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Laplacian Polynomial and Number of Spanning Trees in Terms of Characteristic Polynomial of Induced Subgraphs

H. B. Walikar*
Department of Mathematics
Karnatak University
Dharwad - 580003, India.
e-mail: *walikarhb@yahoo.co.in*
and
H. S. Ramane
Department of Mathematics
Gogte Institute of Technology
Udyambag, Belgaum - 590008, India.
e-mail: *hsramane@yahoo.com*

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Abstract

In this paper, we express the Laplacian polynomial of a graph in terms of the characteristic polynomials of its induced subgraphs. Further the Laplacian polynomial of a regular graph is expressed in terms of derivatives of its characteristic polynomial. In the sequel we obtain the Laplacian polynomial of a complement of a graph in terms of the characteristic polynomial of induced subgraphs of a graph. Using these we obtain the number of spanning trees of a graph.

Keywords: Laplacian polynomial, characteristic polynomial, number of spanning trees, complement of a graph.

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1. Introduction

The Laplacian polynomial and the characteristic polynomial of a graph and their eigenvalues can be used in several areas of mathematical research and have physical interpretation in various physical and chemical theories. More details about this can be obtained

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in [2, 3, 4, 5, 11]. In this paper we obtain the relationship between the Laplacian polynomial and the characteristic polynomial of a graph and also for the complement of a graph. Further using these results we find the number of spanning trees of a graph and its complement.

Throughout this paper we use undirected graph without loops and multiple edges. We use standard terminology of graph theory [6].

Let G be the graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ where n is the number of vertices of G. Let $d_G(v)$ denote the degree of a vertex v in G. The *adjacency matrix* of a graph G is the matrix $A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The *characteristic polynomial* [4] of a graph G is defined as $\phi(G : \lambda) = \det(\lambda I - A(G))$, where I is an identity matrix of order n. The *degree matrix* of a graph G is the diagonal matrix $D(G) = \mathrm{diag}[d_i]$ where $d_i = d_G(v_i)$. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix.* It is also called as the matrix of admittance due to its role in electrical theory [10]. The *Laplacian polynomial* [9, 11] of a graph G is defined as $c(G : \lambda) = \det(\lambda I - L(G))$ where I is an identity matrix of order n. If G is a regular graph of degree r then $c(G : \lambda) = (-1)^n \phi(G : r - \lambda)$ [4].

2. Laplacian Polynomial in Terms of Characteristic Polynomials

In this section we express the Laplacian polynomial of a graph in terms of characteristic polynomials of its induced subgraphs and for regular graphs, the Laplacian polynomial is expressed in terms of the derivatives of characteristic polynomial.

Define the set $M_k(V) = \{S \mid S \subseteq V(G) \text{and} \mid S \mid = k\}, k = 0, 1, 2, \ldots, n$. The graph $G - S$ is an induced subgraph of G induced by the vertex set $V(G) - S$. Note that $G - V(G) = K_0$, where K_0 is the graph without vertices and hence without edges. $\phi(K_0 : \lambda) = 1 = c(K_0 : \lambda)$. We denote the product of degrees of the vertices of G which belong to S by $P_G(S)$, that is $\prod_{v \in S} d_G(v) = P_G(S)$.

Theorem 2.1. [12] *(k^{th} derivative) Let G be a graph with n vertices then*

$$\frac{d^k}{d\lambda^k} \phi(G : \lambda) = k! \sum_{S \in M_k(V)} \phi(G - S : \lambda), \quad 0 \leq k \leq n. \tag{1}$$

Theorem 2.2. *Let G be a graph with n vertices, then*

$$c(G : \lambda) = (-1)^n \sum_{k=0}^{n} \left\{ \sum_{S \in M_k(V)} (P_G(S)) \phi(G - S : -\lambda) \right\}. \tag{2}$$

Proof. Let $A(G) = [a_{ij}]$, $1 \leq i, j \leq n$ be the adjacency matrix of G. Let $D(G) =$

diag$[d_i]$ where $d_i = d_G(v_i), i = 1, 2, \ldots, n$. Then,

$$\begin{aligned} c(G:\lambda) &= \det(\lambda I - L(G)) \\ &= \det(\lambda I - D(G) + A(G)) \\ &= \begin{vmatrix} \lambda - d_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & \lambda - d_2 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & \lambda - d_3 & \cdots & a_{3n} \\ \vdots & & & \vdots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \lambda - d_n \end{vmatrix} \end{aligned}$$

Splitting above determinant as a sum of two determinants, we get

$$c(G:\lambda) = \begin{vmatrix} \lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & \lambda - d_2 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & \lambda - d_3 & \cdots & a_{3n} \\ \vdots & & & \vdots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \lambda - d_n \end{vmatrix} + \begin{vmatrix} -d_1 & 0 & 0 & \cdots & 0 \\ a_{21} & \lambda - d_2 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & \lambda - d_3 & \cdots & a_{3n} \\ \vdots & & & \vdots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \lambda - d_n \end{vmatrix}$$

Again splitting each of the above determinants as a sum of two determinants and continuing the procedure in succession, we get at n^{th} step,

$$\begin{aligned} c(G:\lambda) &= |\lambda I + A(G)| + \sum_{1 \le i \le n} (-d_i)|\lambda I + A(G - v_i)| \\ &\quad + \sum_{1 \le i < j \le n} (-d_i)(-d_j)|\lambda I + A(G - v_i - v_j)| + \cdots \\ &\quad + \sum_{1 \le i < j < k < \ldots < l \le n} (-d_i)(-d_j)(-d_k)\cdots(-d_l) \\ &\quad |\lambda I + A(G - v_i - v_j - v_k - \cdots - v_l)| \end{aligned}$$

(In above expression I stands for the identity matrix of the appropriate order).

Multiplying each row of the above determinant by -1, we get

$$\begin{aligned} c(G:\lambda) &= (-1)^n| - \lambda I - A(G)| + \sum_{1 \le i \le n} (-d_i)(-1)^{n-1}| - \lambda I - A(G - v_i)| \\ &\quad + \sum_{1 \le i < j \le n} (-d_i)(-d_j)(-1)^{n-2}| - \lambda I - A(G - v_i - v_j)| + \cdots \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{1\le i<j<k<\ldots<l\le n} (-d_i)(-d_j)(-d_k)\cdots(-d_l)(-1)^{n-n} \\
&\quad |-\lambda I - A(G - v_i - v_j - v_k - \cdots - v_l)| \\
= &\ (-1)^n \left\{ |-\lambda I - A(G)| + \sum_{S\in M_1(V)} (P_G(S))|-\lambda I - A(G-S)| \right. \\
&+ \sum_{S\in M_2(V)} (P_G(S))|-\lambda I - A(G-S)| + \cdots \\
&\left. + \sum_{S\in M_n(V)} (P_G(S))|-\lambda I - A(G-S)| \right\} \\
= &\ (-1)^n \sum_{k=0}^{n} \left\{ \sum_{S\in M_k(V)} (P_G(S))|-\lambda I - A(G-S)| \right\} \\
= &\ (-1)^n \sum_{k=0}^{n} \left\{ \sum_{S\in M_k(V)} (P_G(S))\phi(G-S:-\lambda) \right\}.
\end{aligned}
$$

□

Corollary 2.3. *If G is an r-regular graph with n vertices then*

$$
c(G:\lambda) = (-1)^n \sum_{k=0}^{n} \frac{r^k}{k!}\frac{d^k}{dx^k}\phi(G:x)\,|_{x=-\lambda}\ . \tag{3}
$$

Proof. As G is an r-regular graph, $P_G(S) = r^k$ for all $S \in M_k(V)$. Substituting this in Equation (2) gives that

$$
\begin{aligned}
c(G:\lambda) &= (-1)^n \sum_{k=0}^{n} \left\{ \sum_{S\in M_k(V)} r^k \phi(G-S:-\lambda) \right\} \\
&= (-1)^n \sum_{k=0}^{n} \left\{ r^k \sum_{S\in M_k(V)} \phi(G-S:x)\,|_{x=-\lambda} \right\}
\end{aligned}
$$

Using Equation (1), this becomes

$$
c(G:\lambda) = (-1)^n \sum_{k=0}^{n} \frac{r^k}{k!}\frac{d^k}{dx^k}\phi(G:x)\,|_{x=-\lambda}\ .
$$

□

If $\phi(G:\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ is the characteristic polynomial of G, then $\frac{d^k}{d\lambda^k}\phi(G:\lambda)\mid_{\lambda=0} = k!a_{n-k}$, $0 \le k \le n$ and also note that $c(G:0) = 0$, see [4, pp. 39]. Substitution of these in Equation (3), gives that r is an eigenvalue of any r-regular graph.

The next result was first observed by Kelmans [7, 8].

Theorem 2.4. [7, 8] *If $\overline{G}$ denote the complement of a graph G then,*

$$c(\overline{G}:\lambda) = (-1)^{n-1}\frac{\lambda}{n-\lambda}c(G:n-\lambda). \tag{4}$$

Now we obtain the Laplacian polynomial of $\overline{G}$ in terms of the characteristic polynomials of the induced subgraphs of G.

Corollary 2.5. *If $\overline{G}$ is the complement of G, then*

$$c(\overline{G}:\lambda) = \frac{\lambda}{\lambda-n}\sum_{k=0}^{n}\left[\sum_{S\in M_k(V)}(P_G(S))\phi(G-S:\lambda-n)\right] \tag{5}$$

Proof. Follows from the Theorems 2.2 and 2.4. □

Using results (3) and (4) we obtain the following Corollary.

Corollary 2.6. *If G is an r-regular graph and $\overline{G}$ be its complement then*

$$c(\overline{G}:\lambda) = \frac{\lambda}{\lambda-n}\sum_{k=0}^{n}\frac{r^k}{k!}\frac{d^k}{dx^k}\phi(G:x)|_{x=\lambda-n}. \tag{6}$$

3. Relations between Characteristic Polynomials of a Graph and its Complement

Combining Equations (2) and (5) we get following results, which shows the relationship between the characteristic polynomials of induced subgraphs of G and $\overline{G}$.

Theorem 3.1. *If $\overline{G}$ is the complement of G then*

$$(-1)^n\sum_{k=0}^{n}\sum_{S\in M_k(V)}(P_{\overline{G}}(S))\phi(\overline{G}-S:-\lambda)$$

$$= \frac{\lambda}{\lambda-n}\sum_{k=0}^{n}\sum_{S\in M_k(V)}(P_G(S))\phi(G-S:\lambda-n). \tag{7}$$

If G is an r-regular graph then $\overline{G}$ is also regular graph of degree $n-1-r$. Thus $P_G(S)=r^k$ and $P_{\overline{G}}(S)=(n-1-r)^k$ for all $S\in M_k(V)$. Therefore from Theorem 3.1 and Equation (1) we have the following Corollary.

Corollary 3.2. *If G is an r-regular graph, then*

$$(-1)^n\sum_{k=0}^{n}\frac{(n-1-r)^k}{k!}\frac{d^k}{dx^k}\phi(\overline{G}:x)|_{x=-\lambda}=\frac{\lambda}{\lambda-n}\sum_{k=0}^{n}\frac{r^k}{k!}\frac{d^k}{dx^k}\phi(G:x)|_{x=\lambda-n}.$$

4. Number of Spanning Trees

The most renowned application of the Laplacian matrix of a graph is the Matrix Tree Theorem due to Kirchhoff [10].

Theorem 4.1. (Matrix Tree Theorem) *Let $t(G)$ denote the number of spanning trees in a graph G and let $L_{ij}(G)$ be the matrix obtained from $L(G)$ by deleting the i^{th} row and j^{th} column. The absolute value of the determinant of $L_{ij}(G)$ is equal to the number of spanning trees $t(G)$ of the graph G.*

Theorem 4.2. [4] *Let G be a graph with n vertices, then*

$$t(G)=\frac{(-1)^{n-1}}{n}\frac{d}{d\lambda}c(G:\lambda)|_{\lambda=0}. \tag{8}$$

Here we add some more results on the number of spanning trees of a graph G.

Theorem 4.3. *For any graph G,*

$$t(G)=\frac{1}{n}\sum_{k=0}^{n}\sum_{S\in M_k(V)}(P_G(S))\frac{d}{d\lambda}\phi(G-S:\lambda)\ |_{\lambda=0}\ . \tag{9}$$

Proof. From the Theorem 2.2, we have

$$c(G:\lambda)=(-1)^n\sum_{k=0}^{n}\sum_{S\in M_k(V)}(P_G(S))\phi(G-S:-\lambda)$$

Differentiating with respect to λ

$$\frac{d}{d\lambda}c(G:\lambda)=(-1)^n\sum_{k=0}^{n}\sum_{S\in M_k(V)}(P_G(S))(-1)\frac{d}{d\lambda}\phi(G-S:\lambda)\ |_{\lambda=-\lambda}$$

Put $\lambda=0$.

$$\frac{d}{d\lambda}c(G:\lambda)\mid_{\lambda=0}=(-1)^{n+1}\sum_{k=0}^{n}\sum_{S\in M_k(V)}(P_G(S))\frac{d}{d\lambda}\phi(G-S:\lambda)\mid_{\lambda=0} \tag{10}$$

Using Equation (8) in (10) we get the result. □

Corollary 4.4. *For any r-regular graph with n vertices,*

$$t(G)=\frac{1}{n}\sum_{k=1}^{n}\frac{r^{k-1}}{(k-1)!}\frac{d^k}{d\lambda^k}\phi(G:\lambda)\mid_{\lambda=0}.$$

Proof. Since G is an r-regular graph, $d_G(v)=r$ for all $v\in V(G)$. Therefore $P_G(S)=r^k$ for all $S\in M_k(V)$. Substituting this in Equation (9), we get

$$\begin{aligned} t(G) &= \frac{1}{n}\sum_{k=0}^{n}\sum_{S\in M_k(V)} r^k\frac{d}{d\lambda}\phi(G-S:\lambda)\mid_{\lambda=0} \\ &= \frac{1}{n}\sum_{k=0}^{n} r^k\frac{d}{d\lambda}\sum_{S\in M_k(V)}\phi(G-S:\lambda)\mid_{\lambda=0} \\ &= \frac{1}{n}\sum_{k=0}^{n}\frac{r^k}{k!}\frac{d}{d\lambda}\frac{d^k}{d\lambda^k}\phi(G:\lambda)\mid_{\lambda=0}, \quad \text{From Theorem 2.1} \\ &= \frac{1}{n}\sum_{k=0}^{n}\frac{r^k}{k!}\frac{d^{k+1}}{d\lambda^{k+1}}\phi(G:\lambda)\mid_{\lambda=0} \\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{r^{k-1}}{(k-1)!}\frac{d^k}{d\lambda^k}\phi(G:\lambda)\mid_{\lambda=0}. \end{aligned}$$

□

Theorem 4.5. *Let G be the graph with n vertices and $\overline{G}$ be its complement then*

$$t(\overline{G})=\frac{(-1)^n}{n^2}\sum_{k=0}^{n}\sum_{S\in M_k(V)}(P_G(S))\phi(G-S:-n).$$

Proof. From Corollary 2.5,

$$(\lambda-n)c(\overline{G}:\lambda)=\lambda\sum_{k=0}^{n}\sum_{S\in M_k(V)}(P_G(S))\phi(G-S:\lambda-n).$$

Differentiating with respect to λ we get

$$\begin{aligned}(\lambda-n)\frac{d}{d\lambda}c(\overline{G}:\lambda)+c(\overline{G}:\lambda) &= \lambda\sum_{k=0}^{n}\sum_{S\in M_k(V)}(P_G(S))\frac{d}{d\lambda}\phi(G-S:\lambda-n)\\ &+\sum_{k=0}^{n}\sum_{S\in M_k(V)}(P_G(S))\phi(G-S:\lambda-n).\end{aligned}$$

Put $\lambda=0$.

$$(-n)\frac{d}{d\lambda}c(\overline{G}:\lambda)\mid_{\lambda=0}=\sum_{k=0}^{n}\sum_{S\in M_k(V)}(P_G(S))\phi(G-S:-n),\quad \text{since } c(\overline{G}:0)=0.$$

Using Equation (8) in above we get the result. □

Theorem 4.6. *Let G be the r-regular graph with n vertices and $\overline{G}$ be its complement then,*

$$t(\overline{G})=\frac{(-1)^n}{n^2}\sum_{k=0}^{n}\frac{r^k}{k!}\frac{d^k}{d\lambda^k}\phi(G:\lambda)\mid_{\lambda=-n}.$$

Proof. Let G be an r-regular graph. Therefore from Equation (6), we get

$$(\lambda-n)c(\overline{G}:\lambda)=\lambda\sum_{k=0}^{n}\frac{r^k}{k!}\frac{d^k}{dx^k}\phi(G:x)\mid_{x=\lambda-n}.$$

Differentiating with respect to λ , we get

$$\begin{aligned}(\lambda-n)\frac{d}{d\lambda}c(\overline{G}:\lambda)+c(\overline{G}:\lambda) &= \lambda\sum_{k=0}^{n}\frac{r^k}{k!}\frac{d^{k+1}}{dx^{k+1}}\phi(G:x)\mid_{x=\lambda-n}\\ &+\sum_{k=0}^{n}\frac{r^k}{k!}\frac{d^k}{dx^k}\phi(G:x)\mid_{x=\lambda-n}\end{aligned}$$

Put $\lambda=0$.

$$(-n)\frac{d}{d\lambda}c(\overline{G}:\lambda)\mid_{\lambda=0}=\sum_{k=0}^{n}\frac{r^k}{k!}\frac{d^k}{dx^k}\phi(G:x)\mid_{x=-n},\qquad \text{Since } c(\overline{G}:0)=0.$$

Using Equation (8) in above equation we get the required result. □

If $r = 0$ then $G = \overline{K}_n$ and $\phi(\overline{K}_n : \lambda) = \lambda^n$. Therefore taking that $\lim r^0 = 1$, as r tends to zero, we have from Theorem 4.6, $t(K_n) = n^{n-2}$, which is a Cayley formula [1].

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References

[1] A. Cayley, A theorem on trees, *Quart. J. Math.*, **23**(1889), 376–378.

[2] F. R. K. Chung, *Spectral Graph Theory*, CBMS, 92, American Mathematical Society, Providence, Rhode Island, (1997).

[3] D. M. Cvetkovic, M. Doob, I. Gutman and A. Torgasev, *Recent Results in the Theory of Graph Spectra*, North–Holland, Amsterdam, (1988).

[4] D. M. Cvetkovic, M. Doob and H. Sachs, *Spectra of Graphs*, Academic Press, New York, (1980).

[5] I. Gutman and O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer-Verlag, Berlin, (1986).

[6] F. Harary, *Graph Theory*, Narosa Publishing House, New Delhi, (1998).

[7] A. K. Kelmans, The number of trees of a graph I, *Avtomat. i Telemeh*, **26**(1965), 2194–2204 (in Russian), transl. *Automat. Remote Control*, **26**(1965), 2118–2129.

[8] A. K. Kelmans, The number of trees of a graph II, *Avtomat. i Telemeh*, **27**(1966), 56–65 (in Russian), transl. *Automat. Remote Control*, **27**(1966), 233–241.

[9] A. K. Kelmans, Transformations of a graph increasing its Laplacian polynomial and nuber of spanning trees, *Europ. J. Combinatorics*, **18**(1997), 35–48.

[10] G. Kirchhoff, Uber die Auflosung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer strome gefuhrt wird, *Ann. Phys. Chem.*, **72**(1847), Translated by J. B. O'Toole in *I. R. E. Trans. Circuit Theory*, **CT-5** (1958), 4.

[11] B. Mohar, The Laplacian spectrum of graphs, in: Y. Alavi, G. Chartrand, O. R. Ollermann and A. J. Schwenk, *Graph Theory, Combinatorics and Applications*, Wiley, New York, (1991).

[12] H. S. Ramane, Ph. D. Dissertation: *Some Topics in Spectral Graph Theory*, Karnatak University, Dharwad, Oct. 2002.