Abstract

Let \( \ell, m, n, r \) be integers such that \( 2 \leq \ell \leq r \), \( r \geq 4 \) and \( m, n \geq 0 \). Suppose \( G \) is an \( r \)-regular \( \ell \)-edge-connected graph and that \( k \) is an even integer with \( m \leq k \leq \frac{r}{2} \). We say that \( G \) is an \( (m, n; k) \)-factor graph if for each disjoint pair \( E_1, E_2 \subseteq E(G) \) with \( |E_1| = m \) and \( |E_2| = n \), \( G \) has a \( k \)-factor \( F \) such that \( E_1 \subseteq E(F) \) and \( E_2 \cap E(F) = \emptyset \).

In this note we consider when \( G \) is an \( (m, n; k) \)-factor graph and characterize those graphs which fail for certain parameters \( \ell, m, n, r, k \).

Keywords: regular graph, regular factor, edge-connectivity

2000 Mathematics Subject Classification: 05C70

1. Introduction

We consider finite undirected graphs that may have multiple edges. Let \( G \) be a graph with vertex and edge sets \( V(G) \) and \( E(G) \), respectively. For \( x \in V(G) \), we denote by \( \deg_G(x) \) the degree of \( x \) in \( G \) and for \( S \subseteq V(G) \), we write \( \deg_G(S) \) instead of \( \sum_{x \in S} \deg_G(x) \). We denote by \( G[X] \) the subgraph of \( G \) induced by \( X \) for a subset \( X \) of \( V(G) \). The number of components of a graph \( G \) is denoted by \( \omega(G) \). If \( \deg_G(x) = r \) for each \( x \in V(G) \), we call the graph an \( r \)-regular graph. For subsets \( S \) and \( T \) of \( V(G) \), we denote by \( E_G(S,T) \) the set of the edges joining \( S \) and \( T \) and let \( q_G(S,T) = |E_G(S,T)| \). Let \( q_G(S) = q_G(S,S) \). Note that \( q_G(S) \equiv 0 \pmod 2 \) for every \( S \) of \( V(G) \). Let \( f \) be an integer-valued function defined on \( V(G) \), denoted by \( f : V(G) \to \mathbb{Z}^+ \cup \{0\} \) and for \( S \subseteq V(G) \), we write \( f(S) \) instead of \( \sum_{x \in S} f(x) \). A spanning subgraph \( F \) of \( G \) such
that $\deg_F(x) = f(x)$ for each $x \in V(G)$ is called a $f$-factor of $G$. If $f$ is a constant function taking a value $k$, an $f$-factor is called an $k$-factor.

Suppose that $\ell \geq 2$ and $r \geq 4$ are integers with $r \geq \ell$. Let $G(r, \ell)$ be the set of finite $r$-regular, $\ell$-edge-connected graphs. Let $G \in G(r, \ell)$ and $E^* \subseteq E(G)$. Suppose that $(E_1, E_2)$ is a partition of $E^*$ such that:

1. $|E_1| = m, |E_2| = n$;
2. there exists a $k$-factor $F$ of $G$ with $m \leq k \leq \frac{r}{2}$ such that $E_1 \subseteq E(F)$ and $E_2 \subseteq E(G) \setminus E(F)$.

Then we say that $(E_1, E_2)$ is a potential $(m, n; k)$-factor of $G$; otherwise we say that $(E_1, E_2)$ is an $(m, n; k)$-obstruction. If $G$ contains no $(m, n; k)$-obstruction, then $G$ is said to be an $(m, n; k)$-factor graph. Let $F(m, n; k)$ be the set of $(m, n; k)$-factor graphs.

When $r = \ell$ and $G \in G(r, r)$, Aldred, Holton and Sheehan [1] and Egawa and Kotani [2] investigated potential $(m, n; k)$-factors in $G$. Their results are detailed below.

The main result of this paper extends this work to graphs $G \in G(r, \ell)$ with $\ell \leq r$ in the case when $k$ is even (note [1] considered only the case $k = 2$).

We define a graph $G$ to be “nearly” bipartite as follows. Set $E^* \subseteq E(G)$. Let $(S, T)$ be a partition of $V(G)$ with $|S| = s$, $|T| = t$ and let $(E_1, E_2)$ be a partition of $E^*$. Define $q_i(S, T) = |E_i \cap E_G(S, T)|$ and $q_i(S) = q_i(S, S)$ with $1 \leq i \leq 2$. We also require that one of the following holds. Either:

1. $s = t - 1$, $r$ and $k$ are even, and $q_1(S) + q_2(T) + \frac{k}{2} - 1 \geq q_G(T) = q_G(S) + \frac{k}{2}$;
2. $s = t$ and $q_1(S) + q_2(T) - 1 \geq q_G(T) = q_G(S) \geq 1$; or
3. $s = t + 1$, $r$ and $k$ are even and $q_1(S) + q_2(T) - \frac{k}{2} - 1 \geq q_G(T) = q_G(S) - \frac{k}{2}$.

Clearly a “nearly” bipartite graph has an $(m, n; k)$-obstruction for suitably chosen $m, n, k$. Theorems, and our main result show that if $G \in G(r, \ell)$ then $G \in F(m, n; k)$ for suitably restricted $m, n, k$ unless $G$ is “nearly” bipartite.

The following theorem was proved by R. E. L. Aldred, D. A. Holton and J. Sheehan. \textbf{Theorem A.} [1] Let $m, n, r$ be integers with $m, n \geq 0$ and $r \geq 4$, and let $G \in G(r, r)$, and let $E_1, E_2$ be disjoint edge sets in $G$ with $|E_1| = m$ and $|E_2| = n$. Suppose that $m$ and $n$ satisfy one of the following three conditions:

1. $m = 0$ and $n \leq r - 2$;
2. $m = 1$ and $n \leq \lceil \frac{r}{2} \rceil - 1$; or
3. $m = 2$ and $n = 0$.
Then either $G$ has a 2-factor $F$ with $E(F) \supseteq E_1$ and $E(F) \cap E_2 = \emptyset$ or $G$ is “nearly” bipartite.

This was generalized by Egawa and Kotani in the following theorem.

**Theorem B.** [2] Let $m, n, r, k$ be integers with $m, n \geq 0$, $k \geq 2$, and let $G \in \mathcal{G}(r, r)$, and let $E_1, E_2$ be disjoint edge sets in $G$ with $|E_1| = m$ and $|E_2| = n$. Suppose that $m$ and $n$ satisfy one of the following three conditions:

(i) $m = 0$ and $n \leq r - k$;

(ii) $1 \leq m \leq \frac{k}{2}$ and $m + n \leq \lceil \frac{r}{2} \rceil$; or

(iii) $\frac{k}{2} < m \leq k$ and $m + n \leq k$.

Then either $G$ has a $k$-factor $F$ with $E(F) \supseteq E_1$ and $E(F) \cap E_2 = \emptyset$ or $G$ is “nearly” bipartite.

In both Theorems A and B, $G$ is required to be $r$-regular and $r$-edge-connected. Here we weaken the restriction on edge-connectivity when $k$ is any even integer and obtain our main result.

**Theorem 1.** Let $\ell, m, n$ and $r$ be integers with $2 \leq \ell \leq r$, $r \geq 4$, and $m, n \geq 0$, and $k$ be an even integer with $m \leq k \leq \frac{r}{2}$, and let $G \in \mathcal{G}(r, \ell)$, and let $E_1, E_2$ be disjoint edge sets in $G$ with $|E_1| = m$ and $|E_2| = n$. Suppose that $m$ and $n$ satisfy one of the following three conditions:

(i) $m = 0$ and $n \leq \lceil \frac{\ell + r}{2} \rceil - k$;

(ii) $1 \leq m \leq \frac{k}{2}$ and $m + n \leq \lceil \frac{k}{2} \rceil$; or

(iii) $\frac{k}{2} < m \leq k$ and $m + n \leq k + \lceil \frac{\ell - r}{2} \rceil$.

Then either $G$ has a $k$-factor $F$ with $E(F) \supseteq E_1$ and $E(F) \cap E_2 = \emptyset$ or $G$ is “nearly” bipartite.

Our proof of Theorem 1 makes use of Tutte’s $f$-factor theorem. Let $G$ be a graph, $f$ be an integer-valued function defined on $V(G)$ and let $B = (S, T, U)$ be a partition of $V(G)$. We define $\delta_G(B)$ by

$$\delta_G(B) = h_G(B) - f(S) + f(T) - \deg_{G-S}(T),$$

where $h_G(B)$ is the number of components $C$ of $G[U]$ such that $f(V(C)) + q_G(V(C), T)$ is odd. These components are called odd components.

**Theorem C.** [4] Let $G$ be a graph, and let $f$ be an integer-valued function defined on $V(G)$. Then
(i) $\delta_G(B) \equiv f(V(G)) \pmod{2}$ for any partition $B$ of $V(G)$, and

(ii) $G$ has an $f$-factor if and only if $\delta_G(B) \leq 0$ for any partition $B$ of $V(G)$.

We shall also use the lemma below which generalizes a result of Katerinis [3]. In this lemma we make use of the following notation. For each $v \in V(G)$ we denote by $\#_{\{v\}}^{(i)}$, the number of edges in $E_i$ incident with vertex $v$, $(i = 1, 2)$ and for each $X \subseteq V(G)$ we write $\#_X^{(i)}$ instead of $\sum_{v \in X} \#_{\{v\}}^{(i)}$, $(i = 1, 2)$.

**Lemma 1.** Let $\ell,m,n$ be integers with $\ell \geq 2$ and let $G$ be an $\ell$-edge-connected graph. Suppose $B = (S,T,U)$ is a partition of $V(G)$ and $E^* \subseteq E(G)$ is partitioned as that $E^* = (E_1,E_2)$ with $|E_1| = m$ and $|E_2| = n$. Let $G' = G - E^*$ and let $\omega = \omega(G'[U])$. Then

(i) $\deg_{G'}(T) \geq \deg_G(T) - \deg_G(S) - 2(q_1(T) + q_2(T)) - q_1(T,U) - q_2(T,U) + 2q_G(S),$

and

(ii) $2 \deg_{G'}(T) \geq \ell \omega + \deg_G(T) - \deg_G(S) - 2(m + n) + 2\#_S^{(1)}$.

**Proof.** Let $\ell,m,n,G,B,E^*,G'$ and $\omega$ be as stated in the hypotheses of Lemma 1. Suppose $C_1,C_2,\ldots,C_\omega$ are the components of $G'[U]$ and $a_i$ (resp. $b_i$) the number of edges of $E^*$ joining $C_i$ to $S \cup T$ (resp. to the other $C_j$’s). Clearly

\[ \deg_G(S) = 2q_G(S) + q_G(S,T) + q_G(S,U) \]

\[ \geq 2q_G(S) + q_G(S,T). \]

But

\[ q_G(S,T) = \deg_G(T) - \deg_G(S) \]

and

\[ \deg_G(S) = \deg_{G'}(T) + 2(q_1(T) + q_2(T)) + q_1(T,U) + q_2(T,U). \]

Substituting (3) and (4), in (2), we have

\[ \deg_G(S) \geq 2q_G(S) + \deg_G(T) - \deg_{G'}(T) - 2(q_1(T) + q_2(T)) - q_1(T,U) - q_2(T,U). \]

Therefore, Lemma 1 (1) holds.

On the other hand, we have

\[ q_1(S,U) + q_1(T,U) + q_2(T,U) \leq \sum_{i=1}^{\omega} a_i. \]

Further, since $|E^*| = m + n$, $\sum_{i=1}^{\omega} a_i + (\frac{1}{2}) \sum_{i=1}^{\omega} b_i + q_1(S) + q_1(T) + q_2(T) + q_1(S,T) \leq m + n$. Thus

\[ 2(q_1(T) + q_2(T)) \leq 2(m + n) - 2 \sum_{i=1}^{\omega} a_i - \sum_{i=1}^{\omega} b_i - 2(q_1(S) + q_1(S,T)). \]
Substituting (4), (5) and (6) in (3), we have

\[ q_G(S, T) \geq \deg_G(T) - \deg_{G^r-S}(T) - 2(m+n) + \sum_{i=1}^{\omega} (a_i + b_i) + 2(\ell q_1(S) + q_1(S, T)) + q_1(S, U). \]  

From (1)

\[ \deg_G(S) \geq 2q_1(S) + q_G(S, T) + q_{G'}(S, U) + q_1(S, U). \]  

Since \( G \) is \( \ell \)-edge-connected, we have \( \sum_{i=1}^{\omega} q_G(V(C_i), V(G - V(C_i))) \geq \ell \omega \). But

\[ \sum_{i=1}^{\omega} q_G(V(C_i), V(G - V(C_i))) \leq \sum_{i=1}^{\omega} (a_i + b_i) + \deg_{G^r-S}(T) + q_{G'}(S, U). \]

Thus

\[ \sum_{i=1}^{\omega} (a_i + b_i) + \deg_{G^r-S}(T) + q_{G'}(S, U) \geq \ell \omega. \]

Hence \( q_{G'}(S, U) \geq \ell \omega - \sum_{i=1}^{\omega} (a_i + b_i) - \deg_{G^r-S}(T) \) and so (8) implies,

\[ \deg_G(S) - \ell \omega + \sum_{i=1}^{\omega} (a_i + b_i) + \deg_{G^r-S}(T) \geq 2q_1(S) + q_G(S, T) + q_1(S, U). \]  

From (7) and (9),

\[ \deg_G(S) - \ell \omega + 2 \deg_{G^r-S}(T) \geq \deg_G(T) - 2(m+n) + 2(\ell q_1(S) + q_1(S, T) + q_1(S, U)). \]

Therefore

\[ 2 \deg_{G^r-S}(T) \geq \ell \omega + \deg_G(T) - \deg_G(S) - 2(m+n) + 2\#_S^{(1)}. \]

\[ \square \]

2. Proof of Theorem 1

Throughout the proof we assume \( G \in \mathcal{G}(r, \ell) \) and that \( E^* \subseteq E(G) \). We set \( G' = G - E^* \) and suppose that \( (E_1, E_2) \) is a partition of \( E^* \) with \( |E_1| = m \) and \( |E_2| = n \).

Now define a function \( f : V(G) \to \{k - m, \ldots , k\} \) such that \( f(v) = k - \#_v^{(1)} \) for \( v \in V(G) \). An \( f \)-factor is a spanning subgraph of \( G \), in which each vertex \( v \) has degree \( f(v) \). Then following two claims are immediate:

**Claim 1.** \( G' \) has an \( f \)-factor if and only if \( (E_1, E_2) \) is a potential \((m, n; k)\)-factor;

**Claim 2.** \( G' \) does not contain an \( f \)-factor if and only if there is some partition \( B = (S, T, U) \) of \( G' \) such that \( \delta_{G^r}(B) = h_{G^r}(B) - f(S) + f(T) - \deg_{G^r-S}(T) \geq 2 \).
We now consider the cases \((E_1, E_2)\) is an \((m, n; k)\)-obstruction of \(G\) with \(E^* = E_1 \cup E_2\).

From Claims 1 and 2, we may choose a partition \(B = (S, T, U)\) of \(G'\) such that \(\delta(B) \geq 2\).

Set \(s = |S|, \ t = |T|\) and let \(\omega = \omega(G'[U])\). Then we have,

\[
\delta_{G'}(B) = h_{G'}(B) - ks + \#_{S}^{(1)} + kt - \#_{T}^{(1)} - \deg_{G' - S}(T) \geq 2. \tag{10}
\]

Now we prove the following claim.

**Claim 3.** \(\deg_{G' - S}(T) - h_{G'}(B) - \#_{S}^{(1)} + \#_{T}^{(1)} \geq -2m\).

**Proof.** Let \(C\) be an odd component of \(G'[U]\). We denote by \(h_{\text{odd}} (h_{\text{even}})\) the number of odd components in \(G'\) such that \(f(C)\) is odd (even). Note that \(h_{G'}(B) = h_{\text{odd}} + h_{\text{even}}\).

By the definition of odd component, we have \(h_{\text{odd}} \leq 2q_1(U) + q_1(T, U) + q_1(S, U)\) and \(h_{\text{even}} \leq q_{G'}(T, U)\) since \(k\) is even. Thus,

\[
\deg_{G' - S}(T) - h_{G'}(B) - \#_{S}^{(1)} + \#_{T}^{(1)} \geq 2q_{G'}(T) + q_{G'}(T, U) - (2q_1(U) + q_1(T, U) + q_1(S, U)) - q_1(T, S) - 2q_1(S) - q_1(S, T) + 2q_1(T) + q_1(T, U).
\]

Using Lemma 1 (i) and since \(G\) is \(r\)-regular, (13) implies

\[
h_{G'}(B) \geq (r - k)(t - s) + 2 - \#_{S}^{(1)} - \#_{T}^{(1)} - 2 \geq \deg_{G' - S}(T). \tag{13}
\]

From (11) and the assumption \(m \leq k\),

\[
k(t - s) \geq 2 - 2m \geq 2 - 2k
\]

\[
t - s \geq \frac{2}{k} - 2
\]

\[
t - s \geq -1. \tag{12}
\]

From (10),

\[
h_{G'}(B) + k(t - s) + \#_{S}^{(1)} - \#_{T}^{(1)} - 2 \geq \deg_{G' - S}(T). \tag{13}
\]

Using Lemma 1 (i) and since \(G\) is \(r\)-regular, (13) implies

\[
h_{G'}(B) \geq (r - k)(t - s) + 2 - \#_{S}^{(1)} - \#_{T}^{(1)} + q_1(S, T) + q_2(S, T) + 2q_{G}(S). \tag{14}
\]

On the other hand, using Lemma 1 (ii), (13) implies

\[
-4 \geq \ell \omega - 2h_{G'}(B) + (r - 2k)(t - s) - 2(m + n). \tag{15}
\]

We now consider the cases \(\omega \geq 1\) and \(\omega = 0\) separately.
Case 1. \( \omega \geq 1 \).

If \( m = 0 \), \( t - s \geq 1 \) by (11). Then by (15), we have
\[
-4 \geq \ell - 2 + r - 2k - 2n \\
n \geq \frac{\ell + r}{2} - k + 1,
\]
which contradicts our assumption \( \lceil \frac{\ell + r}{2} \rceil - k \geq n \) of Theorem 1(i).

If \( 1 \leq m \leq \frac{k}{2} \), \( t - s \geq 0 \) by (11). Then by (15), we have
\[
-4 \geq \ell - 2 - 2(m + n) \\
m + n \geq \frac{\ell}{2} + 1.
\]
This contradicts our assumption \( \lceil \frac{\ell}{2} \rceil \geq m + n \) of Theorem 1(ii).

If \( \frac{k}{2} < m \leq k \), \( t - s \geq -1 \) by (11). Then by (15), we have
\[
-4 \geq \ell - 2 - r + 2k - 2(m + n) \\
m + n \geq \frac{\ell - r}{2} + k + 1.
\]
This contradicts our assumption \( \lceil \frac{\ell - r}{2} \rceil + k \geq m + n \) of Theorem 1(iii).

Case 2. \( \omega = 0 \).

Then we have
\[
rs = 2q_G(S) + q_G(S, T) \\
rt = 2q_G(T) + q_G(S, T).
\]
Thus, we obtain \( 2q_G(S) = r(s - t) + 2q_G(T) \). Then by (14) we have
\[
0 \geq 2 + (r - k)(t - s) - 2q_1(S) - 2q_2(T) + 2q_G(S) \tag{16}
= 2 - k(t - s) - 2q_1(S) - 2q_2(T) + 2q_G(T). \tag{17}
\]
Assume \( t - s \geq 2 \). By the hypotheses of Theorem 1, we have \( k \leq r - n \). Since \( q_1(S) \leq q_G(S) \) and \( q_2(T) \leq n \), from (16), we have
\[
0 \geq 2r - 2(r - n) + 2 - 2n = 2.
\]
This is a contradiction. Thus we may assume \( t - s \leq 1 \). Hence, we obtain \( |t - s| \leq 1 \) since \( t - s \geq -1 \) by (12). So from (17), we have
\[
q_1(S) + q_2(T) + \frac{k}{2}(t - s) - 1 \geq q_G(T). \tag{18}
\]
We consider the cases \( t - s = 1 \), \( t - s = 0 \) and \( t - s = -1 \) separately.
Case 2-1. $t - s = 1$.

From (18), we have.

$$q_1(S) + q_2(T) + \frac{k}{2} - 1 \geq q_G(T) = q_G(S) + \frac{r}{2}.$$  

Then, $G$ becomes “nearly” bipartite of type [I].

Case 2-2. $t - s = 0$.

If $q_G(T) = 0$, then $q_G(S) = q_1(S) = q_2(T) = 0$. Thus we have $-1 \geq 0$ by (18). This is a contradiction. Thus $q_G(T) \geq 1$. Therefore we obtain

$$q_1(S) + q_2(T) - 1 \geq q_G(T) = q_G(S) \geq 1.$$  

Then, $G$ becomes “nearly” bipartite of type [II].

Case 2-3. $t - s = -1$.

From (18), we have

$$q_1(S) + q_2(T) - \frac{k}{2} - 1 \geq q_G(T) = q_G(S) - \frac{r}{2}.$$  

Then, $G$ becomes “nearly” bipartite of type [III]. This completes the proof of Theorem 1.

3. Sharpness

In this section, we show conditions (i), (ii) and (iii) in Theorem 1 are the weakest possible. First, we establish further properties of the “nearly” bipartite graphs in Theorem 1.

**Corollary 1.** Let $k, \ell, m, n, r, G$ be as in Theorem 1. Suppose $(E_1, E_2)$ is an $(m, n; k)$-obstruction in $G$ and that $(S, T)$ is a partition of $V(G)$ that qualifies $G$ as “nearly” bipartite. Then $q_G(S) + q_G(T) < \frac{r}{2} + \ell - k$.

**Proof.** We can easily check that the following inequality holds since $k \leq \frac{r}{2}$.

$$k + \left\lceil \frac{\ell - r}{2} \right\rceil \leq \left\lfloor \frac{\ell}{2} \right\rfloor \leq \left\lceil \frac{\ell + r}{2} \right\rceil - k.$$  

(19)

Thus the upper bound on $m + n$ provided by Theorem 1 decrease from condition (i) to condition (ii) to condition (iii).

First, we consider the case of “nearly” bipartite of type [I]. Then, we can use the bound on $m + n$ provided by condition (i) in Theorem 1 and (19) to establish the following.

$$q_G(S) + q_G(T) \leq 2(q_1(S) + q_2(T)) + k - 2 - \frac{r}{2} \leq 2(m + n) + k - 2 - \frac{r}{2} \leq \ell + r + 1 - 2k + k - 2 - \frac{r}{2} = \frac{r}{2} + \ell - k - 1.$$
Next, we consider the case of “nearly” bipartite of type [II]. Then, we have
\[ q_1(S) \geq q_G(T) - q_2(T) + 1 > 0. \]  
(20)

Thus, condition (i) of Theorem 1 cannot apply and we can use condition (ii) of Theorem 1 and (19) and (20) to establish the following.
\[ q_G(S) + q_G(T) \leq 2(q_1(S) + q_2(T)) - 2 \leq 2(m + n) - 2 \]
\[ \leq \ell + 1 - 2 = \ell - 1 \leq \frac{r}{2} + \ell - k - 1. \]

Finally, we consider the case of “nearly” bipartite of type [III]. Then, we have
\[ q_1(S) \geq q_G(T) - q_2(T) + \frac{k}{2} + 1 > \frac{k}{2}. \]
(21)

Since neither condition (i) or (ii) of Theorem 1 can apply, we use condition (iii) of Theorem 1 and (19) and (21) to establish the following.
\[ q_G(S) + q_G(T) \leq 2(q_1(S) + q_2(T)) - k - 2 + \frac{r}{2} \leq 2(m + n) - k - 2 + \frac{r}{2} \]
\[ \leq 2k + \ell - r + 1 - k - 2 + \frac{r}{2} = k - \frac{r}{2} + \ell - 1 \leq \frac{r}{2} - k + \ell - 1. \]

Let \( k, \ell, m, n, r \) be as in Theorem 1. Suppose that these integers satisfy one of the following three conditions:

(a) \( m = 0, \ell \geq k \) and \( n = \left\lceil \frac{r + \ell}{2} \right\rceil - k + 1; \)

(b) \( 1 \leq m \leq \frac{k}{2}, n \geq 1 \) and \( m + n = \left\lceil \frac{\ell}{2} \right\rceil + 1; \) or

(c) \( \frac{k}{2} < m \leq k \) and \( m + n = k + \left\lceil \frac{\ell - r}{2} \right\rceil + 1. \)

In each of the enumerated cases above, we can construct a graph \( G \in G(r, \ell) \) such that \( G \notin F(m, n; k) \) and \( G \) is not “nearly” bipartite. For the sake of brevity, we will provide details of such a graph in Case (a) with \( r \) even as all other cases may be addressed with minor modifications. Note when \( r \equiv 0 \) (mod 2) we must also have \( \ell \equiv 0 \) (mod 2). (If there is some \( X \subseteq V(G) \) such that \( q_G(X, V(G) - X) \equiv 1 \) (mod 2), then we have \( q_G(X) \equiv 1 \) (mod 2) since \( r|X| \equiv 0 \) (mod 2) by \( r \) even. However this contradicts the fact \( q_G(X) \equiv 0 \) (mod 2) for every \( X \) of \( V(G) \).) Let \( S = \{s_1, s_2, \ldots, s_{r-1}\}, T = \)
\{t_1, t_2, \ldots, t_r\}, \quad U = \{u^i_j : 1 \leq i \leq 7, 1 \leq j \leq r\} \text{ and let } V(G) = S \cup T \cup U, \text{ that is, let } B = (S, T, U) \text{ be a partition of } V(G). \text{ Now add edges}

\begin{align*}
E_G(S, T) &= \{s_i t_j : 1 \leq i \leq r, 1 \leq j \leq r\} \setminus \{s_i t_i : 1 \leq i \leq \ell - k + 1\}, \\
E_G(U) &= \left\{u^h_i u^h_j : 1 \leq h \leq 7, 1 \leq i < j \leq r\right\} \\
&\cup \left\{u^h_i u^{h+1}_j : 1 \leq h \leq 6, 1 \leq i \leq \frac{r}{2}, j = i + \frac{r}{2}\right\} \\
&\cup \left\{u^h_i u^j_j : 1 \leq i \leq \frac{r}{2}, j = i + \frac{r}{2}\right\} \setminus \{u^1_i u^1_{i+1} : 1 \leq i \leq \ell - 1, i \text{ is odd}\}, \\
E_G(S, U) &= \{s_i u^1_i : 1 \leq i \leq \ell - k + 1\}, \\
E_G(T, U) &= \{t_i u^1_i : 1 \leq i \leq k - 1, j = i + \ell - k + 1\} \text{ and} \\
E_G(T) &= E_2 = \begin{cases} \\
\{t_i t_{i+1} : 1 \leq i \leq r, i \text{ is odd}\} \cup \{t_i t_{i+1} : k \leq i \leq \ell - k, i \text{ is even}\} & \text{if } \ell - k + 1 \geq k - 1; \\
\{t_i t_{i+1} : 1 \leq i \leq \ell - k - 1, i \text{ is odd}\} \cup \{t_{\ell-k+1} t_k\} & \cup \{t_i t_{i+1} : k + 1 \leq i \leq r - 1, i \text{ is odd}\} & \text{otherwise.} \\
\end{cases}
\end{align*}

The resulting graph \( G \) is \( r \)-regular, \( \ell \)-edge-connected. Now let \( G', f \) be as in the proof of Theorem 1. Then applying Theorem C, we obtain \( \delta_{G'}(B) = h_{G'}(B) + k - (k - 1) \geq 2 \) holds. Hence by Claim 2, \( G' \) does not contain an \( f \)-factor, that is, \( G \) does not have a \( k \)-factor avoiding \( E_2 \). But by construction, \( G \) contains at least \( 7\left\lfloor \frac{r}{3} \right\rfloor \) vertex disjoint triangles. If \( \frac{r}{2} + \ell - k > 7\left\lfloor \frac{r}{3} \right\rfloor \), then we have \( 7\left(\frac{r-2}{3}\right) \leq 7\left\lfloor \frac{r}{3} \right\rfloor < \frac{r}{2} + \ell - k \leq \frac{3}{2}r - 2 \) since \( r \geq \ell \). We multiply both sides by 6. Then we have \( 14r - 28 < 9r - 12 \). Thus we have \( r < \frac{16}{3} \). However, this contradicts our assumption \( r \geq 4 \). Thus \( \frac{r}{2} + \ell - k \leq 7\left\lfloor \frac{r}{3} \right\rfloor \). So by Corollary 1, \( G \) is not “nearly” bipartite.

We show a counterexample when \( r = 8, \ell = 4, k = 4, m = 0 \) and \( n = 3 \) in Figure 1. Let \( K_8 \) be a complete graph with 8 vertices and let \( K_8^{-2} \) be a graph that deleting 2 independent edges from \( K_8 \).

![Figure 1: A counterexample under the assumption \( r = 8, \ell = 4, k = 4, m = 0 \) and \( n = 3 \) in Theorem 1 (i).](image)
References


