AKCE J. Graphs. Combin., 5, No. 1 (2008), pp. 61-71

k-FACTORS IN REGULAR GRAPHS AND EDGE-CONNECTIVITY

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Communicated by: Mariko Hagita
Received 8 February 2008; revised 10 April 2008; accepted 28 April 2008

Abstract

Let ℓ, m, n, r be integers such that $2 \leq \ell \leq r$, $r \geq 4$ and m, $n \geq 0$. Suppose G is an r-regular ℓ-edge-connected graph and that k is an even integer with $m \leq k \leq \frac{r}{2}$. We say that G is an $(m, n; k)$-factor graph if for each disjoint pair $E_1, E_2 \subseteq E(G)$ with $|E_1| = m$ and $|E_2| = n$, G has a k-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \emptyset$. In this note we consider when G is an $(m, n; k)$-factor graph and characterize those graphs which fail for certain parameters ℓ, m, n, r, k.

Keywords: regular graph, regular factor, edge-connectivity

2000 Mathematics Subject Classification: 05C70

1. Introduction

We consider finite undirected graphs that may have *multiple edges*. Let G be a graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. For $x \in V(G)$, we denote by $\deg_G(x)$ the *degree* of x in G and for $S \subseteq V(G)$, we write $\deg_G(S)$ instead of $\sum_{x \in S} \deg_G(x)$. We denote by $G[X]$ the subgraph of G induced by X for a subset X of $V(G)$. The number of components of a graph G is denoted by $\omega(G)$. If $\deg_G(x) = r$ for each $x \in V(G)$, we call the graph an r*-regular graph*. For subsets S and T of $V(G)$, we denote by $E_G(S, T)$ the set of the edges joining S and T and let $q_G(S, T) = |E_G(S, T)|$. Let $q_G(S) = q_G(S, S)$. Note that $q_G(S) \equiv 0 \pmod 2$ for every S of $V(G)$. Let f be an integer-valued function defined on $V(G)$, denoted by $f : V(G) \to Z^+ \cup \{0\}$ and for $S \subseteq V(G)$, we write $f(S)$ instead of $\sum_{x \in S} f(x)$. A spanning subgraph F of G such that $\deg_F(x) = f(x)$ for each $x \in V(G)$ is called a f-*factor* of G. If f is a constant function taking a value k, an f-factor is called an k-*factor*.

Suppose that $\ell \geq 2$ and $r \geq 4$ are integers with $r \geq \ell$. Let $\mathcal{G}(r,\ell)$ be the set of finite r-regular, ℓ-edge-connected graphs. Let $G \in \mathcal{G}(r,\ell)$ and $E^* \subseteq E(G)$. Suppose that (E_1, E_2) is a partition of E^* such that:

(1) $|E_1| = m$, $|E_2| = n$;

(2) there exists a k-factor F of G with $m \leq k \leq \frac{r}{2}$ such that $E_1 \subseteq E(F)$ and $E_2 \subseteq E(G) \setminus E(F)$.

Then we say that (E_1, E_2) is a *potential* $(m,n;k)$-factor of G; otherwise we say that (E_1, E_2) is an $(m,n;k)$-*obstruction.* If G contains no $(m,n;k)$-obstruction, then G is said to be an $(m,n;k)$-*factor graph.* Let $F(m,n;k)$ be the set of $(m,n;k)$-factor graphs. When $r = \ell$ and $G \in \mathcal{G}(r,r)$, Aldred, Holton and Sheehan [1] and Egawa and Kotani [2] investigated potential $(m,n;k)$-factors in G. Their results are detailed below.

The main result of this paper extends this work to graphs $G \in \mathcal{G}(r,\ell)$ with $\ell \leq r$ in the case when k is even (note [1] considered only the case $k = 2$).

We define a graph G to be "nearly" bipartite as follows. Set $E^* \subseteq E(G)$. Let (S,T) be a partition of $V(G)$ with $|S| = s$, $|T| = t$ and let (E_1, E_2) be a partition of E^*. Define $q_i(S,T) = |E_i \cap E_G(S,T)|$ and $q_i(S) = q_i(S,S)$ with $1 \leq i \leq 2$. We also require that one of the following holds. Either:

(I) $s = t - 1$, r and k are even, and $q_1(S) + q_2(T) + \frac{k}{2} - 1 \geq q_G(T) = q_G(S) + \frac{r}{2}$;

(II) $s = t$ and $q_1(S) + q_2(T) - 1 \geq q_G(T) = q_G(S) \geq 1$; or

(III) $s = t + 1$, r and k are even and $q_1(S) + q_2(T) - \frac{k}{2} - 1 \geq q_G(T) = q_G(S) - \frac{r}{2}$.

Clearly a "nearly" bipartite graph has an $(m,n;k)$-obstruction for suitably chosen m,n,k. Theorems , and our main result show that if $G \in \mathcal{G}(r,\ell)$ then $G \in F(m,n;k)$ for suitably restricted m,n,k unless G is "nearly" bipartite.

The following theorem was proved by R. E. L. Aldred, D. A. Holton and J. Sheehan.

Theorem A. [1] *Let m,n,r be integers with $m,n \geq 0$ and $r \geq 4$, and let $G \in \mathcal{G}(r,r)$, and let E_1, E_2 be disjoint edge sets in G with $|E_1| = m$ and $|E_2| = n$. Suppose that m and n satisfy one of the following three conditions:*

(i) *$m = 0$ and $n \leq r - 2$;*

(ii) *$m = 1$ and $n \leq \lceil \frac{r}{2} \rceil - 1$; or*

(iii) *$m = 2$ and $n = 0$.*

Then either G has a 2-factor F with $E(F) \supseteq E_1$ and $E(F) \cap E_2 = \emptyset$ or G is "nearly" bipartite.

This was generalized by Egawa and Kotani in the following theorem.

Theorem B. [2] *Let m, n, r, k be integers with $m, n \geq 0$, $k \geq 2$ and $r \geq 2k$, and let $G \in \mathcal{G}(r, r)$, and let E_1, E_2 be disjoint edge sets in G with $|E_1| = m$ and $|E_2| = n$. Suppose that m and n satisfy one of the following three conditions:*

(i) $m = 0$ *and* $n \leq r - k$ *;*

(ii) $1 \leq m \leq \frac{k}{2}$ *and* $m + n \leq \lceil \frac{r}{2} \rceil$ *; or*

(iii) $\frac{k}{2} < m \leq k$ *and* $m + n \leq k$ *.*

Then either G has a k-factor F with $E(F) \supseteq E_1$ and $E(F) \cap E_2 = \emptyset$ or G is "nearly" bipartite.

In both Theorems A and B, G is required to be r-regular and r-edge-connected. Here we weaken the restriction on edge-connectivity when k is any even integer and obtain our main result.

Theorem 1. *Let ℓ, m, n and r be integers with $2 \leq \ell \leq r$, $r \geq 4$ and $m, n \geq 0$, and k be an even integer with $m \leq k \leq \frac{r}{2}$, and let $G \in \mathcal{G}(r, \ell)$, and let E_1, E_2 be disjoint edge sets in G with $|E_1| = m$ and $|E_2| = n$. Suppose that m and n satisfy one of the following three conditions:*

(i) $m = 0$ *and* $n \leq \lceil \frac{\ell + r}{2} \rceil - k$ *;*

(ii) $1 \leq m \leq \frac{k}{2}$ *and* $m + n \leq \lceil \frac{\ell}{2} \rceil$ *; or*

(iii) $\frac{k}{2} < m \leq k$ *and* $m + n \leq k + \lceil \frac{\ell - r}{2} \rceil$ *.*

Then either G has a k-factor F with $E(F) \supseteq E_1$ and $E(F) \cap E_2 = \emptyset$ or G is "nearly" bipartite.

Our proof of Theorem 1 makes use of Tutte's f-factor theorem. Let G be a graph, f be an integer-valued function defined on $V(G)$ and let $B = (S, T, U)$ be a partition of $V(G)$. We define $\delta_G(B)$ by

$$\delta_G(B) = h_G(B) - f(S) + f(T) - \deg_{G-S}(T),$$

where $h_G(B)$ is the number of components C of $G[U]$ such that $f(V(C)) + q_G(V(C), T)$ is odd. These components are called *odd* components.

Theorem C. [4] *Let G be a graph, and let f be an integer-valued function defined on $V(G)$. Then*

(i) $\delta_G(B) \equiv f(V(G)) \pmod 2$ *for any partition* B *of* $V(G)$, *and*

(ii) G *has an* f*-factor if and only if* $\delta_G(B) \le 0$ *for any partition* B *of* $V(G)$.

We shall also use the lemma below which generalizes a result of Katerinis [3]. In this lemma we make use of the following notation. For each $v \in V(G)$ we denote by $\#^{(i)}_{\{v\}}$, the number of edges in E_i incident with vertex v, $(i = 1, 2)$ and for each $X \subseteq V(G)$ we write $\#^{(i)}_X$ instead of $\sum_{v \in X} \#^{(i)}_{\{v\}}$, $(i = 1, 2)$.

Lemma 1. *Let* ℓ, m, n *be integers with* $\ell \ge 2$ *and let* G *be an* ℓ*-edge-connected graph. Suppose* $B = (S, T, U)$ *is a partition of* $V(G)$ *and* $E^* \subseteq E(G)$ *is partitioned as that* $E^* = (E_1, E_2)$ *with* $|E_1| = m$ *and* $|E_2| = n$. *Let* $G' = G - E^*$ *and let* $\omega = \omega(G'[U])$. *Then*

(i) $\deg_{G'-S}(T) \ge \deg_G(T) - \deg_G(S) - 2(q_1(T) + q_2(T)) - q_1(T, U) - q_2(T, U) + 2q_G(S)$, *and*

(ii) $2\deg_{G'-S}(T) \ge \ell\omega + \deg_G(T) - \deg_G(S) - 2(m + n) + 2\#^{(1)}_S$.

Proof. Let $\ell, m, n, G, B, E^*, G'$ and ω be as stated in the hypotheses of Lemma 1. Suppose $C_1, C_2, \ldots, C_\omega$ are the components of $G'[U]$ and a_i (resp. b_i) the number of edges of E^* joining C_i to $S \cup T$ (resp. to the other C_j's). Clearly

$$\begin{aligned} \deg_G(S) &= 2q_G(S) + q_G(S, T) + q_G(S, U) && (1) \\ &\ge 2q_G(S) + q_G(S, T). && (2) \end{aligned}$$

But

$$q_G(S, T) = \deg_G(T) - \deg_{G-S}(T) \qquad (3)$$

and

$$\deg_{G-S}(T) = \deg_{G'-S}(T) + 2(q_1(T) + q_2(T)) + q_1(T, U) + q_2(T, U). \qquad (4)$$

Substituting (3) and (4), in (2), we have

$$\deg_G(S) \ge 2q_G(S) + \deg_G(T) - \deg_{G'-S}(T) - 2(q_1(T) + q_2(T)) - q_1(T, U) - q_2(T, U).$$

Therefore, Lemma 1 (1) holds.

On the other hand, we have

$$q_1(S, U) + q_1(T, U) + q_2(T, U) \le \sum_{i=1}^{\omega} a_i. \qquad (5)$$

Further, since $|E^*| = m + n$, $\sum_{i=1}^{\omega} a_i + (\frac{1}{2}) \sum_{i=1}^{\omega} b_i + q_1(S) + q_1(T) + q_2(T) + q_1(S, T) \le m + n$. Thus

$$2(q_1(T) + q_2(T)) \le 2(m + n) - 2\sum_{i=1}^{\omega} a_i - \sum_{i=1}^{\omega} b_i - 2(q_1(S) + q_1(S, T)). \qquad (6)$$

Substituting (4), (5) and (6) in (3), we have

$$q_G(S,T) \geq \deg_G(T) - \deg_{G'-S}(T) - 2(m+n) + \sum_{i=1}^{\omega}(a_i+b_i) + 2(q_1(S)+q_1(S,T)) + q_1(S,U). \tag{7}$$

From (1)

$$\deg_G(S) \geq 2q_1(S) + q_G(S,T) + q_{G'}(S,U) + q_1(S,U). \tag{8}$$

Since G is ℓ-edge-connected, we have $\sum_{i=1}^{\omega} q_G(V(C_i), V(G-V(C_i))) \geq \ell\omega$. But

$$\sum_{i=1}^{\omega} q_G(V(C_i), V(G-V(C_i))) \leq \sum_{i=1}^{\omega}(a_i+b_i) + \deg_{G'-S}(T) + q_{G'}(S,U).$$

Thus

$$\sum_{i=1}^{\omega}(a_i+b_i) + \deg_{G'-S}(T) + q_{G'}(S,U) \geq \ell\omega.$$

Hence $q_{G'}(S,U) \geq \ell\omega - \sum_{i=1}^{\omega}(a_i+b_i) - \deg_{G'-S}(T)$ and so (8) implies,

$$\deg_G(S) - \ell\omega + \sum_{i=1}^{\omega}(a_i+b_i) + \deg_{G'-S}(T) \geq 2q_1(S) + q_G(S,T) + q_1(S,U). \tag{9}$$

From (7) and (9),

$$\deg_G(S) - \ell\omega + 2\deg_{G'-S}(T) \geq \deg_G(T) - 2(m+n) + 2(2q_1(S) + q_1(S,T) + q_1(S,U)).$$

Therefore

$$2\deg_{G'-S}(T) \geq \ell\omega + \deg_G(T) - \deg_G(S) - 2(m+n) + 2\#_S^{(1)}.$$

□

2. Proof of Theorem 1

Throughout the proof we assume $G \in \mathcal{G}(r,\ell)$ and that $E^* \subseteq E(G)$. We set $G' = G - E^*$ and suppose that (E_1,E_2) is a partition of E^* with $|E_1| = m$ and $|E_2| = n$.

Now define a function $f : V(G) \to \{k-m,\dots,k\}$ such that $f(v) = k - \#_{\{v\}}^{(1)}$ for $v \in V(G)$. An f-factor is a spanning subgraph of G, in which each vertex v has degree $f(v)$. Then following two claims are immediate:

Claim 1. *G' has an f-factor if and only if (E_1,E_2) is a potential $(m,n;k)$-factor;*

Claim 2. *G' does not contain an f-factor if and only if there is some partition $B = (S,T,U)$ of G' such that $\delta_{G'}(B) = h_{G'}(B) - f(S) + f(T) - \deg_{G'-S}(T) \geq 2$.*

We now assume that (E_1, E_2) is an $(m,n;k)$-obstruction of G with $E^* = E_1 \cup E_2$. From Claims 1 and 2, we may choose a partition $B=(S,T,U)$ of G' such that $\delta(B) \geq 2$. Set $s=|S|$, $t=|T|$ and let $\omega = \omega(G'[U])$. Then we have,

$$\delta_{G'}(B) = h_{G'}(B) - ks + \#_S^{(1)} + kt - \#_T^{(1)} - \deg_{G'-S}(T) \geq 2. \tag{10}$$

Now we prove the following claim.

Claim 3. $\deg_{G'-S}(T) - h_{G'}(B) - \#_S^{(1)} + \#_T^{(1)} \geq -2m$.

Proof. Let C be an odd component of $G'[U]$. We denote by h_{odd} (h_{even}) the number of odd components in G' such that $f(C)$ is odd(even). Note that $h_{G'}(B) = h_{odd} + h_{even}$. By the definition of odd component, we have $h_{odd} \leq 2q_1(U) + q_1(T,U) + q_1(S,U)$ and $h_{even} \leq q_{G'}(T,U)$ since k is even. Thus,

$$\begin{aligned}
\deg_{G'-S}(T) - h_{G'}(B) - \#_S^{(1)} + \#_T^{(1)} &\geq 2q_{G'}(T) + q_{G'}(T,U) - (2q_1(U) + q_1(T,U) \\
&\quad + q_1(S,U) + q_{G'}(T,U)) - q_1(S,T) - 2q_1(S) \\
&\quad - q_1(S,U) + q_1(S,T) + 2q_1(T) + q_1(T,U) \\
&= 2q_1(T) + 2q_{G'}(T) - (2q_1(S) + 2q_1(U) \\
&\quad + 2q_1(S,U)) \\
&\geq -2m.
\end{aligned}$$

□

Thus using Claim 3 and (10),

$$\begin{aligned}
k(t-s) &\geq 2 + \deg_{G'-S}(T) - h_{G'}(B) - \#_S^{(1)} + \#_T^{(1)} \\
&\geq 2 - 2m.
\end{aligned} \tag{11}$$

From (11) and the assumption $m \leq k$,

$$\begin{aligned}
k(t-s) &\geq 2 - 2m \geq 2 - 2k \\
t-s &\geq \frac{2}{k} - 2 \\
t-s &\geq -1.
\end{aligned} \tag{12}$$

From (10),

$$h_{G'}(B) + k(t-s) + \#_S^{(1)} - \#_T^{(1)} - 2 \geq \deg_{G'-S}(T). \tag{13}$$

Using Lemma 1 (i) and since G is r-regular, (13) implies

$$h_{G'}(B) \geq (r-k)(t-s) + 2 - \#_S^{(1)} - \#_T^{(2)} + q_1(S,T) + q_2(S,T) + 2q_G(S). \tag{14}$$

On the other hand, using Lemma 1 (ii), (13) implies

$$-4 \geq \ell\omega - 2h_{G'}(B) + (r-2k)(t-s) - 2(m+n). \tag{15}$$

We now consider the cases $\omega \geq 1$ and $\omega = 0$ separately.

Case 1. $\omega \geq 1$.

If $m = 0$, $t - s \geq 1$ by (11). Then by (15), we have

$$\begin{aligned} -4 &\geq \ell - 2 + r - 2k - 2n \\ n &\geq \frac{\ell + r}{2} - k + 1, \end{aligned}$$

which contradicts our assumption $\lceil \frac{\ell+r}{2} \rceil - k \geq n$ of Theorem 1(i).

If $1 \leq m \leq \frac{k}{2}$, $t - s \geq 0$ by (11). Then by (15), we have

$$\begin{aligned} -4 &\geq \ell - 2 - 2(m + n) \\ m + n &\geq \frac{\ell}{2} + 1. \end{aligned}$$

This contradicts our assumption $\lceil \frac{\ell}{2} \rceil \geq m + n$ of Theorem 1(ii).

If $\frac{k}{2} < m \leq k$, $t - s \geq -1$ by (11). Then by (15), we have

$$\begin{aligned} -4 &\geq \ell - 2 - r + 2k - 2(m + n) \\ m + n &\geq \frac{\ell - r}{2} + k + 1. \end{aligned}$$

This contradicts our assumption $\lceil \frac{\ell-r}{2} \rceil + k \geq m + n$ of Theorem 1(iii).

Case 2. $\omega = 0$.

Then we have

$$\begin{aligned} rs &= 2q_G(S) + q_G(S, T) \\ rt &= 2q_G(T) + q_G(S, T). \end{aligned}$$

Thus, we obtain $2q_G(S) = r(s - t) + 2q_G(T)$. Then by (14) we have

$$\begin{aligned} 0 &\geq 2 + (r - k)(t - s) - 2q_1(S) - 2q_2(T) + 2q_G(S) &(16)\\ &= 2 - k(t - s) - 2q_1(S) - 2q_2(T) + 2q_G(T). &(17) \end{aligned}$$

Assume $t - s \geq 2$. By the hypotheses of Theorem 1, we have $k \leq r - n$. Since $q_1(S) \leq q_G(S)$ and $q_2(T) \leq n$, from (16), we have

$$0 \geq 2r - 2(r - n) + 2 - 2n = 2.$$

This is a contradiction. Thus we may assume $t - s \leq 1$. Hence, we obtain $|t - s| \leq 1$ since $t - s \geq -1$ by (12). So from (17), we have

$$q_1(S) + q_2(T) + \frac{k}{2}(t - s) - 1 \geq q_G(T). \tag{18}$$

We consider the cases $t - s = 1$, $t - s = 0$ and $t - s = -1$ separately.

Case 2-1. $t - s = 1$.

From (18), we have.

$$q_1(S) + q_2(T) + \frac{k}{2} - 1 \geq q_G(T) = q_G(S) + \frac{r}{2}.$$

Then, G becomes "nearly" bipartite of type [I].

Case 2-2. $t - s = 0$.

If $q_G(T) = 0$, then $q_G(S) = q_1(S) = q_2(T) = 0$. Thus we have $-1 \geq 0$ by (18). This is a contradiction. Thus $q_G(T) \geq 1$. Therefore we obtain

$$q_1(S) + q_2(T) - 1 \geq q_G(T) = q_G(S) \geq 1.$$

Then, G becomes "nearly" bipartite of type [II].

Case 2-3. $t - s = -1$.

From (18), we have

$$q_1(S) + q_2(T) - \frac{k}{2} - 1 \geq q_G(T) = q_G(S) - \frac{r}{2}.$$

Then, G becomes "nearly" bipartite of type [III]. This completes the proof of Theorem 1. □

3. Sharpness

In this section, we show conditions (i), (ii) and (iii) in Theorem 1 are the weakest possible. First, we establish further properties of the "nearly" bipartite graphs in Theorem 1.

Corollary 1. *Let k, ℓ, m, n, r, G be as in Theorem 1. Suppose (E_1, E_2) is an $(m, n; k)$-obstruction in G and that (S, T) is a partition of $V(G)$ that qualifies G as "nearly" bipartite. Then $q_G(S) + q_G(T) < \frac{r}{2} + \ell - k$.*

Proof. We can easily check that the following inequality holds since $k \leq \frac{r}{2}$.

$$k + \left\lceil \frac{\ell - r}{2} \right\rceil \leq \left\lceil \frac{\ell}{2} \right\rceil \leq \left\lceil \frac{\ell + r}{2} \right\rceil - k. \tag{19}$$

Thus the upper bound on $m + n$ provided by Theorem 1 decrease from condition (i) to condition (ii) to condition (iii).

First, we consider the case of "nearly" bipartite of type [I]. Then, we can use the bound on $m + n$ provided by condition (i) in Theorem 1 and (19) to establish the following.

$$\begin{aligned} q_G(S) + q_G(T) &\leq 2(q_1(S) + q_2(T)) + k - 2 - \frac{r}{2} \leq 2(m+n) + k - 2 - \frac{r}{2} \\ &\leq \ell + r + 1 - 2k + k - 2 - \frac{r}{2} = \frac{r}{2} + \ell - k - 1. \end{aligned}$$

Next, we consider the case of "nearly" bipartite of type [II]. Then, we have

$$q_1(S) \geq q_G(T) - q_2(T) + 1 > 0. \tag{20}$$

Thus, condition (i) of Theorem 1 cannot apply and we can use condition (ii) of Theorem 1 and (19) and (20) to establish the following.

$$\begin{aligned} q_G(S) + q_G(T) &\leq 2(q_1(S) + q_2(T)) - 2 \leq 2(m+n) - 2 \\ &\leq \ell + 1 - 2 = \ell - 1 \leq \frac{r}{2} + \ell - k - 1. \end{aligned}$$

Finally, we consider the case of "nearly" bipartite of type [III]. Then, we have

$$q_1(S) \geq q_G(T) - q_2(T) + \frac{k}{2} + 1 > \frac{k}{2}. \tag{21}$$

Since neither condition (i) or (ii) of Theorem 1 can apply, we use condition (iii) of Theorem 1 and (19) and (21) to establish the following.

$$\begin{aligned} q_G(S) + q_G(T) &\leq 2(q_1(S) + q_2(T)) - k - 2 + \frac{r}{2} \leq 2(m+n) - k - 2 + \frac{r}{2} \\ &\leq 2k + \ell - r + 1 - k - 2 + \frac{r}{2} = k - \frac{r}{2} + \ell - 1 \leq \ell - 1 \leq \frac{r}{2} - k + \ell - 1. \end{aligned}$$

□

Let k, ℓ, m, n, r be as in Theorem 1. Suppose that these integers satisfy one of the following three conditions:

(a) $m = 0$, $\ell \geq k$ and $n = \lceil \frac{r+\ell}{2} \rceil - k + 1$;

(b) $1 \leq m \leq \frac{k}{2}$, $n \geq 1$ and $m + n = \lceil \frac{\ell}{2} \rceil + 1$; or

(c) $\frac{k}{2} < m \leq k$ and $m + n = k + \lceil \frac{\ell - r}{2} \rceil + 1$.

In each of the enumerated cases above, we can construct a graph $G \in \mathcal{G}(r, \ell)$ such that $G \notin F(m, n; k)$ and G is not "nearly" bipartite. For the sake of brevity, we will provide details of such a graph in Case (a) with r even as all other cases may be addressed with minor modifications. Note when $r \equiv 0 \pmod 2$ we must also have $\ell \equiv 0 \pmod 2$. (If there is some $X \subseteq V(G)$ such that $q_G(X, V(G) - X) \equiv 1 \pmod 2$, then we have $q_G(X) \equiv 1 \pmod 2$ since $r|X| \equiv 0 \pmod 2$ by r even. However this contradicts the fact $q_G(X) \equiv 0 \pmod 2$ for every X of $V(G)$.) Let $S = \{s_1, s_2, \dots, s_{r-1}\}$, $T =$

$\{t_1, t_2, \ldots, t_r\}$, $U = \{u_j^i : 1 \le i \le 7, 1 \le j \le r\}$ and let $V(G) = S \cup T \cup U$, that is, let $B = (S, T, U)$ be a partition of $V(G)$. Now add edges

$$E_G(S,T) = \{s_i t_j : 1 \le i \le r-1, 1 \le j \le r\} \setminus \{s_i t_i : 1 \le i \le \ell - k + 1\},$$
$$E_G(U) = \left\{u_i^h u_j^h : 1 \le h \le 7, 1 \le i < j \le r\right\}$$
$$\cup \left\{u_i^h u_j^{h+1} : 1 \le h \le 6, 1 \le i \le \frac{r}{2}, j = i + \frac{r}{2}\right\}$$
$$\cup \left\{u_i^7 u_j^1 : 1 \le i \le \frac{r}{2}, j = i + \frac{r}{2}\right\} \setminus \left\{u_i^1 u_{i+1}^1 : 1 \le i \le \ell - 1, i \text{ is odd}\right\},$$
$$E_G(S,U) = \left\{s_i u_i^1 : 1 \le i \le \ell - k + 1\right\},$$
$$E_G(T,U) = \left\{t_i u_j^1 : 1 \le i \le k-1, j = i + \ell - k + 1\right\} \text{ and}$$
$$E_G(T) = E_2 = \begin{cases} \{t_i t_{i+1} : 1 \le i \le r, i \text{ is odd}\} \cup \{t_i t_{i+1} : k \le i \le \ell - k, i \text{ is even}\} \\ \qquad \text{if } \ell - k + 1 \ge k - 1; \\ \{t_i t_{i+1} : 1 \le i \le \ell - k - 1, i \text{ is odd}\} \cup \{t_{\ell-k+1} t_k\} \\ \quad \cup \{t_i t_{i+1} : k+1 \le i \le r-1, i \text{ is odd}\} \quad \text{otherwise.} \end{cases}$$

The resulting graph G is r-regular, ℓ-edge-connected. Now let G', f be as in the proof of Theorem 1. Then applying Theorem C, we obtain $\delta_{G'}(B) = h_{G'}(B) + k - (k-1) \ge 2$ holds. Hence by Claim 2, G' does not contain an f-factor, that is, G does not have a k-factor avoiding E_2. But by construction, G contains at least $7\lfloor \frac{r}{3} \rfloor$ vertex disjoint triangles. If $\frac{r}{2} + \ell - k > 7\lfloor \frac{r}{3} \rfloor$, then we have $7(\frac{r-2}{3}) \le 7\lfloor \frac{r}{3} \rfloor < \frac{r}{2} + \ell - k \le \frac{3}{2}r - 2$ since $r \ge \ell$. We multiply both sides by 6. Then we have $14r - 28 < 9r - 12$. Thus we have $r < \frac{16}{5}$. However, this contradicts our assumption $r \ge 4$. Thus $\frac{r}{2} + \ell - k \le 7\lfloor \frac{r}{3} \rfloor$. So by Corollary 1, G is not "nearly" bipartite.

We show a counterexample when $r = 8, \ell = 4, k = 4, m = 0$ and $n = 3$ in Figure 1. Let K_8 be a complete graph with 8 vertices and let K_8^{-2} be a graph that deleting 2 independent edges from K_8.

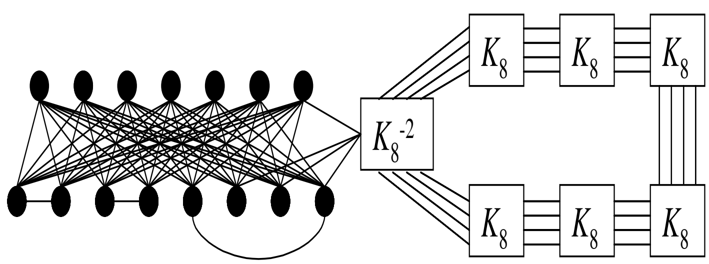

Figure 1: A counterexample under the assumption $r = 8, \ell = 4, k = 4, m = 0$ and $n = 3$ in Theorem 1 (i).

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