AKCE J. Graphs. Combin., 5, No. 1 (2008), pp. 57-60

Orthogonal σ-labellings of graphs

R. Sampathkumar and V. Sriram
Department of Mathematics
Annamalai University
Annamalainagar - 608 002,Tamilnadu, India.
e-mail: *sampathmath@gmail.com*

Communicated by: S. Arumugam
Received 27 July 2006; accepted 11 February 2008

Abstract

We introduce a special kind of orthogonal labelling, called, orthogonal σ-labelling, and we find some applications of it to orthogonal labellings of graphs.

Keywords: Cyclic orthogonal double covers of graphs.

2000 Mathematics Subject Classification: 05C70, 05C78.

1. Introduction

All the graphs we deal with are undirected, finite and simple. Let K_n be the complete graph on an n-element vertex set V. A collection $\mathcal{G} = \{G_i : i \in V\}$ of n subgraphs of K_n is an *orthogonal double cover* (briefly *ODC*) of K_n if it has the following properties:
1. *Double cover property*:
Every edge of K_n belongs to exactly two of the subgraphs in $\mathcal{G}$.
2. *Orthogonality property*:
For any two distinct subgraphs G_i and G_j in $\mathcal{G}$, $\mid E(G_i) \cap E(G_j) \mid= 1$.

Since the number n of subgraphs in an ODC is the same as the number of vertices of K_n, it follows that each G_i must have $n-1$ edges. If all the subgraphs G_i, $i \in V$, are isomorphic to some graph G, then $\mathcal{G}$ is called an *ODC of* K_n *by* G.

Gronau, Mullin and Rosa conjectured the following:

Conjecture 1.1. [1] *If T is an arbitrary tree with n vertices, $n \geq 2$, other than the path P_4 with 3 edges, then there exists an ODC of K_n by T.*

An ODC $\mathcal{G}$ of K_n by G is called *cyclic* (*CODC*) if the cyclic group of order n is a subgroup of the automorphism group of $\mathcal{G}$.

Let $V(K_n) = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, the set of integers modulo n. The *length* of an edge xy, $x, y \in \mathbb{Z}_n$, is defined as $\ell(xy) = min\{\mid x-y \mid, n- \mid x-y \mid\}$. Consider two edges $e_1 = \{x_1, y_1\}$ and $e_2 = \{x_2, y_2\}$, $x_1, y_1, x_2, y_2 \in \mathbb{Z}_n$, with $\ell(e_1) = \ell(e_2)$. Their *rotation-distance* $r(e_1, e_2)$ is defined to be the shorter one of the two rotation mappings e_1 onto e_2 that is $r(e_1, e_2) = min\{r_1, r_2 : \{(x_1 + r_1) mod\, n, (y_1 + r_1) mod\, n\} = \{x_2, y_2\}, \{(x_2 + r_2) mod\, n, (y_2 + r_2) mod\, n\} = \{x_1, y_1\}\}$.

A 1-1 mapping $\phi : V \to \mathbb{Z}_n$ of a simple graph $G = (V, E)$ with $| E |= n - 1$ is called an *orthogonal labelling* (OL) of G if the following conditions are satisfied:

1. For every $k \in \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$, G contains exactly two edges of length k, and exactly one edge of length $\frac{n}{2}$ if n is even.
2. The set of all rotation-distances form a permutation of $\{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$.

The following theorem of Gronau, Mullin and Rosa relates $CODCs$ and OLs.

Theorem 1.2. [1] *A CODC of K_n by a graph G exists if and only if there exists an OL of G.*

In this paper, we introduce a special kind of OL, called orthogonal σ-labelling, a concept similar to the σ-valuation introduced by Rosa [5] in 1966. In Section 2, we find applications of orthogonal σ-labellings of graphs to OLs of graphs, and in Section 3, we find OLs for some caterpillars of diameter 4.

Let $G = (V, E)$ be a simple graph. For $v \in V$ and for any positive integer r, we denote by $G(v; r)$, the simple graph obtained from G by adding r new vertices $v_1, \ldots, v_r$ and adding r new edges $vv_1, \ldots, vv_r$. For $u, v \in V$, $u \neq v$, and for any positive integer r, we denote by $G(u, v; r)$, the simple graph obtained from G by adding $2r$ new vertices $u_1, \ldots, u_r, v_1, \ldots, v_r$ and adding $2r$ new edges $uu_1, \ldots, uu_r, vv_1, \ldots, vv_r$.

2. Orthogonal σ-labelling

An OL ϕ of G is called an *orthogonal σ-labelling* (OσL) if $\ell(xy) =| x - y |$ for each edge xy in G and the calculated rotation-distances are not modular, that is, if $r(e_1, e_2) = r_1$, then $\{(x_1 + r_1) mod\, n, (y_1 + r_1) mod\, n\} = \{x_1 + r_1, y_1 + r_1\}$ and if $r(e_1, e_2) = r_2$, then $\{(x_2 + r_2) mod\, n, (y_2 + r_2) mod\, n\} = \{x_2 + r_2, y_2 + r_2\}$.

In this paper, we define only the labelling and the verification that it is an OL (resp. OσL) is omitted.

Theorem 2.1. *Let $G = (V, E)$ be a simple graph with $n =| V |=| E | +1$ and let $\phi : V \to \mathbb{Z}_n$ be an $O\sigma L$ of G. Then, for any positive integer r, the following conditions hold:*

1. *If n is odd and $\phi(v) = \frac{n-1}{2}$ for some $v \in V$, then $G(v; r)$ has an OL.*
2. *If n is even and there is an edge $e = uv$ with $\ell(e) = \frac{n}{2}$ and $\phi(v) \in \{\frac{n}{2} - 1, \frac{n}{2}\}$, then $G(v; r)$ has an OL.*
3. *If n is even and there is an edge $e = uv$ with $\ell(e) = \frac{n}{2}$ and $\phi(v) \in \{\frac{n}{2} - 2, \frac{n}{2} + 1\}$, then there exists a vertex v^* such that $G(v, v^*; r)$ has an OL.*

Proofs of 1 *and* 2. Define $\Psi : V(G(v;r)) \to \mathbb{Z}_{n+r}$ by $\Psi(x) = \phi(x)$ if $x \in V$ and for every $i \in \{1,\dots,r\}$, $\Psi(v_i) = n+i-1$.
Proof of 3. By hypothesis, $\phi(v) \in \{\frac{n}{2}-2, \frac{n}{2}+1\}$. We consider two cases.
Case 1. $\phi(v) = \frac{n}{2}-2$.

Then $\phi(u) = n-2$, since ϕ is an OσL. There exists a vertex v^* such that $\phi(v^*) = \frac{n}{2}$, since ϕ is onto. Define $\Psi : V(G(v,v^*;r)) \to \mathbb{Z}_{n+2r}$ by $\Psi(x) = \phi(x)$ if $x \in V$ and for every $i \in \{1,\dots,r\}$, $\Psi(v_i) = n+2i-2$ and $\Psi(v_i^*) = n+2i-1$.
Case 2. $\phi(v) = \frac{n}{2}+1$.

Then $\phi(u) = 1$, since ϕ is an OσL. There exists a vertex v^* such that $\phi(v^*) = \frac{n}{2}-1$, since ϕ is onto. Define $\Psi : V(G(v,v^*;r)) \to \mathbb{Z}_{n+2r}$ by $\Psi(x) = \phi(x)$ if $x \in V$ and for every $i \in \{1,\dots,r\}$, $\Psi(v_i) = n+2i-1$ and $\Psi(v_i^*) = n+2i-2$. □

3. Orthogonal labellings of caterpillars of diameter 4

The caterpillar $C(n_1,n_2,\dots,n_r)$ is the tree obtained from the path $P_r := u_1,u_2,\dots,u_r$ by joining the vertex u_i to n_i new vertices $u_{i,1},u_{i,2},\dots,u_{i,n_i}$ for each i. We assume that $n_i \geq 0$ for $i \in \{2,3,\dots,r-1\}$, $n_1 \geq 1$ and $n_r \geq 1$. Clearly, $C(n_1,n_2,\dots,n_r)$ is of diameter $r+1$.

If T is a caterpillar of diameter 2, then $r=1$ and hence it is a star. Any bijection $\phi : V(C(n_1)) \to \mathbb{Z}_{n_1+1}$ is an OL of $C(n_1)$.

In [2], Leck and Leck showed that the caterpillar $C(n_1, n_2)$ of diameter 3 has an OL if and only if n_1+n_2+2 and n_1n_2 are not relatively prime. Gronau, Mullin and Rosa [1], have constructed ODCs for all caterpillars of diameter 3, except P_4.

Gronau, Mullin and Rosa [1], obtained an OL for the caterpillar $C(2k,1,2k)$. They also observed that $C(2,2,2)$ and $C(2,3,2)$ have no OL. Sampathkumar and Simaringa [4], obtained OLs for $C(1,2(k+t)+1,2k)$, $C(4,4k+1,4)$ and $C(2k,4k-3,2k)$. Leck and Leck [3], obtained an OL for $C(n_1,n_2,n_3)$ if $\mid n_1-n_3 \mid \geq n_2$. In this section, we extend the class of caterpillars of diameter 4 for which an OL is known to exist. In [3], Leck and Leck have constructed ODCs for all caterpillars of diameter 4.

1. There exists an OL of $C(1,n_2,n_3)$ for $n_2 > n_3$ and $n_3 \geq 2$.

Proof. $\phi : V(C(1,n_3+1,n_3)) \to \mathbb{Z}_{2n_3+5}$ defined by $\phi(u_1) = 2$, $\phi(u_{1,1}) = n_3+4$, $\phi(u_2) = n_3+2$, $\phi(u_{2,1}) = 0$, $\phi(u_{2,2}) = 1$, $\phi(u_{2,i}) = i$ for $i \in \{3,4,\dots,n_3+1\}$, $\phi(u_3) = n_3+3$ and $\phi(u_{3,i}) = n_3+4+i$ for $i \in \{1,2,\dots,n_3\}$ is an OσL of $C(1,n_3+1,n_3)$. By Theorem 2.1 (1), $(C(1,n_3+1,n_3))(u_2;n_2-n_3-1) \cong C(1,n_2,n_3)$ has an OL. □

2. There exists an OL of $C(2,n_2,2)$ for $n_2 \geq 4$.

Proof. $\phi : V(C(2,4,2)) \to \mathbb{Z}_{11}$ defined by $\phi(u_1) = 2$, $\phi(u_{1,1}) = 0$, $\phi(u_{1,2}) = 6$, $\phi(u_2) = 5$, $\phi(u_{2,1}) = 1$, $\phi(u_{2,2}) = 4$, $\phi(u_{2,3}) = 7$, $\phi(u_{2,4}) = 10$, $\phi(u_3) = 8$, $\phi(u_{3,1}) = 3$ and $\phi(u_{3,2}) = 9$ is an OσL of $C(2,4,2)$. By Theorem 2.1 (1), $(C(2,4,2))(u_2;n_2-4) \cong C(2,n_2,2)$, $n_2 \geq 5$, has an OL. □

3. There exists an OL of $C(2,n_2,3)$ for $n_2 \geq 3$.

Proof. $\phi : V(C(2,3,3)) \to \mathbb{Z}_{11}$ defined by $\phi(u_1) = 4$, $\phi(u_{1,1}) = 6$, $\phi(u_{1,2}) = 9$, $\phi(u_2) = 5$, $\phi(u_{2,1}) = 0$, $\phi(u_{2,2}) = 1$, $\phi(u_{2,3}) = 2$, $\phi(u_3) = 7$, $\phi(u_{3,1}) = 3$, $\phi(u_{3,2}) = 8$ and $\phi(u_{3,3}) = 10$ is an OσL of $C(2,3,3)$. By Theorem 2.1 (1), $(C(2,3,3))(u_2; n_2 - 3) \cong C(2, n_2, 3)$, $n_2 \geq 4$, has an OL. □

4. There exists an OL of $C(1, n_2, 1)$ for $n_2 \geq 1$.

Proof. $\phi : V(C(1,1,1)) \to \mathbb{Z}_6$ defined by $\phi(u_1) = 2$, $\phi(u_{1,1}) = 4$, $\phi(u_2) = 3$, $\phi(u_{2,1}) = 5$, $\phi(u_3) = 0$ and $\phi(u_{3,1}) = 1$ is an OσL of $C(1,1,1)$. By Theorem 2.1 (2), $(C(1,1,1))(u_2; n_2 - 1) \cong C(1, n_2, 1)$, $n_2 \geq 2$, has an OL. □

5. There exists an OL of $C(1, 4, n_3)$ for $n_3 \geq 4$.

Proof. $\phi : V(C(1,4,4)) \to \mathbb{Z}_{12}$ defined by $\phi(u_1) = 3$, $\phi(u_{1,1}) = 8$, $\phi(u_2) = 5$, $\phi(u_{2,1}) = 1$, $\phi(u_{2,2}) = 2$, $\phi(u_{2,3}) = 4$, $\phi(u_{2,4}) = 7$, $\phi(u_3) = 6$, $\phi(u_{3,1}) = 0$, $\phi(u_{3,2}) = 9$, $\phi(u_{3,3}) = 10$ and $\phi(u_{3,4}) = 11$ is an OσL of $C(1,4,4)$. By Theorem 2.1 (2), $(C(1,4,4))(u_3; n_3 - 4) \cong C(1, 4, n_3)$, $n_3 \geq 5$, has an OL. □

6. There exists an OL of $C(1, k, k)$ for $k \geq 2$.

Proof. $\phi : V(C(1,2,2)) \to \mathbb{Z}_8$ defined by $\phi(u_1) = 1$, $\phi(u_{1,1}) = 0$, $\phi(u_2) = 5$, $\phi(u_{2,1}) = 2$, $\phi(u_{2,2}) = 7$, $\phi(u_3) = 3$, $\phi(u_{3,1}) = 4$ and $\phi(u_{3,2}) = 6$ is an OσL of $C(1,2,2)$. By Theorem 2.1 (3), $(C(1,2,2))(u_2, u_3; k - 2) \cong C(1, k, k)$, $k \geq 3$, has an OL. □

In conclusion, we pose the following problem:

Problem. Which caterpillars $C(n_1, n_2, n_3)$ admit an OL?

References

[1] H.-D.O.F. Gronau, R.C. Mullin and A. Rosa, Orthogonal double covers of complete graphs by trees, *Graphs Combin.*, **13** (1997), 251-262.

[2] U. Leck and V. Leck, On orthogonal double covers by trees, *J. Combin. Designs,* **5** (1997), 433-441.

[3] U. Leck and V. Leck, Orthogonal double covers of complete graphs by trees of small diameter, *Discrete Appl. Math.,* **95** (1999), 377-388.

[4] R. Sampathkumar and M. Simaringa, Orthogonal labellings of caterpillars of small diameter, *Bulletin of the ICA,* **44** (2005), 29-36.

[5] A. Rosa, *On certain valuations of the vertices of a graph, Theory of Graphs* (Internat. Symposium, Rome, July 1966), Gordon and Breach, N.Y. and Dunod Paris (1967), 349-355.