\{1,2,\cdots,n\}$. As before, let r_{ij} denote the resistance distance between i and j. We set

$$\tau_i(G) = 2 - \Sigma_{j:j\sim i}\frac{r_{ij}}{w(i,j)}, i=1,2,\cdots,n, \tag{1}$$

where $j \sim i$ denotes that j is adjacent to i and $w(i,j)$ is the weight of the edge (i,j). Let $\tau(G)$ be the column vector with components $\tau_1(G),\dots,\tau_n(G)$. We will write τ instead of $\tau(G)$ when the graph is clear from the context. We denote by $t(G)$ the sum of the weights of all spanning trees of G. The column vector of appropriate dimension having all ones is denoted by $\mathbf{1}$.

In what follows, we continue to denote the Laplacian of G by $L(G)$ and the resistance matrix of G by $R(G)$. The following result [3] generalises the formulae for the determinant and the inverse of the distance matrix of a tree obtained in [11, 10,4].

Theorem 3.1. *Let G be a connected, weighted graph with vertex set $\{1, 2, \cdots, n\}$. Then the following assertions hold:*

(i) $\mathbf{1}'\tau(G) = 2$

(ii) $R(G)$ *is nonsingular and* $R(G)^{-1} = -\frac{1}{2}L(G) + \frac{1}{\tau(G)'R(G)\tau(G)}\tau(G)\tau(G)'$

(iii) $det R(G) = (-1)^{n-1}2^{n-3}\frac{\tau(G)'R(G)\tau(G)}{t(G)}$

We now state and prove one of our main results.

Theorem 3.2. *Let G be a connected, weighted graph with vertex set $\{1, 2, \cdots, n\}$. Let $G_1, G_2, \cdots, G_r$ be the blocks of G. Then the following assertions hold:*

(i) $cof R(G) = \frac{(-1)^{n-1}2^{n-1}}{t(G)}$

(ii) $\tau(G)'R(G)\tau(G) = \sum_{i=1}^{r}\tau(G_i)'R(G_i)\tau(G_i)$

Proof. Clearly if $n = 1$ then the result is trivial. Therefore let $n \geq 2$
We first prove (i). By Theorem 3.1 we have,

$$\begin{aligned} cof R(G) &= \mathbf{1}'R(G)^{-1}\mathbf{1} \times det R(G) \\ &= \frac{1}{\tau(G)'R(G)\tau(G)}\mathbf{1}'\tau(G)\tau(G)'\mathbf{1} \times (-1)^{n-1}2^{n-3}\frac{\tau(G)'R(G)\tau(G)}{t(G)} \\ &= 4 \times (-1)^{n-1}2^{n-3}\frac{1}{t(G)} \text{ (since } \mathbf{1}'\tau(G) = 2) \\ &= \frac{(-1)^{n-1}2^{n-1}}{t(G)} \end{aligned}$$

Hence (i) is proved.

We now prove (ii). Let the number of vertices in G_i be $n_i, i = 1, 2, \ldots, r$. By Corollary 2.3 we have

$$det R(G) = \sum_{i=1}^{r} det R(G_i) \cdot (\prod_{j \neq i} cof R(G_j)) \tag{2}$$

By Theorem 3.1,

$$det R(G) = (-1)^{n-1}2^{n-3}\frac{\tau(G)'R(G)\tau(G)}{t(G)} \tag{3}$$

and

$$det R(G_i) = (-1)^{n_i-1}2^{n_i-3}\frac{\tau(G_i)'R(G_i)\tau(G_i)}{t(G_i)}. \tag{4}$$

A block of G cannot have a single vertex and hence $n_i \geq 2$. If $n_i = 2$ then it is an edge with weight w (say)

$$\begin{aligned} detR(G_i) &= (-1)^1 2^{-1} \frac{2w}{\frac{1}{w}} \\ &= -w^2 \end{aligned}$$

and hence (4) is valid.
By (i) of the present theorem,

$$cofR(G_j) = \frac{(-1)^{n_j-1} 2^{n_j-1}}{t(G_j)}. \tag{5}$$

It is a simple exercise (see, for example, [15], p.158) to verify that

$$\sum_{i=1}^{r} n_i = n + r - 1 \tag{6}$$

and that

$$\prod_{i=1}^{r} t(G_i) = t(G). \tag{7}$$

Substituting (3), (4), (5), (6) and (7) in (2) the result is proved. □

4. Resistance matrix of some graphs

Let G be a graph with blocks $G_1, \ldots, G_r$. It is possible to construct $\tau(G)$ from $\tau(G_i), i = 1, 2, \ldots, r$. This is explained as follows. Suppose vertex j of G is not a cut-vertex and lies in block G_i, then the components of $\tau(G)$ and of $\tau(G_i)$ corresponding to j are the same. If vertex j of G is a cut-vertex and lies in blocks $G_{i_1}, G_{i_2}, \ldots, G_{i_k}$, $k \geq 2$, then the component of $\tau(G)$ corresponding to j is the sum of the components of $\tau(G_1), \ldots, \tau(G_r)$ corresponding to j, minus $2(k-1)$. This observation follows readily from the definition of $\tau(G)$ and Theorem 2.2.

Let G be a graph with resistance matrix $R(G)$ and let $\tau(G)$ be defined as before. According to Theorem 3.1, it is possible to determine $detR(G)$ and $R(G)^{-1}$ once we know $\tau(G), \tau(G)'R(G)\tau(G)$ and $t(G)$, the number of spanning trees. It is of course assumed that $L(G)$ is readily available.

We refer to $\tau(G), \tau(G)'R(G)\tau(G)$ and $t(G)$ as the resistance parameters of G. Now suppose G is graph with blocks $G_1, \ldots, G_r$. If we know the resistance parameters of $G_i, i = 1, 2, \ldots, r$, then we can use Theorem 3.2 to get the resistance parameters of G, and hence determine the determinant and the inverse of $R(G)$.

We illustrate this idea with an example. We consider a connected graph G in which each block is either a cycle or an edge. Such a graph is called a cactus ([15],p.160). A formula for the determinant of $R(G)$ will be obtained. We also indicate how one can arrive at the inverse of $R(G)$.

We first state some preliminary results. We first show that the classical formula for the determinant of the distance matrix of a tree [11,5] follows from Theorems 3.2 and 3.1. Note that for a tree, the classical distance matrix and the resistance matrix coincide.

Lemma 4.1. *Let G be a weighted tree on n vertices with edge weights $w_1, w_2, \dots, w_{n-1}$. Let $\delta_1, \delta_2, \dots, \delta_n$ be the vertex degrees. Then the following assertions hold:*

(i) $\tau_i(G) = 2 - \delta_i, i = 1, 2, \dots, n$

(ii) $\tau(G)'R(G)\tau(G) = 2\sum_{i=1}^{n-1} w_i$

(iii) $detR(G) = (-1)^{n-1}2^{n-2}(\sum_{i=1}^{n-1} w_i)\prod_{i=1}^{n-1} w_i$

Proof. (i) If vertices i and j are adjacent with edge k joining them, then $r_{ij} = w_k$. Hence it follows from the definition of $\tau(G)$ that $\tau_i(G) = 2 - \delta_i$.

(ii) The blocks of G, denoted by $G_1, \dots, G_{n-1}$, are simply the edges of G. Thus

$$R(G_i) = \begin{bmatrix} 0 & w_i \\ w_i & 0 \end{bmatrix}$$

Hence $\tau(G_i)'R(G_i)\tau(G_i) = 2w_i$. The result follows from Theorem 3.2.

(iii) As in (ii), let the blocks of G be $G_1, \dots, G_{n-1}$. Clearly, $t(G_i) = \frac{1}{w_i}, i = 1, 2, \dots,$ n-1 and hence $\frac{1}{t(G)} = \prod_{i=1}^{n-1} w_i$. Now the result follows from (ii) and Theorem 3.1, (iii). □

The following result has been obtained in [3].

Lemma 4.2. *Let C_n be the cycle on n vertices. Then the following assertions hold:*

(i) $\tau(C_n) = \frac{2}{n}\mathbf{1}$

(ii) $\tau(C_n)'R(C_n)\tau(C_n) = \frac{2(n^2-1)}{3n}$

(iii) $detR(C_n) = \frac{(-1)^{n-1}2^{n-2}(n^2-1)}{3n^2}$

The case of a cycle with a constant weight on the edges is a trivial consequence of Lemma 4.2 and is stated next.

Corollary 4.3. *Let C_n be the cycle on n vertices with each edge carrying the weight s. Then the following assertions hold:*

(i) $\tau(C_n) = \frac{2}{n}\mathbf{1}$

(ii) $\tau(C_n)'R(C_n)\tau(C_n) = \frac{2s(n^2-1)}{3n}$

(iii) $detR(C_n) = \frac{(-1)^{n-1}2^{n-2}s^n(n^2-1)}{3n^2}$

Theorem 4.4. *Let G be a cactus with n vertices, having m cycles, of length $l_1, l_2, \cdots, l_m$, and suppose each edge of the i-th cycle has weight $s_i, i = 1, 2, \ldots, m$. Let the remaining k edges have weights $w_1, w_2, \cdots, w_k$. Then the following assertions hold:*

(i) $k + \sum_{i=1}^{m} l_i = m + n - 1$

(ii) $\tau(G)'R(G)\tau(G) = \sum_{i=1}^{m} \frac{2s_i(l_i^2-1)}{3l_i} + 2\sum_{i=1}^{k} w_i$

(iii) $detR(G) = (-1)^{n-1}2^{n-2} \prod_{i=1}^{k} w_i \prod_{i=1}^{m} \frac{s_i^{l_i-1}}{l_i} (\sum_{i=1}^{m} \frac{s_i(l_i^2-1)}{3l_i}) + \sum_{i=1}^{k} w_i)$

Proof. (i) If we remove m edges from G, one edge from each cycle, then we get a tree and it must have $n-1$ edges. Thus the number of edges in G must be $m+n-1$ which also equals $k + \sum_{i=1}^{m} l_i$.

(ii) Let the blocks of G be $G_1, \ldots, G_{m+k}$, where $G_1, \ldots, G_m$ are the cycles and $G_{m+1}, \ldots, G_{m+k}$ are the edges. By Corollary 4.3,

$$\tau(G_i)'R(G_i)\tau(G_i) = \frac{2s_i(l_i^2-1)}{3l_i}, i = 1, 2, \ldots, m. \tag{8}$$

Also, by Lemma 4.1,

$$\tau(G_i)'R(G_i)\tau(G_i) = 2w_i, i = m+1, \ldots, m+k. \tag{9}$$

The result follows from (8),(9) and Theorem 3.2.

(iii) Clearly, $t(G_i) = \frac{l_i}{s_i^{l_i-1}}, i = 1, 2, \ldots, m$ and $t(G_i) = \frac{1}{w_i}, i = m+1, \ldots, m+k$. Thus

$$t(G) = \prod_{i=1}^{m} \frac{l_i}{s_i^{l_i-1}} \prod_{i=1}^{k} \frac{1}{w_i}.$$

Now the result follows from (ii) and Theorem 3.1. □

We consider another application of Theorems 3.1 and 3.2. We now consider unweighted graphs. Recall that a graph G is vertex-transitive if for any two vertices i and j of G, there exists an automorphism of G that maps i to j.

Theorem 4.5. *Let G be a vertex-transitive graph with n vertices. Then the following assertions hold:*

(i) $\tau(G) = \frac{2}{n}\mathbf{1}$

(ii) $\tau(G)'R(G)\tau(G) = \frac{4}{n}\alpha(G)$

(ii) $detR(G) = (-1)^{n-1}\frac{\alpha(G)2^{n-1}}{nt(G)}$, *where* $\alpha(G)$ *is a row-sum of* $R(G)$.

Proof. Since G is vertex-transitive, $\tau(G)$ is a scalar multiple of $\mathbf{1}$ and the result follows since $\mathbf{1}'\tau(G) = 2$, by Theorem 3.1. Thus (i) holds. In (ii), we first remark that since G is vertex-transitive, the row-sums of $R(G)$ are all equal. The result follows using (i). Again, (iii) follows using (ii) and Theorem 3.1. □

In Table 1 we indicate the determinant of the resistance matrix of some vertex-transitive graphs, including the platonic graphs. The value of $\alpha(G)$ for the platonic graphs has been given in [12].

Graph	$\alpha(G)$	$t(G)$	$detR(G)$
Tetrahedron	$\frac{3}{2}$	16	$-\frac{3}{16}$
Octahedron	$\frac{13}{6}$	384	$-\frac{13}{432}$
Cube	$\frac{29}{6}$	384	$-\frac{29}{144}$
Icosahedron	$\frac{14}{3}$	5184000	$-\frac{14}{91125}$
Dodecahedron	$\frac{274}{15}$	5184000	$-\frac{70144}{759375}$
Cycle(length=n)	$\frac{n^2-1}{6}$	n	$\frac{(-1)^{n-1}2^{n-2}(n^2-1)}{3n^2}$
K_n	$\frac{2(n-1)}{n}$	n^{n-2}	$(-1)^{n-1}(n-1)(\frac{2}{n})^n$
$K_{n,n}$	$\frac{4n-3}{n}$	$n^{2(n-1)}$	$-(\frac{2}{n})^{2n}(n-\frac{3}{4})$

We conclude by finding the determinant of the resistance matrix of the graph G in Figure 1.

The blocks of the graph are: K_4, a cycle on four vertices C_4, a cube and 11 edge blocks. The number of spanning trees in K_4, C_4 and the cube are $16, 4$ and 384 respectively. Thus $t(G) = 16 \times 4 \times 384 = 24576$. The values of $\tau' R\tau$ for K_4, C_4, the cube and the edge are given, respectively, by $\frac{3}{2}, \frac{5}{2}, \frac{29}{12}$ and 2. It follows from Theorem 3.2 that

$$\tau(G)'R(G)\tau(G) = \frac{3}{2} + \frac{5}{2} + \frac{29}{12} + 2 \times 11 = \frac{341}{12}.$$

Hence by Theorem 3.1,

$$detR(G) = (-1)^{24}2^{22}\frac{\frac{341}{12}}{24576} = \frac{43648}{9}.$$

It is easy to find R^{-1} as well, as we just have to calculate $\tau(G)$ by combining the τ vectors of the blocks as explained at the beginning of this section.

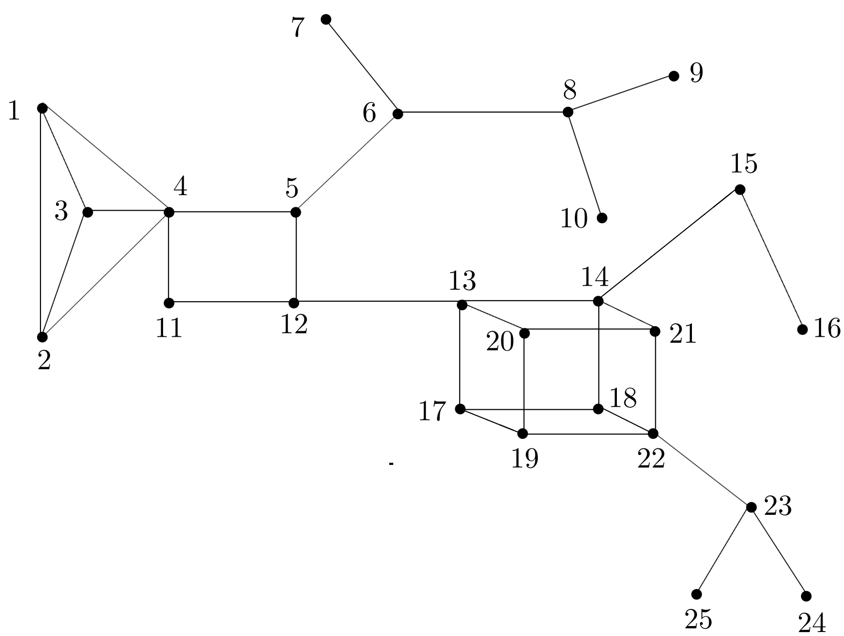

Figure 1: A graph composed of multiple blocks

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